Part 4

WORK, ENERGY AND POWER

Work-Energy Theorem

Power

Potential Energy and Energy Conservation

Conservative and Non-conservative Forces

Problem Solving Strategy

Final Comments on Energy Conservation

Energy Problems

WORK AND ENERGY

In many ways, the ideas of work and energy are the central themes of physics. Although one must know Newton's laws and their applications in order to say they have mastered the material of this course, it is the idea of *energy* that is the all pervading concept that extends to all branches of science and engineering - and is central to understanding how the universe works. The concept of energy, of course, is still deeply rooted in Newton's laws, and they must be understood in order to solve problems involving work and energy.

Central to understanding the energy concept is the definition of work and the work-energy theorem – *i.e.,* how work done on a system changes its kinetic energy. Those ideas lead to the definitions of *potential energy* and then the *total mechanical energy* of a system and the idea that the flow of energy can always be traced - more commonly stated as *The Principle of Conservation of Energy*.

Ultimately, you will <u>use</u> energy conservation (or the work-energy theorem) to solve problems that involve velocity changes and position changes, but for which you are not interested in the accelerations and the times involved in the problem (or may not even be able to solve for them). That is, there are many problems that you just do not need to know the details of the motion, but do want to find the final velocity of something or the amount of energy that was lost or some other quantity that does not require a complete solution of the position as a function of time.

KINETIC ENERGY

Energy appears in many different forms - but we will begin with the energy associated with motion called the *kinetic energy*. It is defined for an object of mass m and speed v as

$$K = \frac{1}{2}mv^2$$

We have already encountered the kinetic energy - the energy associated with the motions of objects - in the discussion of collision problems. We *defined* elastic collisions in terms of whether the collision conserved the total kinetic energy of the objects that collided. That is, K became a useful quantity because it was conserved in certain kinds of collisions. The kinetic energy of an object thus depends on its mass and its speed (but not the direction of its travel). As we will see, its definition follows from an application of Newton's second law to the changing motion of an object. It will be useful to talk about the kinetic energy of a system of particles as well as of individual particles. And the kinetic energy of a *system* is simply the sum of all the individual kinetic energies associated with all the moving components of the system. That is, if the m_i and v_i are the masses and speeds of each of the objects or particles that make up a system, the total kinetic energy is just the sum of the individual kinetic energies:

$$K_{system} = \sum_i \frac{1}{2}m_i v_i^2$$

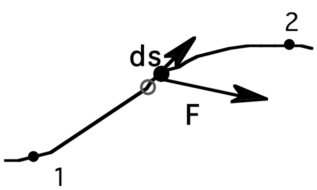
It should be noted that since the kinetic energy does not have direction, it is a scalar and not a vector quantity, so the total kinetic energy is a simple summation and does not require vector addition (as does summing velocities or forces or momenta).

In a discussion of thermodynamics when a system of many particles is being considered - say the particles that make up a mole of an ideal gas (*i.e.*, Avogadro's number of molecules), the total of all the energies of all the particles will be called the *internal energy* of the system.

Here, we are concerned with how that kinetic energy can be changed. What we will find is that the kinetic energy of a system can be changed through the action of the forces that act on the system in the form of *work* - a quantity that will need to be carefully defined. In a later discussion of thermodynamics, we will also discover that the internal energy of a system - the total kinetic energy of all the molecules that make up the system - can also be changed through exchanges of heat between the system and its surroundings in addition to any work done on the system. But *that* discussion will be postponed for now.

WORK

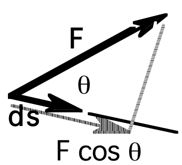
Work is a transfer of energy from one object to another via a force acting over some displacement. It is a way to change the energy of an object or system.



When a force acts on an object which moves from one point to another along some path as shown, the work done by the force is given by the equation

$$W = \int_1^2 \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{s}}$$

where the integral is over the path taken by the object, $\vec{\mathbf{F}}$ is the force on the object, and $\mathbf{d}\vec{\mathbf{s}}$ is an incremental displacement along that path. This general *line integral*, as it is called, will simplify considerably for all problems we will try to consider.



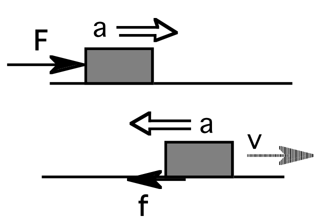
The scalar product of the vector force and vector displacement is just the product of the component of the force parallel to the displacement and the displacement itself. Ie,

$$W = \int_1^2 \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{s}} = \int_1^2 F \cos\theta \, ds = \int_1^2 F_s \, ds$$

where F_s Is the component of the vector force $\vec{\mathbf{F}}$ in the direction of the displacement $\mathbf{d}\vec{\mathbf{s}}$. Notice that the work W is thus a scalar quantity.

The work done on an object by a force can only be calculated directly in special cases. We will consider one dimensional problems-where the displacement is always along a straight line, constant force problems-where the integral reduces to a product of force times distance, and things like spring problems-where the integral is particularly easy to do.

The work is positive when the force increases the energy of the object – *i.e.,* when the force has a component in the direction of the displacement. It is negative when the force opposes the displacement and hence absorbs energy from the object .



If a force $\vec{\mathbf{F}}$ accelerates a block on a table, the block gains energy and the calculation of the work would be positive ($W = \int F dx > 0$). As the block slides along the table, friction slows it down. The block loses energy and the calculated work done by the friction is negative because the force is opposing the motion of the block, *i.e.,*

$$W = \int (-f) dx < 0$$

Energy is transferred to an object by a force only if the object moves. That is, the work done on an object which does not move is zero. And it is only the component of the force that is in the direction of the displacement that transfers energy to the object. That is, the work done by **F** depends only on the component of the force parallel to the displacement.

In One Dimension Calculating the work by the integral depends on knowing the force as a function of the position x. That can be known in some special cases and the calculation is easy.

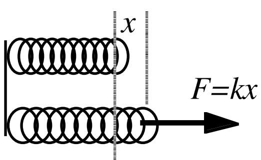
$$W = \int_{x_1}^{x_2} F_x dx$$

Constant Force Problems If the force is a constant, then it can come outside the integral and the work reduces to the simpler form of *work = (force)x(displacement*

$$W = \int_{x_1}^{x_2} F_x dx = F_x \int_{x_1}^{x_2} dx = F_x\left(x_2 - x_1\right) = F_x S$$

where S is the total displacement of the object subject to the constant force F_x. Many problems are of this form, and the work calculation is just a simple force time distance.

