Part 2

KINEMATICS

Motion in One and Two Dimensions

Projectile Motion

Circular Motion

Kinematics Problems

KINEMATICS
The Description of Motion

Physics is much more than just the description of motion. But being able to describe the motion of an object mathematically is an important component to the study of physics. Much of the general description will seem like it should be intuitive. But that is largely because we are already familiar with the words used for the description - *ie*, they are a part of our every day vocabulary. But "displacement", "velocity", and "acceleration" have very specific meanings - and the connection to the mathematical description with which you will need to be very conversant is important. After a brief discussion of the ideas involved, the more formal mathematical description will be developed. That is, starting from the fundamental definitions of velocity and acceleration in one-dimension in terms of time derivatives of the position coordinates, expressions can be obtained for the position and velocity as functions of time if there is a known constant acceleration. Those equations will be very useful in some special case problems. We will then extend the ideas to problems involving motion in two and three dimensions. Both projectile motion and circular motion are important examples of two-dimensional motion.

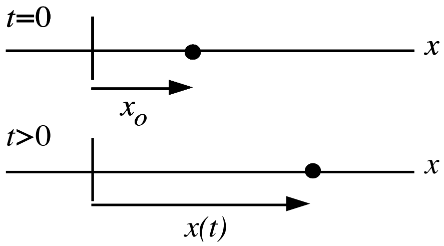
Upon completing this brief study, you should be comfortable with the general description, the mathematical description, and the graphical description of motion, and should be able to go back and forth between those descriptions. (And if you cannot, you should work carefully through the detailed descriptions in the text.)

ONE DIMENSIONAL MOTION

When an object moves in one dimension - for example a bead which is constrained to move along a stretched wire - its motion can be described in a number of different ways. For example, one could just describe in words what the bead *does*, *ie*, where it starts, what direction it moves, how quickly it moves, whether it speeds up or slows down or changes direction of comes to a stop. One could even sketch a series of pictures showing the location of the bead at successive times (think of the individual frames of a movie - or of individual images of a "flip-book"). But as we will see, a *graph* of its position as a function of time can convey a much more precise description of the object's positional history than either the words or the individual "snap-shots" or pictures of its position at different times. And, when appropriate, a formal mathematical description can be more precise yet - as a specific function which represents the position as a function of time can contain all of the information that can be known about the object's motion.

Part of the difficulty - if that is the right word - of kinematics involves the precise use of the words and ideas, which are also a part of our colloquial language. For example "speed" and "velocity" are not interchangeable terms for describing how quickly something moves - partly because "velocity" also contains information about the direction of travel. In addition, it is often necessary to distinguish between ***average*** and ***instantaneous*** values of those quantities. (To say that a race car at Indianapolis averages 225 mph for four laps during qualifying does not mean its speed was constant at 225 mph. It's instantaneous speed varies - slower in the turns and faster on the straights. Its *velocity* is continuously changing as its direction also changes. And its *average* velocity is *zero* for the four laps - because its net displacement is zero if it finishes exactly where it started, at the Start-Finish line!)

In *any* description of motion, it is necessary to establish a coordinate system - say the ***x***-axis, with a well defined origin to that coordinate system. The object's *displacement* is then just its position with respect to the origin, and can be represented by the coordinate $x(t)$ representing its distance along the axis



from the origin as a function of time. Often, it is useful to describe the object's initial position as x_o, where the subscript simply implies that to be the value of the displacement from the origin of coordicates at the time t=0, *ie,* when the problem description starts. The function then simply states where the object is at any other time.

When described in this way, it is common that positive displacements will be to the right and negative displacements will be to the left of the origin. Changes in displacement are defined as $\Delta x = x_2 - x_1$, and can be either positive or negative depending on whether x_1 or x_2 is the larger value. The *average* velocity will always be the change in position divided by the time interval during which the change occurred, that is

Average velocity: $$v_{avg} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t}$$

and the average velocity over the time interval $\Delta t = t_2 - t_1$ can be either positive or negative depending on whether the change in position is positive or negative. This should not be at all confusing, however, because it simply says that an object moving to the right - *ie,* in the positive $\boldsymbol{x}$-direction has a positive velocity and the velocity is negative if it moves to the left.

An instantaneous velocity can then be defined in terms of the average velocity given above for an infinitesimally small time interval - *ie*, as the limit of Δt goes to zero. But that simply definess the derivative of $x(t)$ with respect to time t.

Instantaneous velocity: $$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

Accelerations are similarly defined, that is

Average acceleration: $$a_{avg} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}$$

Instantaneously acceleration: $$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2 x}{dt^2}$$

The acceleration can also be either positive or negative, depending on whether the velocity is increasing or decreasing. (But notice that simply saying an object is slowing does not mean its acceleration is negative. For example, the velocity of an object that is moving to the left but slowing is becoming *less negative* - hence its acceleration is positive.)

The concept of acceleration is typically much less intuitive than that of velocity. Part of the reason for that is that our *sense* of motion is very visual - and we can *see* an object moving and can detect whether it is moving quickly or slowly. That is, our visual sense of an object's motion is very sensitive to its *velocity* - but it is much more difficult to sense acceleration - a quantity that is defined in terms of the rate of change of velocity. It is for that reason that we need to be careful in dealing with the concept of acceleration. And it is acceleration that is ultimately connected to Newton's laws that relate forces to motion.

The Mathematics of One-Dimensional Motion

If *x(t)* is a function that represents position as a function of time, then the velocity and acceleration are given by the derivatives

$$v = \frac{dx}{dt} \quad \text{and} \quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \qquad \text{by definition.}$$

Note that a positive velocity means the direction of motion is in the positive $\boldsymbol{x}$ direction. If the acceleration is positive, it means that the *change* in velocity is positive - *ie*, either a positive velocity is increasing or a velocity is becoming less negative.

The inverse relations can also be written. That is, given the acceleration as a function of time, the velocity and the position as functions of time can be derived:

$$v(t) - v_o = \int_o^t a(t)\,dt \quad \text{and} \quad x(t) - x_o = \int_o^t v(t)\,dt = \int_o^t \left[v_o + \int_o^t a(t)\,dt \right] dt$$

The integrals are particularly easy to do when the acceleration is a constant - *ie*, when $a(t) = a$ – and they lead to the kinematics equations that are commonly used in introductory problems describing motion in one or two dimensions.