Spring Problems



For springs, the force required to either stretch or compress the spring an amount x from its relaxed length is given by $F = kx$ (which is known as Hooke's law). The *work* required to stretch or compress the spring is then obtained by integrating the (non-constant) force over the amount the spring has been stretched (Δx):

$$W = \int_0^{\Delta x} F_x dx = \int_0^{\Delta x} kx\ dx = k\int_0^{\Delta x} x\ dx = \frac{1}{2}k\Delta x^2$$

THE WORK-ENERGY THEOREM

Consider the work done by all the forces that act on an object. That is, determine the net work done (W_{net}) by the vector sum of all the forces $\vec{\mathbf{F}}_{\mathbf{net}} = \sum \vec{\mathbf{F}} = m\vec{\mathbf{a}}$. Or,

$$W_{net} = \int_1^2 \vec{\mathbf{F}}_{\mathbf{net}} \bullet \mathbf{d}\vec{\mathbf{s}}$$

For one-dimensional problems, since F_{net}=*ma,* this reduces to

$$W_{net} = \int_{x_1}^{x_2} F_{net} dx = \int_{x_1}^{x_2} ma\ dx = m\int_{x_1}^{x_2} \frac{dv}{dt}\ dx$$

By change of variable, dx can be rewritten *(dx/dt)dt* which is just *vdt*. Integrating *v(dv/dt)dt* over time is the equivalent of integrating *dv* over velocity (another change of variable), *i.e.*:

$$W_{net} = m\int_{x_1}^{x_2} \frac{dv}{dt}\ dx = m\int_{t_1}^{t_2} \frac{dv}{dt} \cdot \frac{dx}{dt}\ dt = m\int_{t_1}^{t_2} v\ \frac{dv}{dt}\ dt = m\int_{v_1}^{v_2} v\ dv = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

Since the kinetic energy of an object is defined as $K = \tfrac{1}{2}mv^2$, then the net work is numerically equal to the change in kinetic energy, or

$$\boxed{W_{net} = K_{final} - K_{initial} = \Delta K}$$

Work-Energy Theorem: The work done by the sum of all of the forces that act on an object equals the change in the object's kinetic energy.

This theorem is a direct consequence of Newton's second law. What that means is that the results that are based on the application of the work-energy theorem and are necessarily consistent with Newton's laws. (Or said differently, there is no *new* information in the work-energy theorem, just an extension of the ideas already encountered.)

<u>Note:</u> For one dimensional problems with a constant force acting this reduces to:

$$W_{net} = \int_{x_1}^{x_2} F\,dx = F\int_{x_1}^{x_2} dx = ma(x_2 - x_1) = \tfrac{1}{2}m\left(v_2{}^2 - v_1{}^2\right)$$

which is the same as $v_2{}^2 - v_1{}^2 = 2a(x_2 - x_1)$ from the kinematics equations for one-dimensional motion with constant acceleration, since a constant force results in a constant acceleration.

<u>Examples:</u>

- The work done by gravity on an object which is dropped a distance h:

  $W_g = \int(mg)dy = mgh$. By work-energy theorem, $W_g = mgh = \Delta K$ or $mgh = \tfrac{1}{2}mv^2$

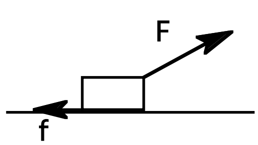
- The work by a force F to lift something a height h:

  If the object starts and ends with zero velocity, there is no change in kinetic energy. So the net work must be zero. So $W_{net} = 0 = \int(F - mg)dy$ integrated from y=0 to h. That is, the work done by the force F, even if that force is not constant, is given by

  $$W_F = \int F dy = mgh.$$

  So the work to lift an object of mass m a height h is just mgh.

- When a block is dragged along a horizontal surface with friction:

  The work done by the force F, is $W_F = (F\cos\theta)\,S$ -- since the force is constant and the integral just results in "force times the displacement". The work done by the friction is $W_f = -f\,S$ -- since the force f is opposite in direction to the displacement so the work done by friction is negative.

  So the net work done by both forces is $W_{net} = (F\cos\theta - f)S$ which equals the change in kinetic energy $\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2$ by the work-energy theorem. This is the same result one gets if one uses the net force $(F\cos\theta - f)$ to determine the acceleration of the mass then determines the change in speed when it accelerates over a distance S and then determines the change in the kinetic energy.

These simple examples show how the formal definition of work (*i.e.*, as a line integral of $\vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{s}}$ over a displacement simplifies when the motion is one-dimensional and the forces involved are constant.

POWER **The rate at which work is done. Or, equivalently, the rate at which energy is transferred from one object to another.**

The work done by a force on an object is given by $W_F = \int_1^2 \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{s}}$

But as we have seen, the integral over the displacement $\mathbf{d}\vec{\mathbf{s}}$ can be changed to an integral over time by a change of variable. That is, $\mathbf{d}\vec{\mathbf{s}}$ can be replaced by

$$\mathbf{d}\vec{\mathbf{s}} = \left(\frac{d\vec{\mathbf{s}}}{dt}\right) dt = \vec{\mathbf{v}}\, dt$$

So the work done by $\vec{\mathbf{F}}$ can be written

$$W_F = \int_1^2 \vec{\mathbf{F}} \bullet \vec{\mathbf{v}}\, dt$$

But integrals and derivatives are inverse relationships, so

$$W_F = \int_1^2 \vec{\mathbf{F}} \bullet \vec{\mathbf{v}}\, dt = \int_1^2 \left(\frac{dW_F}{dt}\right) dt$$

Hence we can then define power as:

$$\boxed{\text{Power } = \frac{dW_F}{dt} = \vec{\mathbf{F}} \bullet \vec{\mathbf{v}}}$$

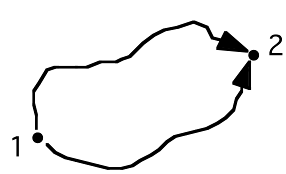
The power delivered by a force at any given time is equal to the scalar product of force and the velocity at that moment – or the *component* of the force in the direction of the displacement and the *speed* of the object at that moment. The units of power are joules per second - or "watts". That is 1 joule/sec = 1 watt

POTENTIAL ENERGY AND ENERGY CONSERVATION

The concepts of work, conservative and non-conservative forces, and potential energy are particularly useful in problem solving - but the ideas are rather subtle (maybe even obscure) when you first try to relate them. This discussion will try to connect the ideas - then specialize to problems involving gravity and/or springs.

Conservative Forces

A force is *conservative* if the work done on an object by the force as it moves from one position to another does not depend on the path taken.



The work done by a force $\vec{\mathbf{F}}$ on an object which moves from point 1 to 2 along some path is the integral of $\vec{\mathbf{F}}$ over the displacement along that path. The work has the effect of changing the energy of the object - either the energy increases (if the work is positive) or decreases (if the work is negative).

$$W_F = \int_1^2 \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{s}}$$

An integral of this form is called a *line integral*. In general, the value of the integral depends on the path over which the integral is evaluated - and can only be calculated for certain special cases. There are some types of forces for which the work does not depend on the path. Those forces for which the work is path independent are called *conservative forces***.** (We shall see why shortly.) Both gravitational and electrical forces are conservative. So in any problem involving the gravitational force, $F_g=mg,$ that force can be considered a conservative force. Since the forces between atoms are electrical in nature, and the tension in a spring is a result of the forces between atoms, the spring force can also be considered to be a conservative force. All the other forces we will consider in this course are *non-conservative* because they will depend on the path the object takes while subject to the force.

The work-energy theorem can then lead to a particularly powerful concept. The net work done by all the forces acting on an object can be broken into two contributions, the work by all the conservative forces and that done by all the non-conservative forces. That is,

$$W_{net} = \int_1^2 \vec{\mathbf{F}}_{net} \bullet \mathbf{d}\vec{\mathbf{s}} = \int_1^2 \vec{\mathbf{F}}_{conservative} \bullet \mathbf{d}\vec{\mathbf{s}} + \int_1^2 \vec{\mathbf{F}}_{non-cons} \bullet \mathbf{d}\vec{\mathbf{s}} = W_{conservative} + W_{non-cons}$$

But the net work done by all the forces acting on an object is just the change in the kinetic energy, according to the work-energy theorem, so

$$W_{net} = W_{conservative} + W_{non-cons} = \Delta K$$

Potential Energy

We now define *potential energy* – given the symbol U – so that the change in the potential energy is the negative of the work done by the conservative forces. That is,

$$\Delta U = -W_{conservative} = -\int_1^2 \vec{\mathbf{F}}_{conservative} \bullet \mathbf{d}\vec{\mathbf{s}}$$

Potential energy is defined so that the work done by conservative force always decreases the potential energy of an object or system. It is defined in such a way that one can assign a value to the potential energy of an object (U) and then replace the work done by a conservative force with the corresponding change in the potential energy. Conceptually, potential energy can just be thought of as energy stored in the "system" - or energy that can be converted to kinetic energy as a result of the action of the conservative forces. Potential energy is only associated with conservative forces.

This somewhat strange and abstract definition allows one to account for any work done by conservative forces just in terms of the corresponding change in potential energy. It is a subtle concept, however, in that the *value* of the potential energy is not uniquely determined – only its *change* can be uniquely determined. For example, the potential energy of a mass on a table can be assigned any value. But if it is then lifted a distance h above the table, its potential energy has *increased* by mgh because the work gravity did was $-mgh$ while it was lifted. If it then falls back to the table, gravity would do $+mgh$ of work on it, so its potential energy would decrease by that same amount.

For our purposes, we will only consider the gravitational force and forces exerted by ideal springs as being conservative forces. So we can define a potential energy associated with those forces and use the corresponding potential energy functions in problem solving. But it can be shown that other forces we will encounter in this course (friction, string tension, forces that you exert when you push or pull an object, etc.) do not satisfy the requirement that the work done is path independent, so they are not conservative forces and a potential energy cannot be associated with those forces.

> **Both gravitational and spring forces are conservative forces. We can thus write potential energy functions associated with those forces.**
>
> **Consider all forces other than springs and gravitation as non-conservative.**

Gravitational Potential Energy

The most common example of potential energy is due to the work done by gravity. When an object of mass m falls a distance h:

$$W_g = \int_h^0 (-mg)dy = mgh = -\Delta U$$

and we say the potential energy of the object decreased an amount mgh. Conversely, if you lift an object of mass m through a distance h, then gravity does an amount of work equal to $-mgh$ (since the displacement is opposite to the force due to gravity) so the potential energy of the object has been increased by mgh. In general, near the surface of the earth, where $r=R_{earth}$ and hence $F_g = mg$, a change in the gravitational potential energy can thus be written

$$\Delta U = U_2 - U_1 = -\int_{y_1}^{y_2} (-mg)dy = mg(y_2 - y_1) = mgh$$

where h is just the vertical displacement y_2-y_1. It is then convenient to just *define* the gravitational potential energy as $U_g = mgy$, where U is assigned the value zero when y is zero. Notice that *where* one sets $U=0$ is an arbitrary choice -and can be selected by convenience for a given problem.

When the vertical displacement Δy is not small compared to R_{earth} , then the gravitational force depends on r and the more general form of the gravitational force would have to be used. In that case:

$$\vec{\mathbf{F}}_g(r) = -G\frac{mM}{r^2}\hat{\mathbf{r}} \quad \text{and} \quad U(r) = -G\frac{mM}{r}$$

Notice that in this form, the gravitational potential energy of m is zero when its separation from the mass M is infinitely large (*i.e.,* when the force is also zero) – and $U(r)$ becomes more negative as the two masses get closer together. It is left as a problem to show that this form for gravitational potential energy function is consistent with $U=mgh$ where h is the distance above Earth's surface as long as h is very small compared to the radius of the earth. That is, there is no inconsistency in these two forms – one applies to objects close to Earth's surface and the other to the gravitational interaction of any two objects and is dependent on their separation.