The kinematics equations for constant acceleration:

If the acceleration is constant, the time integrals simplify since $\boldsymbol{a}$ comes outside the integrals:

$$v(t) - v_o = a\,t \quad \text{and} \quad x(t) - x_o = \int_o^t (v_o + a\,t)dt = v_o\,t + \tfrac{1}{2}a\,t^2$$

That is, *if the acceleration is a constant* and equal to the value of $\boldsymbol{a}$, then the velocity and position functions are given by

$$\boxed{v(t) = v_o + a\,t \qquad \text{and} \qquad x(t) = x_o + v_o\,t + \tfrac{1}{2}a\,t^2}$$

A third useful equation relates displacement, velocity, and the constant acceleration by eliminating the time variable from the above equations. This third kinematics equation is obtained by solving the $\boldsymbol{v(t)}$ equation for $\boldsymbol{t}$, substituting that expression into the position equation $\boldsymbol{x(t)}$, and then rearranging terms. It is useful, but is not independent of the other two.

$$v^2 - v_o^2 = 2a(x - x_o)$$

Many problems in both one and two dimensional motion will utilize these equations. They will be the most important equations you have to describe the motion of an object. But you should also realize that they are <u>special case</u> - that is, they <u>only</u> apply when the acceleration is a constant.

Examples:

- A car accelerates from rest with an acceleration a. Determine the distance travelled and the speed after a time t. [Notice that v_o =0 and the distance travelled is just x-x_o. So the $x(t)$ equation contains just the right information and you can solve for the distance.]

- To find the distance the car requires to stop from a given speed, it is the final speed that is zero – and the acceleration would be negative. Notice that if the *time* to stop is not needed, just using the equation $v^2 - v_o^2 = 2a(x - x_o)$ is all that is needed.

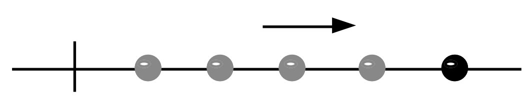
- An object is thrown straight up with an initial speed v_o having been released at a height H above the ground. Determine the maximum height the object achieves, the total time it is in the air, and the speed with which it hits the ground, assuming that the only acceleration after the object is released is due to gravity. [Rewrite the kinematics equations in terms of $y(t)$. Let y_o=H and the acceleration a=-g (since the acceleration is in the negative y-direction).]

  To solve for the maximum height, you need to know how long it takes to reach that height. Set $vy=0$ and solve for the time. Substitute that value of time into the y-equation to find y_{max}. (Ie, you must maximize the function $y(t)$.) To solve for the total time it is in the air, you find the time that it hits the ground - ie, set $y(t)$ =0 and solve for t. Finding the speed with which it hits the ground just requires substituting that value of t into the velocity equation.

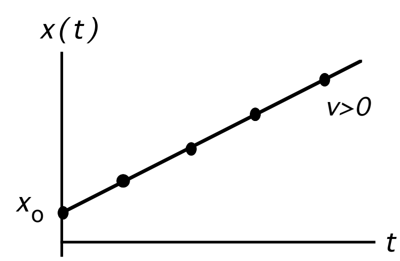
Graphing position, velocity, and acceleration.

The kinematics - or motion - of objects can be described by the equations that locate an object as a function of time - or by simply graphing the position as a function of time. Seeing the graph of an objects position can be very helpful in deducing what the object is actually *doing, ie,* how it is moving just as the equations can be helpful in precisely determining the location of an object at any particular moment in time.

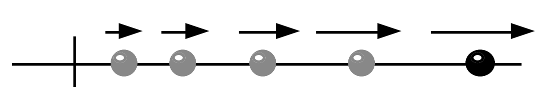
Constant Velocity: $\boldsymbol{x(t) = x_o + v_o t}$



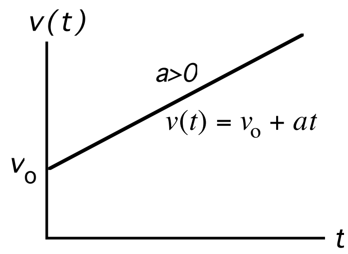
When an object is moving with a constant velocity in one dimension, it's position changes equal amounts in equal increments of time. Hence, a graph of it's position $x(t)$ will be a linear function of time, *ie*, a straight line with the slope of the graph equalling the object's constant velocity. If the slope is positive, the object is moving in the positive x-direction and if negative, then in the negative x-direction.

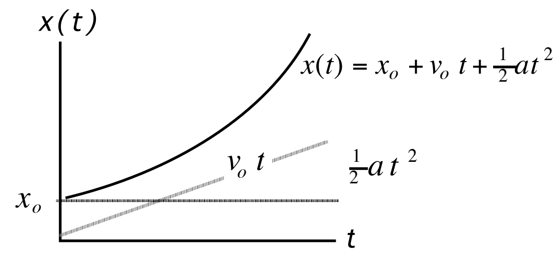


Constant Acceleration: $\boldsymbol{x(t) = x_o + v_o t + \frac{1}{2}at^2}$



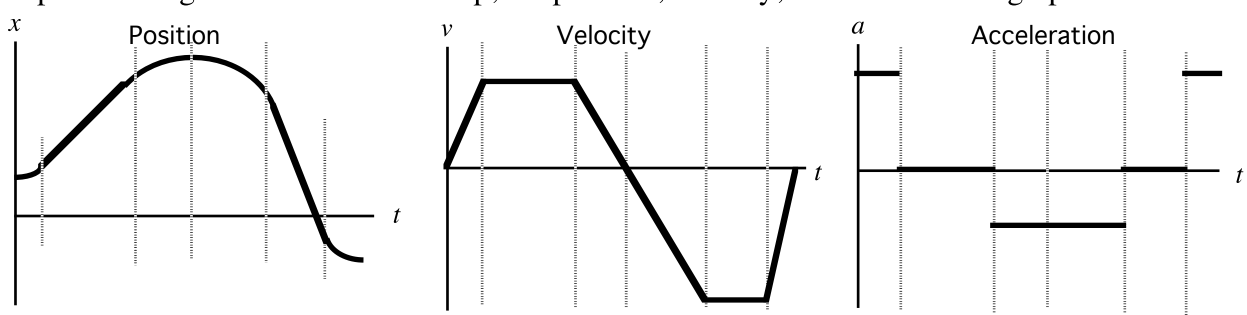
When an object undergoes a constant acceleration in one dimension, its velocity changes an equal amount in equal increments of time. That is, it changes at a constant rate. Hence, graphing the velocity $v(t)$ as a function of time would result in a linear graph whose slope (either positive or negative) represents the value of the acceleration (*ie*, $a=dv/dt$). Similarly, a constant acceleration would mean that the *slope* of the position vs time graph would constantly be changing - resulting in a parabolic function for x *vs* $\boldsymbol{t}$.