Elastic Potential Energy

Spring forces are one dimensional of the form $F(x)$. Since the force to extend or compress a spring is, by Hooke's law, $F(x) = kx$ where x is the amount of extension or compression from its relaxed length, the spring exerts an equal and opposite force $F_{spring} = -kx$. The potential energy of a spring is then just the negative of the work *the spring* does while it is being extended or compressed, which means it equals the work done on the spring in extending or compressing it. That is

$$U_{elas} = -W_{spring} = -\int^x F_{spring}dx = -\int^x (-kx)dx = \int^x F_{applied}dx = \int^x (kx)dx = \tfrac{1}{2}kx^2$$

It is then convenient to just define the elastic potential energy associated with a Hooke's law spring as

$$U_{elas} \text{ or } U_{spring} = \tfrac{1}{2}kx^2$$

Notice that even if the spring does not "obey" Hooke's law, *i.e.,* if $F_{spring} = -kx + bx^2$, or something, it would still be conservative since it depends only on x - it would just have a different potential energy function.

Total Mechanical Energy

Defining a change in the potential energy as the negative of the work done by conservative forces, as peculiar as that seems, leads to a particularly useful result:

$$W_{net} = W_{conservative} + W_{non-cons} = -\Delta U + W_{non-cons} = \Delta K$$

or the change in the kinetic and potential energies equals the work done by all the non-conservative forces. We will then define the *total mechanical energy* of an object as the sum of its kinetic and potential energies - and that can only be changed by the work of non-conservative forces.

If the total mechanical energy of an object is defined as the sum of its kinetic and potential energies, then

$$\boxed{W_{non-cons} = \Delta K + \Delta U = \Delta E_{total} \quad \text{where} \quad E_{total} = \frac{1}{2}mv^2 + U}$$

This states that only non-conservative forces can change the total energy of an object or system if the total energy is defined as the sum of the kinetic and potential energies.

If only conservative forces act, then the total energy is a constant.

If non-conservative forces act, then the total energy will change by an amount equal to the work done by the non-conservative forces.

Energy Conservation

The ***Principle of Energy Conservation*** is essentially contained in the above two statements. That is, the statements declare that either total energy is conserved - or that it can at least be accounted for by the action of identifiable non-conservative forces. This principle is central to all of physics. It states that one can always trace the energy of an object or system. Energy can never be created spontaneously, nor can it ever be destroyed. But energy can be transferred from one object to another (ie, work can change the energy of a system) and it can change form (from potential to kinetic, for example) - but it can always be accounted for. These statements make it clear what one means by a "conservative" force. It is a force that will not change the total mechanical energy of an object (or system). "Non-conservative" forces, on the other hand necessarily change the total energy.

The idea of potential energy comes from the attempt to focus on the object or system being described and not on the forces that act on the system - all as a way to simplify the problem solving. By assigning a *potential energy* to account for the work done by *conservative forces* (just gravity and springs, in this course), the effects of those forces can be included in the *total mechanical energy* associated with the problem. In many problems, that allows the solution to be obtained very simply by just keeping track of all the energy in the problem. In those problems, energy is just a book-keeping tool that lets you decide what gained and what lost energy and in what form they gained or lost energy. If only conservative forces act, total energy is a constant. That simple statement lets you write an expression for the total energy initially and set it equal to an expression for the total final energy - and often solve for whatever unknown is asked for in the problem. It total energy is not conserved, the same expression can then include whatever energy gains or losses occur due to non-conservative forces.

$$\tfrac{1}{2}mv_o^2 + U_o + E_{gained} = \tfrac{1}{2}mv_f^2 + U_f + E_{lost}$$

Many problems can be solved using energy principles that would be very difficult or even impossible using F=ma - even though the energy principle is based on Newton's laws. Problems for which the energy approach is particularly useful are those in which there is no interest in the acceleration or times within the problem, but you do need to know how velocities changed, how high something went, how much a spring was compressed, etc. Energy conservation is the first of three ***conservation laws*** that are among the central themes of theoretical physics.

Notes on Conservative Forces and Potential Energy:

- In one dimension *any* force that depends only on x is conservative. That is, any force of the form $F(x)$ is a conservative force and the corresponding change in potential energy is

  $$\Delta U = -\int F(x)dx$$

  (Notice that the direction of frictional forces depend on whether the object moves in the $+x$ or $-x$ direction, so friction cannot be conservative since it depends on something other than position). Since spring forces are of the form: $F_{spring} = F(x) = -kx$, they are conservative.

- In three dimensions any *central force* or force of the form $F(r)\,\mathbf{r}$ that is directed toward a center and depends only on the distance from the center is also conservative.

  $$\Delta U = -\int F(r)dr$$

  The gravitational force is conservative since fundamentally it is of the form

  $$\vec{\mathbf{F}}_{\mathbf{g}} = -G\frac{mM}{r^2}\hat{\mathbf{r}}$$

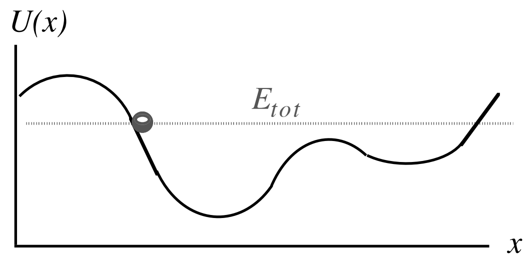
- Potential energy is only defined for conservative forces. So there is a potential energy associated with gravitational forces and with spring forces, but not with forces that YOU apply or friction or tension, etc.

- The term *conservative force* comes from the result that when objects are subject only to conservative forces, the effect of the forces on the energy of the object can be accounted for in terms of changes in the potential energy and kinetic energy - and total energy is conserved.

- Finally, if the potential energy is defined in terms of an integral over a conservative force, then if the potential energy is known as a function of position, the force associated with it can be obtained from its derivative with respect to position. That is,

  $$F(x) = -\frac{dU(x)}{dx}$$

There are certain types of problems for which the potential energy may be easier to obtain than the forces themselves - and this states that the force can then be derived from $U(x)$.



These ideas give value to the idea of the potential energy. Even though, technically, potential energy can always be avoided by considering the work done by conservative forces. But it is the *potential energy function* that is often used by physicists in the description of problems - and that function often gives great insight into the behavior of some object or system.

In a graph of energy as a function of position, the potential energy function limits the range of motion of a particle since the total energy must always be *at least* as great as the potential energy (the kinetic energy can never be negative). For example, if a ball can roll on the hilly surface shown, the *y*-value becomes a measure of the potential energy (ie, *U(x)* has the shape of the surface).

If the above ball were released at the point shown, the dotted line would represent its total mechanical energy. The difference between the total energy and the potential energy would be the ball's kinetic energy. So it is easy to visualize how the ball would move, where it is fast, where it is slow, where it reverses direction, and where it would be in equilibrium just by looking at the potential energy function.

WORK-ENERGY-POWER PROBLEM SOLVING STRATEGY

When faced with a problem that involves energy concepts, there are some specific things that you should do to navigate your way through the choices you need to make in order to resolve the problem.

- Draw a figure - or preferably a series of figures - that make it clear what is happening in the problem.

- Carefully show the forces that act on the object you are trying to describe while the object is moving. Resolve those forces into components along the direction of the motion.

- Decide, specifically, what question is being asked - and what information you are given in the problem either explicitly or implicitly. There are several common types of problems:

(1) The work is to be determined directly, knowing the forces involved using the definition of work:

$$W = \int_1^2 \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{s}} = \int_1^2 F\cos\theta\, ds = F\cos\theta \bullet S \qquad \text{(if } F \text{ is const)}$$

(2) The work done by a force, or a sum of forces, is to be deduced knowing the change in kinetic energy or the initial and final speeds:

$$W_{net} = \Delta K = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_o^2$$

(3) The final speed or kinetic energy of an object can be determined knowing the work that is done by all the forces involved *or* by knowing the changes in potential energy and how much energy is lost (or work is done by non-conservative forces).

$$W_{net} = \Sigma W_i = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_o^2 \qquad \Delta K = -\Delta U - E_{lost} \quad \text{or} \quad \tfrac{1}{2}mv_f^2 + U_f = \tfrac{1}{2}mv_o^2 + U_o - E_{lost}$$

(4) If the motion is circular, the kinetic energy can be related to the centripetal acceleration (even though they are measures of different quantities). That is,

$$K = \tfrac{1}{2}mv^2 \quad \text{and for circular motion} \quad ma_c = \frac{mv^2}{R}$$

That is, mv^2 is common to both the kinetic energy and the centripetal component of the net force. The force that turns the object into a circular path can be related to the kinetic energy. And this can be an important idea in problems involving circular motion.

(5) The power delivered by a force is to be determined by knowing the work done, or the change in kinetic energy, and the time over which the that amount of work is done - and equals **F•v.**

$$Power = \frac{dW}{dt} = \frac{dK}{dt} = \vec{\mathbf{F}} \bullet \vec{\mathbf{v}}$$

In problem solving:

- Determine whether you prefer to consider the problem a work-energy theorem problem, where you can set $W_{net} = \Delta K$, or an energy conservation problem where $E_{tot} = \tfrac{1}{2}mv^2 + U$ is either constant or is changed by the work done by non-conservative forces (friction, for example). Stay consistent with your choice - the net work does not change the total energy of the system, since changes in the potential energy corresponds to work by conservative forces.

- Decide whether energy is conserved.

  Is there friction in the problem? Is energy lost as a result of the friction involved?
  Is some force other than either gravity or a spring force acting on the object or system? If so, does that force change the energy of the object being described?

  If energy is *conserved*, then you can just set $E_{tot} = \tfrac{1}{2}mv^2 + U = const$ or, equivalently, the initial and final energies can be set equal:

$$E_{tot} = \tfrac{1}{2}mv_1^2 + U_1 = \tfrac{1}{2}mv_2^2 + U_2$$

  If energy is *not* conserved, then set the *change* in total energy equal to the work done by non-conservative forces - or, equivalently, $W_{non\text{-}cons} = E_{final} - E_{initial}$ and then determine the energy gained or lost due to the work done by the non-conservative forces or from the change in the total energy, depending on what you know in the problem.

SUMMARY OF THE PRINCIPLE OF ENERGY CONSERVATION

If only conservative forces act, then the work energy theorem states that

$$W_{net} = \int \mathbf{F}_{net} \bullet \mathbf{ds} = -\Delta U = -(U_2 - U_1) = \Delta K = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

So if we define the ***total mechanical energy*** as the sum of the kinetic and potential energies, so that the action of conservative forces only leaves the total energy unchanged. That is,

$$\Delta E_{tot} = \Delta K + \Delta U = 0 \qquad \text{or} \qquad \tfrac{1}{2}mv_1^2 + U_1 = \tfrac{1}{2}mv_2^2 + U_2$$

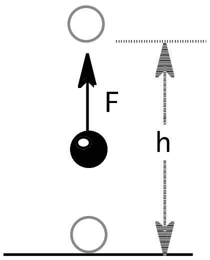
If *non*-conservative forces also act, then the work done by the non-conservative forces will change the total mechanical energy of the object or system. That is,

$$W_{net} = W_{conservative} + W_{non-cons} = -\Delta PE + W_{non-cons} = \Delta KE \quad \text{and} \quad \boxed{W_{non-cons} = \Delta KE + \Delta PE = \Delta E_{total}}$$

- Work done by conservative forces: Changes the potential energy $W_{cons} = -\Delta U$
- Work done by non-conservative forces: Changes the total energy $W_{non\text{-}cons} = \Delta E_{tot}$
- Work done by the net force: Changes the kinetic energy $W_{net} = \Delta K$

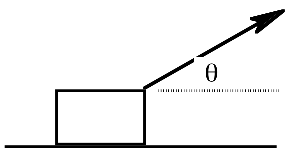
There is still a remaining subtlety to this grand principle of energy conservation. We have developed the idea in terms of the work done by forces acting on an object or system. But since all forces act in equal and opposite pairs - according to Newton's third law - the work done by any force on some object represents a transfer of energy from some other object. Necessarily then, the energy that is gained or lost by some system or object had to have reduced or increased the energy of some other object by the same amount - and that statement does not depend on whether the forces were conservative or non-conservative! That is, if ALL objects are considered in the calculation of energy gained and lost by the action of all the forces, then there can be no change in the total energy of the universe due to the interaction of all the parts of the universe. That, ultimately, is the real meaning of the principle of energy conservation. Energy can always be accounted for in the interaction of objects. Any energy gained by one object had to have been "lost" by another. Energy cannot be spontaneously generated from nothing. It can only be transferred between objects and/or changed from one form to another. But it can always be accounted for. That is the fundamental principle.