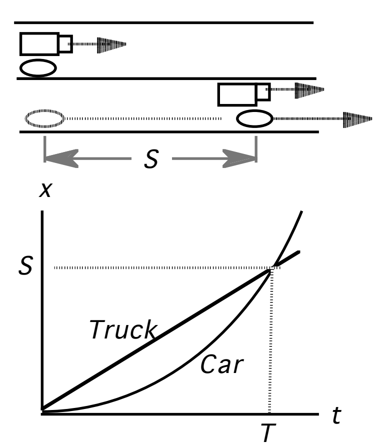
It is important to be able to graph position, velocity, and acceleration as functions of time - either starting from the position function or starting from a known acceleration. If you can plot the position of an object as a function of time, then you can also plot its velocity as a function of time because *v(t)* is just the slope of the *x vs t* graph at each point. Whenever the slope of ***x vs t*** is zero, ***v*** is zero. When the slope of *x vs t* is constant, the velocity curve is flat since that means the velocity is not changing. If the slope of *x vs t* is decreasing, then the velocity is decreasing - either becoming less positive or more negative. A negative velocity just means that ***x*** is decreasing (ie, the object is moving to the left), etc. By the same token, if the velocity is changing, the acceleration is non-zero. If *v vs **t*** is increasing, the acceleration is positive, if it is decreasing, then *a* is negative.

For example, if an object started with zero velocity at some position *x* relative to the origin of coordinates, accelerated up to some constant speed moving to the right, then slowed to as stop and reversed direction all in one smooth transition, accelerated up to a constant speed back toward the origin, went past the origin and slowed to a stop, the position, velocity, and acceleration graphs would look like:



One should also be able to start with the graph of the acceleration versus time and generate the velocity and position curves, since a constant acceleration means a velocity that changes linearly (either increasing or decreasing, depending on the sign of *a*), etc. Note that if *a* is positive, *v vs t* is linearly increasing and *x vs t* curves <u>up</u> (as a parabola). When the acceleration is negative, *x vs t* curves down.

Notice that in a problem involving two objects - like a car and a truck - graphing both as functions of time on the same axes can often give a clear indication of *how* a specific problem should be solved. For example: Suppose a truck is traveling at constant speed. And just as it passes a car which has stopped, the car accelerates at a constant rate. How long will it take before the car passes the truck - and how far will they have traveled during that time?



Graphing both truck and car shows how to solve the problem. The car overtakes the truck when the two functions that describe their motion intersect. So the problem can be solved by setting the equations equal and finding both *t* when the car passes and *S*, how far they've traveled.

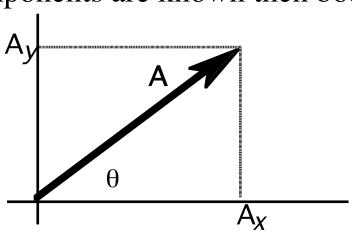
MOTION IN TWO AND THREE DIMENSIONS

The complete description of motion in two and three dimensions requires being able to locate objects in three-dimensional space (rather than just along one axis). Rather than locating an object by a single coordinate, one must use vectors to keep track of two or three coordinates simultaneously – and be able to descibe how those coordinates change in time. What follows is a brief review of the essential properties of vectors.

VECTORS - How to describe quantities which have direction

Any quantity that requires both a magnitude and a direction for its complete description is said to be a ***vector*** quantity. Examples include the displacement of an object from a point of reference (ie, the origin of its coordinate system), or the velocity of an object (with its speed representing the magnitude of the vector and some angle giving its direction), or a force on an object (where the direction of the force is as important as the "strength" of the force).

The complete description of the vector $\vec{\mathbf{A}}$ is given either by its components (A_x, A_y) or by its magnitude and direction ($\boldsymbol{A},\theta$) where the angle θ is measured with respect to the x-axis. If the components are known then both the magnitude and angle can be determined.



$$A = \sqrt{A_x^2 + A_y^2} \quad \text{and} \quad \theta = \arctan\left(\frac{A_y}{A_x}\right)$$

If the magnitude and direction are known, then the components are:

$$A_x = A\cos\theta \qquad \text{and} \qquad A_y = A\sin\theta$$

A particularly useful notation to describe a vector quantity is the **ijk** notation where $\hat{\mathbf{i}}$ is a "unit vector" in the direction of the x-axis, $\hat{\mathbf{j}}$ is a unit vector in the y-direction, and $\hat{\mathbf{k}}$ is a unit vector in the z-direction. That is, the above vector $\vec{\mathbf{A}}$ would be given by:

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} = (A\cos\theta)\hat{\mathbf{i}} + (A\sin\theta)\hat{\mathbf{j}}$$

A Note on Vector Notation:

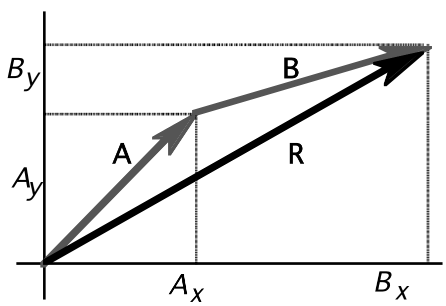
It is useful to note that when referring to a vector quantity – whatever it is – the quantity is identified as a vector by making symbol in boldface type – and including a small arrow over the symbol. Scalar quantities – including the components of vectors – are indicated by italicized and normal type. In your own work, you should always show vector signs over quantities which have both magnitude and direction. That is, the vector statement $\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ is *not* the same as $R=A+B$, since the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are not necessarily collinear, so their magnitudes do not necessarily add.

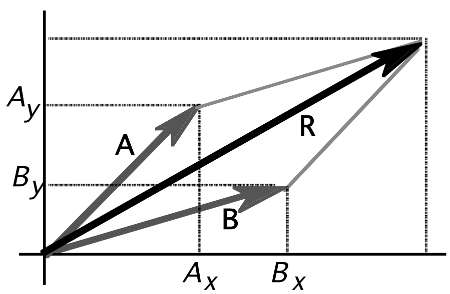
Adding and Subtracting Vectors

Many physics problems require adding vectors. For example, consider the total displacement of an object that first moves an amount described by the displacement vector $\vec{\mathbf{A}}$ followed by a second displacement given by the vector $\vec{\mathbf{B}}$. The total displacement - or "net" displacement - could be given by the vector $\vec{\mathbf{R}}$ (for "resultant vector") which is written:

$$\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}} = \left(A_x + B_x\right)\hat{\mathbf{i}} + \left(A_y + B_y\right)\hat{\mathbf{j}}$$

That is, the x and y components of the total displacement are just the sum of the x components and of the y components of the vectors that were added.





There are two ways to display the vector sum graphically. Either show both vectors drawn in the order of the summation (the so-called "tip-to-tail" method) or show both vectors drawn from the origin and form the parallelogram defined by the two vectors. The two methods are necessarily equivalent.

The parallelogram method may seem more appropriate when two forces are acting on an object simultaneously - and you want to consider the total force or net force that is acting.