WORK AND ENERGY – Questions and Problems

1. 

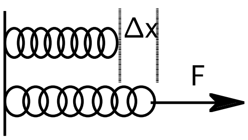
Suppose you lift an object of mass M from a table, first accelerating it upward, then slowing it to a stop a height h above the table. Determine the total work you have to do to accomplish that fete.

[Note: You do not know much about the variable applied force used to lift the weight. Explain carefully how the work-energy theorem lets you determine the work done.]

2. 

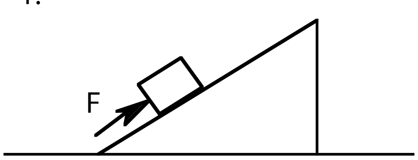
Suppose you drag a block of mass M across a horizontal surface a distance S by pulling with force F at an angle θ, as shown. Then you return the block to the starting point in the same manner. If the coefficient of kinetic friction is given by μ, determine the total amount of work you do in making the round trip.

Determine the angle θ which requires the least force to drag the block assuming a coefficient of friction μ – then determine the work to pull the block a distance S at that angle. [Question: Is that the minimum amount of *work* required to slide the block that distance?]

3. 

Obtain an expression for the work required to extend the length of a Hooke's law spring an amount Δx beyond its relaxed length.

Sketch graphs of both the force required and the work done as functions of the extension of the spring.

4. 

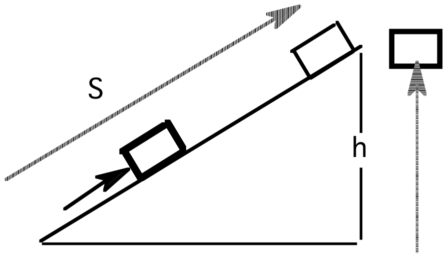
Each of the following statements seems to be correct. Either explain any incorrect answer or explain the apparent contradiction if all three are true statements.

a. Pushing an object up an incline requires that work be done on the object by a force.

b. When an applied force does work on an object, the kinetic energy change is equal to the work done by the force.

c. When an object is pushed up an incline at constant speed, there is no change in kinetic energy.

5. In comparing the work required to push a block up a frictionless incline to that needed to simply lift the block the same height: SELECT THE CORRECT ANSWER AND JUSTIFY.

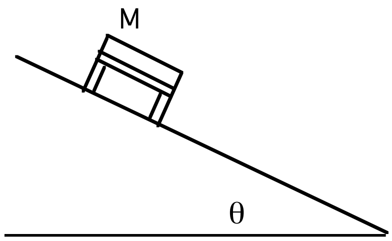


a. More work is needed to push the block up the incline since the force acts through a greater distance (S rather than h).

b. Less work is needed on the incline, since the force required is less (mg sin q rather than mg).

c. Exactly the same amount of work is required if there is no friction.

6. Suppose a piano slides a distance S down an incline. Assume a coefficient of friction of μ_k.

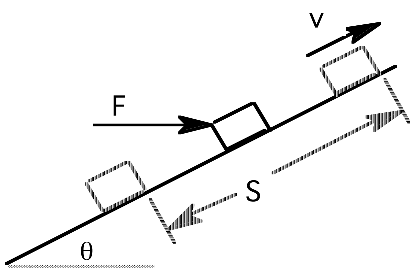


Write expressions for how much work was done by gravity and how much work was done by friction.

What would be the speed of the piano after sliding the distance S?

How much work would have to be done to bring it to a stop?

7.

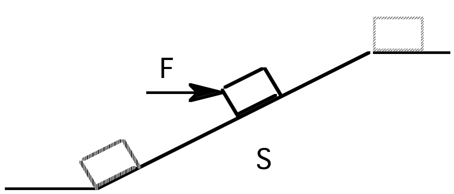


Consider that a <u>horizonta</u>l force F pushes a block of mass M a distance S up an incline. The coefficient of friction between the block and incline is μ.

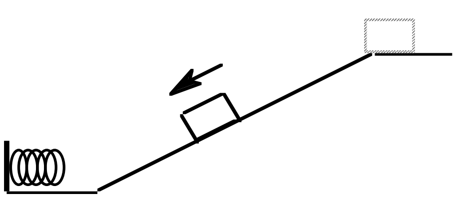
Write the expressions (in terms of the quantities M, g, F, μ, and θ) for the work done by each of the following: The force **F**; the gravitational force; the force due to friction; the normal force.

Write an expression relating the total work done by all the forces to the final speed of the block, if the block started at rest.

8.



a. Determine the work done by a horizontal force to push a block of mass M a distance S up an incline which makes an angle θ with the horizontal. Assume there is no friction.
EXPLAIN YOUR REASONING.

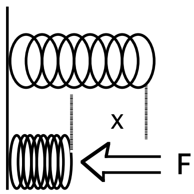


b. Now consider that the block slides back down the incline to a spring of spring constant k. Determine the speed of the block just as it strikes the spring and the amount Δx that the spring is compressed in bringing the block to a stop. [Your answers should be expressed in terms of "known" quantities – M, g, S, k, and θ.]

How would the two parts of this problem change if there were friction between the block and incline?

9. Suppose a particular *non-linear* spring (which does not obey Hooke's law) requires a force

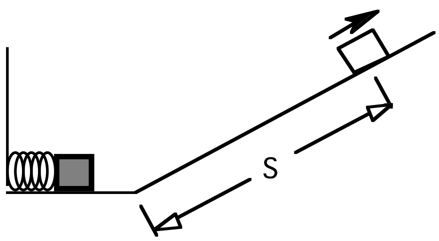
$$F(x) = k_1\, x + k_2\, x^2$$



to compress it an amount x from its relaxed length.
(The x^2 term means the spring gets <u>stiffer</u> as its compressed.)

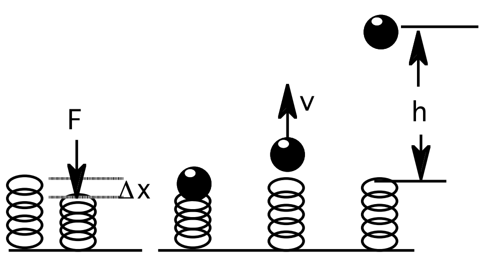
Determine an expression for the potential energy of the spring when it has been compressed an amount *X*. Explain your reasoning.

10. Suppose a block of mass M is set against a compressed spring as shown, and then the spring is released. Obtain an expression for the speed the block will have after it has travelled a distance S up an incline. Assume the horizontal surface is frictionless, but the incline has a kinetic coefficient of μ_k.



Using the expression you have found for the speed the block has when it is a distance S along the incline, determine the maximim distance S_{max} it will slide before coming to a stop.

11. Suppose a vertical spring is compressed an amount Δx. A ball of mass m is then placed on the spring. The spring is then released, shooting the ball vertically upward.

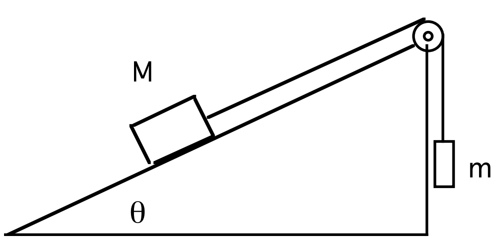


a. Write an expression which relates k, m, g, Δx and the speed of the ball as it leaves the spring. Explain.

b. Write an expression which relates the speed of the ball as a function of height y above the spring. Explain.

c. Write an expression for the maximum height h above the top of the spring (assuming you know the values of m, k, Δx, and g).

12. A block of mass M is on an incline which makes an angle θ with the horizontal. It is connected to a second block of mass m as shown. The coefficient of kinetic friction is μ_k .

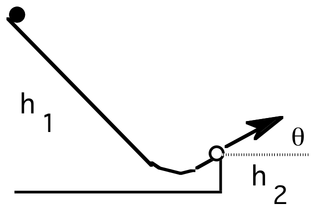


Set up the equations that would allow you to determine the speed of the blocks after the small block falls a distance h. EXPLAIN YOUR REASONING

Now suppose the angle is 10°, m is 200 g, and M is 400 g, the coefficient of friction is 0.3, and the block *m* falls a distance 0.5 m.

Determine the final speed of the blocks.

13. A marble is released at point A and rolls down a track as shown. It leaves the track at point B at an angle θ with respect to the horizontal. Assume no frictional losses along the track.



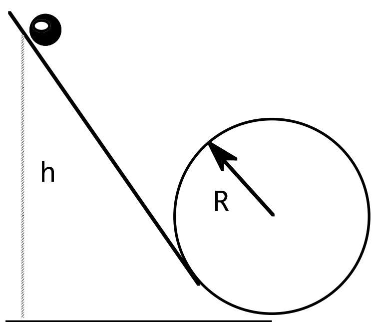
m = 10 gm h = 1 m h = 0.2 m θ = 30°

a. Determine the speed of the marble as it leaves the track.

b. Determine the marble's height above the table at the highest point in its trajectory.

c. Determine the speed of the marble when it hits the table.

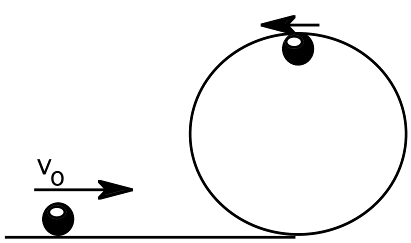
14. Consider that a ball can roll down a ramp and around a vertical circular track, as shown.



Obtain an expression for the minimum height from which the ball must be released to be able to go all the way around the track without losing contact. Explain your reasoning carefully.

[Note: For reasons that will not be clear until after we have covered rotational motion, the *actual* minimum height is about 40% more than what is found in this problem.]

15. Consider that a ball rolls along a horizontal track which then has a vertical loop of radius R as shown.

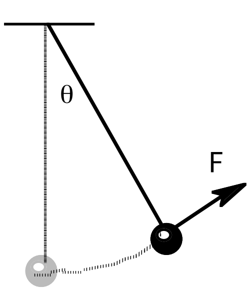


Obtain an expression for the minimum initial speed the ball must have in order to complete the loop without losing contact assuming there are no frictional losses. (Express your answer in terms of m, g, and R.) Explain your arguments carefully.

If the ball has a mass of 50 grams and the radius of the loop is 50 cm, determine the initial speed required.

[Note: For the same reasons as in the previous problem, the actual speed is greater.]

16.

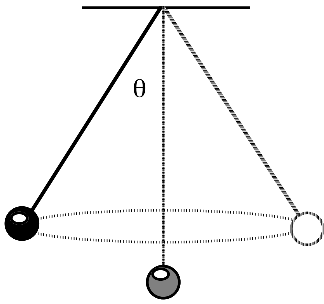


Suppose a pendulum of length L is displaced through an angle θ by some force which is applied tangentially to the arc as shown. Derive an expression for the minumum work done by F in displacing the pendulum through the angle θ.

[HINT: Assume the force F is equal in magnitude to the tangential component of the gravitational force and calculate $W=\int F dS$ where $dS=Ld\theta$. Express your answer in terms of the quantities: M, g, L, θ.]

If the pendulum is released from having been displaced through the angle θ, obtain an expression for its speed as it swings through equilibrium. JUSTIFY.

17. Consider a mass M on the end of a string of length L.

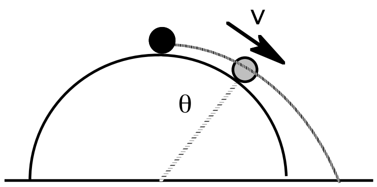


Determine the work required to make the mass "orbit" in a horizontal circle at an angle θ with respect to the vertical.

Set up the problem carefully and explain your reasoning. Assume you know the values of M, g, L, and θ - so your expression for work must be in terms of those quantities.

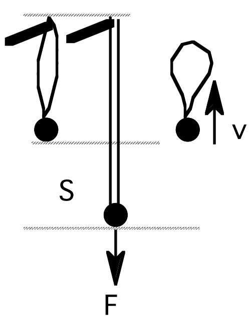
[Hint: The total work you must do can be accounted for in the total energy the pendulum has when it is in its circular orbit.]

18. Suppose a ball is placed on top of a hemisphere, then nudged so that it would roll off.



Use energy conservation and circular motion arguments to determine the angle at which the ball loses contact with the hemisphere.