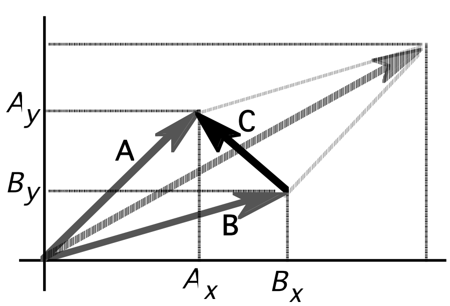
The magnitude of the resultant vector is given by:

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{\left(A_x + B_x\right)^2 + \left(A_y + B_y\right)^2}$$

where the components are just the sums of the components of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. And the direction of the vector $\vec{\mathbf{R}}$ with respect to the x-axis is just $\theta = \arctan\left(R_y / R_x\right)$

Subtracting vectors - just like subtracting numbers - is just the addition of two objects, one of them negative! That is, defining the vector $\vec{\mathbf{C}} = \vec{\mathbf{A}} - \vec{\mathbf{B}}$ is the same as just adding the vectors $\vec{\mathbf{A}}$ and $-\vec{\mathbf{B}}$.
In vector notation:

$$\vec{\mathbf{C}} = \left(A_x - B_x\right)\hat{\mathbf{i}} + \left(A_y - B_y\right)\hat{\mathbf{j}}$$



Notice that the difference of two vectors is represented graphically as the vector from the "tip" of the second vector to the "tip" of the first - *ie*, from $\vec{\mathbf{B}}$ to $\vec{\mathbf{A}}$. Also notice that the vector is just the "other" diagonal of the parallelogram defined by the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

Adding and subtracting vectors will be very important in dealing with motions in two and three dimensions, or with the net force on an object subject to several forces or the total momentum of a system of particles. Be sure you know how to combine the components of vectors.

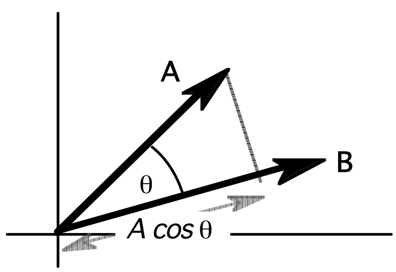
Multiplying Vectors

There are two ways to multiply vectors. Neither method will be important until later in the course. But when they methods are important, you will need to be able to distiguish between them and to interpret the results of multiplying vectors.

Scalar Product

The scalar or "dot" product of two vectors yields a ***scalar*** quantity rather than another vector. The interpretation of the scalar product is that it represents the product of the magnitude of one of the vectors and the component of the other vector parallel to the first.

That is: $$C = \vec{\mathbf{A}} \bullet \vec{\mathbf{B}} = A(B\cos\theta) = B(A\cos\theta)$$



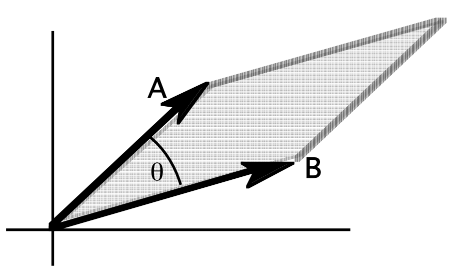
The scalar product is important in the definition of work and potential energy and in the development of the work-energy theorem and ultimately the idea of energy conservation. Notice that the scalar product depends on the components of the vectors parallel to each other. It is a maximum when the two vectors are parallel and zero when **A** and **B** are perpendicular.

Vector Product

The vector or "cross" product of two vectors yields another ***vector*** quantity and is defined in a particularly way. That is:

$$\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$$

where the magnitude of **C** is given by C=$A(B \sin\theta)$ or $B(A \sin\theta)$



The magnitude of the cross product of two vectors is just the area of the rectangle formed or defined by the two vectors being multiplied. The ***direction*** of the cross product vector is perpendicular to the parallelogram. Notice that the vector product is greatest when the two vectors being multiplied are perpendicular and is zero when they are parallel. The vector product is important in the definition of torque and angular momentum.

TWO DIMENSIONAL MOTION PROBLEMS – USING VECTORS

Describing motion in two and three dimensions just requires combining the ideas of motion in one dimension and the descriptions and properties of vectors. The essential "principle" is this: The ***x, y,*** and ***z*** directions can be treated independently. That is, the equations that describe the motion in each direction can be written as if they were a one-dimensional problem. (There are exceptions to this "rule" - although they are rare. If some force acts in the ***y***-direction, but depends on the value of ***x*** , for example, the ***x*** and ***y*** equations cannot be treated independently.) We will also assume that although a problem might be three-dimensional in nature, the motion that occurs can be completely described in two dimensions (for example, when you shoot a basketball, its motion once it leaves your hand is all in a plane regardless of where you were on the court when you took the shot) .

There are really only two types of problems in this chapter. Projectile motion problems are identical to the free-fall problems in the chapter on one dimensional motion with the added complication that there is a horizontal component to the motion. But the acceleration is only due to the gravitational force, so only appears in the ***y***-equations. In all such problems, always begin with the kinematics equations for each coordinate. Assuming that the acceleration is constant, with components ***ax*** and ***ay***., the ***x*** and ***y*** kinematics equations become:

$$\boxed{\begin{array}{ll} x(t) = x_o + v_{ox}\, t + \tfrac{1}{2} a_x\, t^2 & \qquad y(t) = y_o + v_{oy}\, t + \tfrac{1}{2} a_y\, t^2 \\ v_x(t) = v_{ox} + a_x\, t & \qquad v_y(t) = v_{oy} + a_y\, t \end{array}}$$

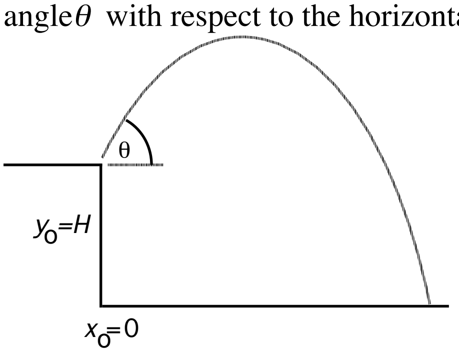
Solving problems then becomes an exercise in using the above kinematics equations by making them specific to the problem given (for example by incorporating given values of the initial position, velocity, and the acceleration components into the equations).

Circular motion problems involve an idea that is difficult to see - but once seen, it will always be true for any object which moves in a circle (hence is worth learning!).

Projectile Motion

Describing the motion of a projectile is a good example of two dimensional motion. We will assume that the only force that acts on the projectile after it is launched (with some initial velocity **vo**) is the gravitational force - which implies that the only acceleration is ***g*** in the negative ***y***-direction. The vertical and horizontal motions are then independent of each other - and can be solved separately.