19. Consider a "rocket ball" – a rubber ball with a strong rubber band attached. Suppose the force to stretch the rubber band an amount x from its rest length is given by $F = k\sqrt{x}$



a. Sketch a graph of the force required to stretch the rubber band as a function of x. On the graph of F *vs* x, how would the *work* to stretch the rubber band be represented? Explain.

b. Determine an expression for the potential energy stored in the rubber band when it is stretched it a total amount S.

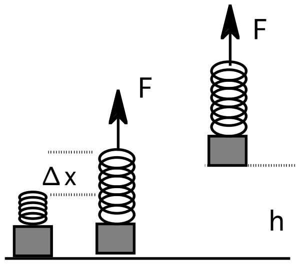
c. Determine an expression for the maximum speed the ball attains when it is released. (Assume v_{max} occurs when the ball passes the point where the rubber band is "relaxed".

d. Determine an expression for the maximum height the ball can be shot if it is stretched an amount S and released straight up, assuming no energy losses in the rubber band or in air resistance.

20. Suppose you throw a ball straight up. If its maximum height is y_{max}, determine how high will it be when its speed is <u>half</u> its initial speed assuming no air resistance. Explain your reasoning.

If there IS air resistance and you throw a ball straight up with initial speed v_o and it reaches a maximum height of y_{max},, set up the expression that would let you determine the work due to air resistance. Explain your reasoning.

21. Consider that you wish to lift a block of mass M with a Hooke's law spring of spring constant k.



a. Starting from the definition of work, obtain an expression (in terms of M, g, and k) for the work required to just barely lift it from the table – *i.e.*, so that the table no longer supports the mass. Justify your reasoning.

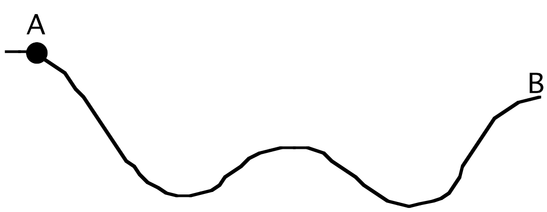
b. Starting with the work-energy theorem, obtain an expression (in terms of M, g, k, and h) for the total work you must do to lift the block to a height h starting from the relaxed spring. Justify your reasoning.

22.

A 50 kg skier starts from rest at the top of a hill and skis down a slope which has a total elevation change of 80 meters. Her final speed at the bottom of the slope is 30 m/s. Determine the amount of work done by friction and air resistance. SHOW YOUR WORK AND EXPLAIN YOUR REASONING. (Let g=10 m/s^2)

Sketch a graph which shows (a) the potential energy, (b) the kinetic energy, and (c) the total energy as a function of time. [Sketch the three graphs on the same set of axes, and identify the graphs.] Explain your reasoning.

23. Suppose a bead slides from A to B without friction along the wire shown. Identify where it has maximum values for the following quantities:

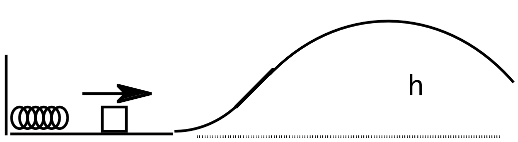


speed acceleration potential energy

kinetic energy total energy

If it is released at point A with initial speed 1 m/s and it arrives at B (one meter below A) with speed 4 m/s, calculate how much work was done by friction. Assume a mass of 10 grams.

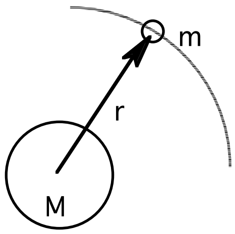
24.



Suppose an ice cube is pushed against a spring, held, and then released. It slides toward a hill as shown.

If the spring has spring constant k and is initially compressed an amount Δx, explain how you would determine whether it would slide over the top of the hill. If it did, show how would you determine its speed at the top.

If the block were not an ice cube, *ie*, if there was friction, how would you determine the work done by friction if you measured the speed at the top?

25. 

Consider that magnitude of the gravitational force between any two objects of masses M and m is given by

$$F_g = G\frac{mM}{r^2}$$

where G is the Universal Gravitational Constant and r is the distance between the centers of the two masses. The value of G is $6.7\text{x}10^{-11}$ N-m^2/kg^2.

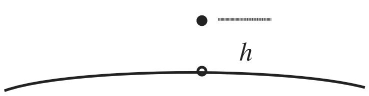
Now consider that the potential energy of interaction at some separation r can be defined in terms of the negative integral of the force over the path from a where the interaction force is zero to where the separation is r – that is,

$$U(r) = -\int_{\infty}^{r} F(r)\,dr$$

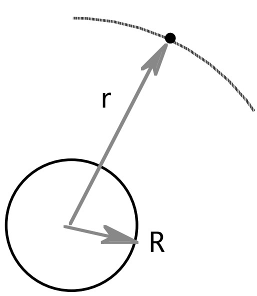
[Note: This is consistent with how U was introduced – *i.e.*, so that $\Delta U = -W_{cons}$.]

a. Show that the general expression for the gravitational potential energy between the two masses when their separation is r is given by:

$$U(r) = -\frac{GmM}{r}$$

b. 

Now consider that M is the earth and an object of mass m is lifted from the surface of the earth (radius R_E) to a height h above the surface of the earth (*i.e.*, from r to $r=R_E+h$ where $h<<R_E$). Starting from the above gravitational potential energy expression, show that the *change* in the potential energy is just mgh (that is, as long as $h<<R_E$).

26. 

a. Obtain an expression for the total energy required to orbit a satellite of mass m at a radius r from the Earth's center. That is, determine the work required to both get the satellite from the surface of the earth to that radius $r>R_E$ and also to give it sufficient speed to stay in orbit at that radius.

[HINT: The energy required would be the sum of the increase in potential energy and the kinetic energy required to remain in that stable orbit at radius r. Recall circular motion arguments.]

b. Determine a value for the minimum energy required to put a 1 kg object in orbit at twice the earth's radius.

27. Determine a value for the minimum energy required for a 1 kg object to escape from earth's gravitational pull. What speed would you have to give that mass for it to escape from earth? Does the minimum escape speed depend on the mass of the object?