Suppose a projectile is launched from an initial height ***H***, with an initial velocity **vo** which is at an angle θ with respect to the horizontal. The kinematics equations then become:



$$a_x = 0 \qquad a_y = -g \qquad v_{ox} = v_o \cos\theta \qquad v_{oy} = v_o \sin\theta$$

$$y(t) = y_o + v_{oy}\, t + \tfrac{1}{2} a_y t^2 \Rightarrow y(t) = H + (v_o \sin\theta)\, t - \tfrac{1}{2} g t^2$$

$$v_y(t) = v_{oy} + a_y\, t \qquad \Rightarrow \qquad v_y(t) = v_o \sin\theta - g\, t$$

$$x(t) = x_o + v_{ox}\, t \qquad \Rightarrow \qquad x(t) = (v_o \cos\theta)\, t$$

$$v_x(t) = v_{ox} = v_o \cos\theta$$

There are now a number of quantities that can be solved for: To find the maximum height of the trajectory, you need to find the ***time*** at which it reaches that point of its path. But that is when the ***y***-velocity goes to zero. Then substitute into the ***y(t)*** equation. To find the how long the projectile is in the air, find the ***time*** that it hits the ground - ie, find ***t*** when ***y***=0. The ***range*** or total horizontal distance it travels while it is in the air is just the ***x(t)*** evaluated at the time at which it hits the ground. The velocity it has at that moment is then just found from the two velocity components evaluated at that time - the vector expression being written in vector notation as

$$\vec{\mathbf{v}}(t) = v_x(t)\,\hat{\mathbf{i}} + v_y(t)\,\hat{\mathbf{j}}$$

where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors in the x and y directions, respectively.

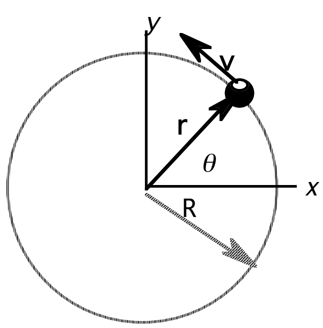
The ***trajectory*** of the projectile (ie, the equation of the path it takes) can be obtained by thinking of the x and y equations as ***parametric*** equations. Solving $x(t)$ for t and substituting into the $y(t)$ equation yields an equation for y in terms of x. Graphing that equation would yield a curve which represents the path that the projectile would follow.

There are many variants on the above projectile motion problem, of course, but they all have the same essential elements.

Circular Motion and Motion in Curved Paths

The other ***class*** of problem that will appear throughout the course are those in which the object you are trying to describe travels in a circle (or just a portion of a circle). Think of all orbit problems, a ball on a string which you whirl around like a slingshot, a pendulum, a car going through a constant radius turn, a roller coaster ride, a test-tube in a centrifuge. All of these problems involve some object subject to various forces moving in an arc or circle. The principles discussed here will apply to all those problems and more.

The essential point to make in this discussion is that even if the speed of the object is constant, the acceleration is not. The reason, of course, is that acceleration is a *vector*. And as the object moves through its path, the direction of its velocity vector keeps changing. And a changing velocity implies a non-zero acceleration - even if it is just the direction of the velocity that is changing. This point will become more clear when we examine the forces that <u>cause</u> the object being described to move in a circle.



Consider an object which moves in a circle of radius R at constant speed. It crosses the x axis at time t=0 and the angle θ increases at a constant rate - that is the angle can be expressed $\theta=\omega t$, where ω is a constant. [The quantity ω is called the angular velocity and is just the rate at which the angle changes - ie, $\omega=d\theta/dt$.] When the object is at the angle θ with respect to the x axis, its coordinates are:

$$x = R\cos\theta = R\cos(\omega t) \qquad \text{and} \qquad y = R\sin\theta = R\sin(\omega t)$$

The vector $\mathbf{r}(t)$ is then written in vector notation as:

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} = R\cos(\omega\, t)\,\hat{\mathbf{i}} + R\sin(\omega\, t)\,\hat{\mathbf{j}}$$

<u>**Velocity**</u>: By definition: $$\mathbf{v}(t) = \frac{d\,\mathbf{r}}{d\,t} = \frac{d\,x}{d\,t}\,\hat{\mathbf{i}} + \frac{d\,y}{d\,t}\,\hat{\mathbf{j}}$$

The velocity vector is then obtained by taking the derivative of the position vector with respect to time. The derivatives of the components of $\mathbf{r}$ just involve taking the derivatives of the sine and cosine functions, since the radius of the circular path R does not change. By chain rule, the derivatives are:

$$\frac{d}{d\,t}\cos(\omega t) = -\omega\sin(\omega t) \qquad \text{and} \qquad \frac{d}{d\,t}\sin(\omega t) = \omega\cos(\omega t)$$

So the velocity of the object in uniform circular motion can be expressed in vector form as

$$\mathbf{v}(t) = -\omega R \sin(\omega t)\,\hat{\mathbf{i}} + \omega R \cos(\omega t)\,\hat{\mathbf{j}}$$

Although it may not be obvious from the mathematical description, the velocity vector is tangential to the circle. Notice that the magnitude of the velocity vector is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{\omega^2 R^2\left(\sin^2(\omega t) + \cos^2(\omega t)\right)}$$

But since $\sin^2\theta + \cos^2\theta = 1$, the tangential velocity $\boldsymbol{v}$ can be related directly to the angular velocity $\boldsymbol{\omega}$. That is,

$$V = \omega R \qquad \text{or} \qquad \omega = v/R.$$

This result will be very useful later in the course. Also notice that $v = \omega R = 2\pi R/T$ where T is the time to complete one rotation (called the ***period*** of the rotation) - that is, the tangential speed is just the circumference of the circle divided by the time to complete one rotation, which makes sense.

<u>Acceleration</u>: By definition:

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\,\hat{\mathbf{i}} + \frac{dv_y}{dt}\,\hat{\mathbf{j}}$$

Since we have the components of the velocity vector, the derivatives can be taken just as was done with the $\mathbf{r}(t)$ vector to obtain $\mathbf{v}(t)$. That is,

$$v_x = -\omega R \sin(\omega t) \qquad \text{and} \qquad v_y = \omega R \cos(\omega t)$$

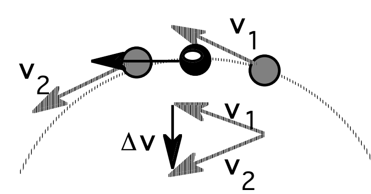
So the acceleration vector is:

$$\mathbf{a} = -\omega^2 R\cos(\omega t)\,\hat{\mathbf{i}} - \omega^2 R\sin(\omega t)\,\hat{\mathbf{j}} = -\omega^2\left(R\cos(\omega t)\,\hat{\mathbf{i}} + R\sin(\omega t)\,\hat{\mathbf{j}}\right)$$

Notice that the expression in parentheses is just the original $\mathbf{r}(\boldsymbol{t})$ vector. That is,

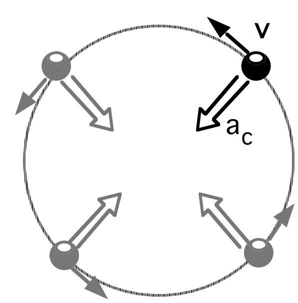
$$\mathbf{a} = -\omega^2\mathbf{r} = -\left(\omega^2 R\right)\hat{\mathbf{r}} = -\left(\frac{v^2}{R}\right)\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is the unit vector pointing from the origin toward the object, so the **r**-vector is written $\mathbf{r}(t) = R(t)\hat{\mathbf{r}}$.



The significance of this result is that the acceleration associated with an object in uniform circular motion is opposite in direction to the vector that locates the particle (*ie*, $\vec{\mathbf{a}}$ is in the $-\hat{\mathbf{r}}$ direction). That is, the acceleration is **<u>toward the center of the circle</u>**. Such an acceleration is called the ***centripetal acceleration*** (meaning "center seeking").

Centripetal Acceleration



Since ω can be related to the tangential velocity by $\omega = v/R$, then the centripetal acceleration associated with an object moving at speed v in a circular path of radius R is given by:

$$\vec{\mathbf{a}}_c = -\left(\frac{v^2}{R}\right)\hat{\mathbf{r}}$$

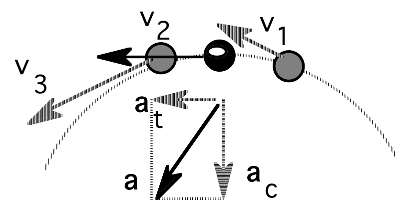
That is, the acceleration of an object moving in a circular path is toward the center and has magnitude

$$a_c = \frac{v^2}{R}$$

If the speed is not constant:

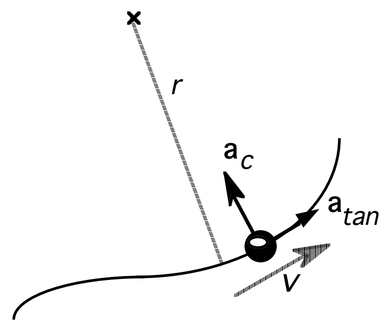
If the tangential speed of the object is not constant, then in addition to the radial component of the acceleration, there will be a tangential component equal to the rate at which the tangential speed is changing - *ie,* $a_{tan} = dv/dt$. That is

$$\mathbf{a} = \mathbf{a}_c + \mathbf{a}_{tan} = -\left(\frac{v^2}{R}\right)\hat{\mathbf{r}} + \left(\frac{dv}{dt}\right)\hat{\boldsymbol{\theta}}$$



where $\hat{\mathbf{r}}$ and $\hat{\theta}$ are unit vectors along the radial and tangential directions.

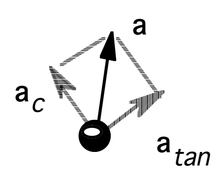
Notice that the description of the acceleration of an object traveling in an arc is based on the derivatives of the position vector - hence it is by definition an ***instantaneous*** acceleration. It does not even depend on the motion being uniform, or even being in a circular path. That is, if an object travels in *any* arc or curved path, the acceleration **a** at any point on the path has, in general, a tangential and a radial (or centripetal) component and is expressed, in general, as a vector sum of the two components.



The tangential component of the acceleration is the rate at which the speed is changing and the centripetal acceleration (which is always perpendicular to the path) is determined by the instantaneous speed and the radius of curvature of the path or trajectory ***at that point***.

The resulting acceleration vector has two components - the centripetal acceleration that is perpendicular to the direction of travel and points toward the center of the radius of curvature, and the tangential acceleration that is associated with any change in speed of the object as it travels around its curved path.

$$\mathbf{a} = \mathbf{a}_c + \mathbf{a}_{tan} \qquad \text{where} \quad a_c = \frac{v^2}{r} \quad \text{and} \quad a_{\mathrm{tan}} = \frac{dv}{dt}$$



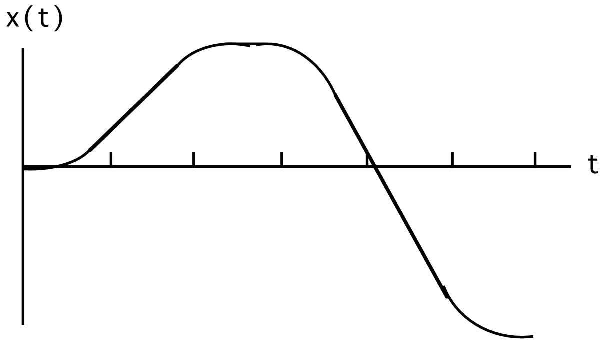
The magnitude and direction of the acceleration can then be found by the usual rules for any vector quantity:

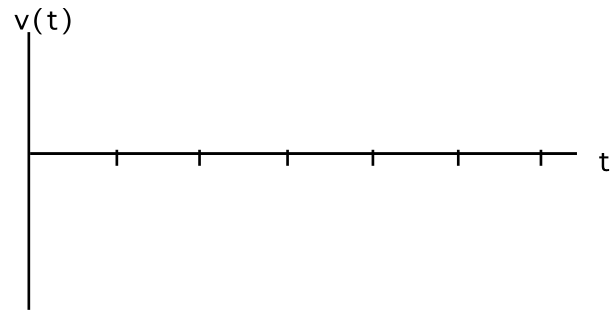
$$a = \sqrt{a_c^2 + a_{tan}^2}$$

Some important problems involving these ideas include cars travelling in curved paths and circular orbit problems. When the earth orbits the sun or the moon orbits the earth, the acceleration of the orbiting object is toward the center of the circular path. Understanding that helped Isaac Newton determine the law of the gravitational force which led to the explanation of the elliptical planetary orbits - and ultimately to the explanation of Kepler's laws for planetary motion.

KINEMATICS – QUESTIONS AND PROBLEMS

1. Consider a bead which can slide along a long straight wire (the x-axis). Suppose that its position as a function of time is shown in the graph of x vs t. Describe in words the motion of the bead. On the second graph, sketch the velocity as a function of time.

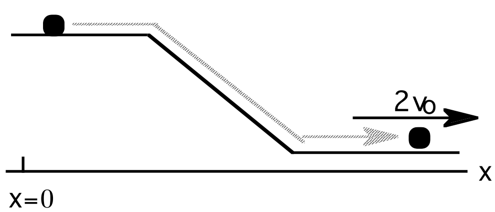




Identify on the position and velocity curves when:

a. The bead is moving to the right at constant speed.
b. The bead is moving to the left at constant speed.
c. The bead is accelerating to the right.
d. The bead is accelerating to the left.

2.



Suppose a marble with initial speed v_o rolls down a ramp, as shown, and doubles its speed.

Sketch graphs of its displacement in the x-direction and of its speed as functions of time.

3. Suppose you drop a golf ball from a height of one meter and it rebounds to a height of 0.8 meters. What is the change in velocity that occurs when it strikes the ground? If it was in contact with the ground for 200 msec, what was its acceleration while it was touching the ground? What was the total time from when it was dropped to when it was “caught” at the 0.8 meter height?

   Sketch the position vs time, velocity vs time, and acceleration vs time graphs.

4. Suppose you drop a water balloon from the top of a building. The water balloon takes four seconds to hit the ground.

   Determine the height of the building. [Set up the equations carefully and solve.]
   Determine the speed of the balloon just before it hits the ground.
   Determine the balloon’s acceleration as it is being stopped by the ground if it comes to a stop in a distance of ten centimeters.

5. Suppose you throw a ball vertically upward with a speed of 10 m/s as it leaves your hand 2 meters above the ground. Determine its maximum height, when it gets to the top of its trajectory, and when it hits the ground.

   Carefully sketch a graph of its height as a function of time from the moment you release it.

6. A modern drag racer can accelerate from rest to cover a quarter mile in five seconds. Assume the acceleration is constant. (Metric units are easier, assume that 1/4 mi=400 m.)

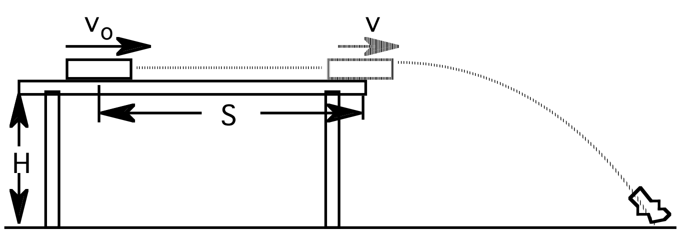
   a. Determine a value for the acceleration. [Express your answer as a constant times "g": a= ( )g]

   b. Assuming constant acceleration, determne the final speed. [Note: 1 m/s ≈ 2 mph]

   c. If the car can stop with an acceleration equal to -g, determine the distance and the time to stop from that speed.

TRAJECTORY PROBLEMS

7. Consider that you give your physics book a shove on a table (a height h above the floor). The book has a speed v_o a distance S from the edge of the table Briefly justify your reasoning in each part below. (You should assume you know the values of H, S, g, and v_o) Set up and solve the problems algebraically.



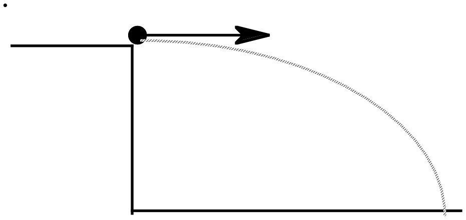
For numerical values, use the values:
S = 1 m v_o = 4 m/s H = 1 m g = 10 m/s2

   a. Obtain an expression for the time the book is in the air after it leaves the table.

   b. Obtain an expression for the horizontal distance the book travels after it leaves the table.

   c. Obtain an expression for the speed with which it hits the ground.

8 .

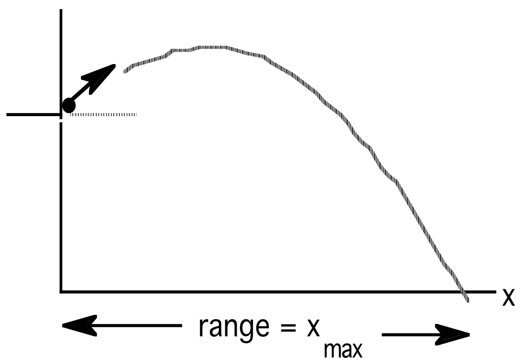


Consider that a marble is shot horizontally from a table with initial speed v_o. The range of the ball is 2 meters if the height of the table is one meter. For this problem, you can assume g = 10 m/s^2

Starting with the x(t) and y(t) equations, determine the initial speed of the marble.

Determine the speed of the marble when it hits the floor.

9.



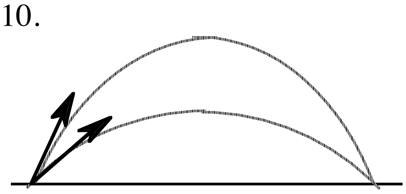
Consider that a projectile is fired from the top of a building of height H with an initial speed of v_o at an angle θ with respect to the horizontal.

Begin with the general expression for y(t) and derive the equation for the trajectory, ie the y(x) that describes the path the projectile takes.

Outline the steps you would take to solve for the range of the projectile.

[NOTE: You do not need to do the algebra - just explain <u>what</u> you would do. Be explicit enough that following the steps should lead to the correct range.]

10.



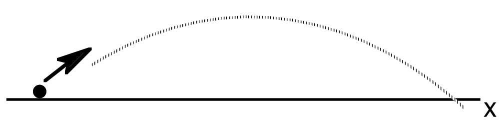
Show that the horizontal range of a projectile is the same if the projectile is launched at angle θ or at angle 90-θ.

Show at what angle θ the horizontal range will be greatest. And show that at that angle, the horizontal range will be four times the maximum height.

11. Suppose Bill throws a water balloon from the top of a building at an angle of 30° with respect to the horizontal. The building is 20 m high and the initial speed of the water balloon is 10 m/s. [Let g=10 m/s-.]

Determine the time Bill's best friend, a physicist, has to calculate whether he is going to get hit by the balloon. If he is 25 m from the building, should he move? (Determine both the time of flight and the distance to the point of impact.)

12.



Consider that a projectile is shot with an initial speed of vo at an angle θ with respect to the horizontal.

Begin with the general expression for y(t) and derive:

(a) The maximum height (in terms of v_o, g, and θ).

(b) The horizontal range of the projectile (in terms of v_o, g, and θ).

(c) An equation for the trajectory, ie the y(x) that describes the path the projectile takes.

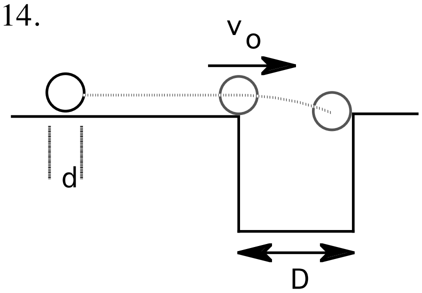
(d) Using your expression for the horizontal range, determine the angle at which the range is maximized.

(e) Show that the horizontal range is the same for angles θ and (90 - θ).

13. Suppose Tiger Woods can launch a golf ball with an initial ball speed of 180 mph at a launch angle of 14°. Starting from the kinematics equations, determine how far he could hit a golf ball. State the assumptions you are making in setting up this problem. [You probably want to convert to SI units – *i.e.*, since one meter is about 1.1 yards and there are 1780 yards per mile and 3600 sec. per hour, 1 mph converts to about 0.45 m/s.]

    Discuss how the assumptions affect the solution - *i.e.*, would you expect the ball to go farther or less far than your calculations suggests. If Woods drives can carry 300 yards, can you think of the effects of the details that we have left out of this analysis and how they might affect the result?

14.

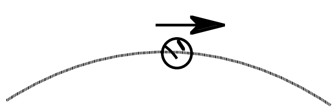


Anyone who plays golf has had a putt, moving too quickly, "skip" across the cup without falling in. It happens when less that half the ball is below the far edge of the cup, so it just bounces up and continues forward. Set up the problem and solve for the maximum speed which a ball can have as it reaches the hole and still drop into the cup. You can do the problem either algebraically in terms of D and d or do it numerically.

Assume the following dimensions (approx.):
Diameter of cup: D=12 cm   Diameter of ball: d=4 cm

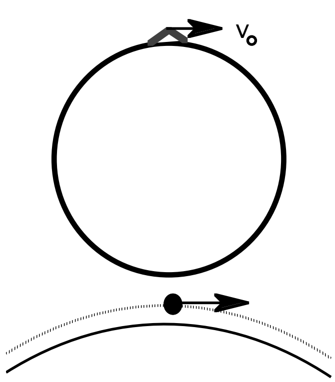
CIRCULAR MOTION

15



Suppose you can throw a baseball with a speed of 30 m/s. The ball actually travels in an arc. At the "top" of its arc (where its motion is horizontal), determine the radius of curvature of its path. Explain your reasoning carefully.

16.

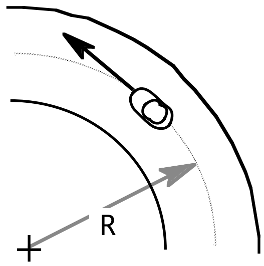


Suppose a bullet is fired from Mt. Everest horizontally with the speed necessary to orbit the earth (assume no air resistance).

If the radius of the earth (and the orbit) is 6.4×10^6 m, determine the speed of the bullet.

Determine the time for one complete orbit.

17. An Indianapolis race car can circulate the 2.5 mile oval at an average speed of 225 mph. Assuming the track is circular (rather than an oval) with a circumference of 2.5 miles and that the speed is constant at 225 mph, determine the average acceleration of the race car. [Express the acceleration in the form a=(const)g - where the constant in parentheses is just the ratio a/g. For example, a=(.5)g means the acceleration is half that of freefall acceleration.] NOTE: 1 mi = 1.6 km

18. 

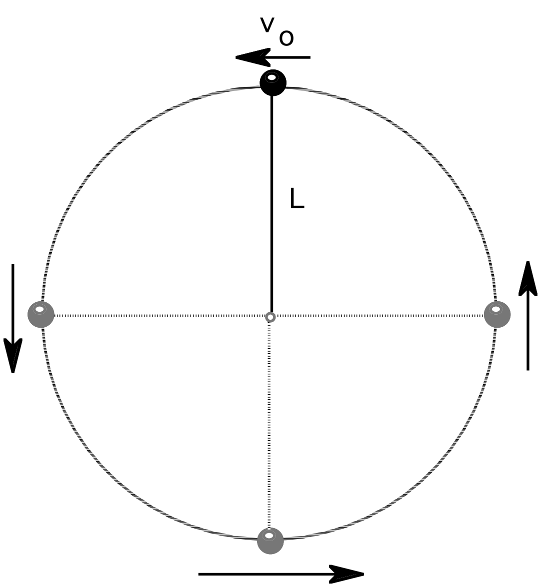
With great tires, your Porsche can generate about (0.9) g of lateral acceleration in a turn. Determine the speed with which the car could go through an unbanked turn with a radius of 40 m.

Determine whether the time to complete a lap around a circular path would increase, decrease, or stay the same as the radius of the circle increases. (For example, could you complete a lap around a 100 m radius circle or a 200 m radius circle in less time if the centripetal acceleration were 0.9 g?)

19. Suppose an object starts at rest and accelerates uniformly while traveling in a circle of radius R. Assume its tangential acceleration is given by a. Show that the object's centripetal acceleration will equal the tangential acceleration after it has completed 2π complete orbits subject to that constant acceleration.

[HINT: The object's tangential speed v depends on its acceleration and how long it has been accelerating. It's centripetal acceleration depends on its tangential speed and the radius. But it's tangential speed also depends on the radius and the period of the "orbit" at that moment – even though that period is continuously changing.]

20. Consider that a ball on a string is in circular motion in a vertical circle of radius R.



a. In order to stay in a circle at the top of the path, the centripetal acceleration must be at least as great as the gravitational acceleration. Using that information, find an expression for the minimum speed at the top.

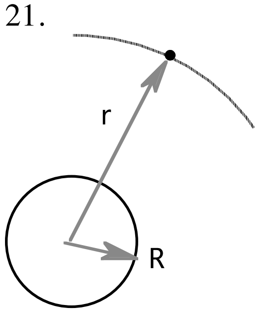
Suppose the string is one meter in length, calculate the value of the speed at the top to just stay in the circular path.

b. As the ball falls to complete the circle, gravity acts to speed the ball up considerably. On the figure showing the ball at several positions, show the acceleration vector at each of those positions. You should show both the horizontal and vertical components of the acceleration as well as the vector sum (ie, *the* acceleration vector). Justify your thinking.

Write an expression for the magnitude of the acceleration at one of those two "side" positions - assuming you know the speed v at that point.

At the very bottom, compare the magnitudes of the centripetal acceleration and the gravitational acceleration. Briefly justify.

$a_c < g$ $\quad$ $a_c = g$ $\quad$ $a_c > g$

21. 

If the gravitational acceleration diminishes by the square of the radius from the center of the earth, determine the relationship between the period of a circular orbit and its radius. That is, obtain an expression for the period of an orbit of radius r (where r is greater than the earth's radius R). Your final expression for the period T should be in terms of R, g, and r.

[HINT: Assume that a_g=g when r=R, where R is earth's radius, so that when r>R, the acceleration is given by the expression $a_g = (R/r)^2\, g$..]

The twenty-four orbiting GPS satellites that make up the network for global positioning each orbit the earth twice per day. Determine the radii of their orbits.