Chapter 6 Two Dimensions

Figure 6.1 The archer on the left has just fired an arrow at the target. The arrow is visible just to the right of the first tree on the left. If it continues in the direction it is currently moving, it will go high over the target. But, during the arrow's flight the force of gravity pulls it downward, changing the direction of the arrow's flight so it hits the target.

Section 6.1 Reference frames

So far, we have only dealt with problems that are one-dimensional, either vertical or horizontal. But we live in a world that is more complicated than that, so let's start to consider what happens when things can start moving in more than one direction. To begin, imagine holding a ball and dropping it, watching it fall to the ground. A motion map for this would look something like Figure 6.2.

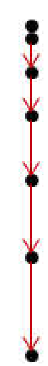


Figure 6.2 A motion map of a ball falling under the influence of gravity

That should look fairly familiar at this point. Let's consider something else that should be familiar. Imagine you are standing on a train that is moving to the right at constant speed, holding a ball in your hand. The motion map for the ball would look something like Figure 6.3.

Figure 6.3 A motion map of a ball on a train moving at constant velocity

There is a conceptual leap between the motion maps in Figure 6.2 and Figure 6.3, and I want to pause for a moment and make sure that you understand the leap. In Figure 6.2, you were holding the ball and then watching it drop. Your point of view, or "reference frame" was yourself. You were standing there watching the ball fall. Perhaps it is our innate egocentricity, but we tend to always see the world from our own reference frame, as if we ourselves are not moving. For instance, if you are sitting in an airplane flying eastward, you don't experience yourself as moving. You see yourself sitting stationary in a seat, and everyone around you is also stationary. You look down, and you see the ground going backwards, westward. But it is possible to also consider someone else's reference frame. Imagine a person on the ground watching you in the airplane above them. From their point of view, they are stationary and you are moving eastward.

Look again at Figure 6.3 and the description just above it. In that figure, the reference frame is not yourself. If you used yourself as a reference point, and all you are doing is holding the ball in your hand, the motion map would just be a dot. No motion! What did we use as a reference frame in Figure 6.3? The ground outside the train—you can think of it as somebody standing outside the train watching you go by with the ball in your hand. Figure 6.3 is drawn from that person's reference frame.

Sometimes switching to a different reference frame can make a situation easier to understand, and the important thing to remember is that ***as long as your reference frame is not accelerating, all of the laws of physics remain true***. If your reference frame is accelerating, "apparent forces" will appear, seeming to change the laws of physics. An example of this is trying to take a drink from a glass while traveling in a car. Probably you are skilled enough to drink from a glass without making too much of a mess under normal circumstances. But what if you are taking a drink from a glass while in a car when the driver suddenly speeds up, slams on the brakes, or makes a sharp turn? When that sudden acceleration happens, all of the expected laws of physics in the reference frame of you sitting in a car are suddenly modified, causing your drink to experience unexpected forces in unusual directions. And all of your years of practice at learning to master the laws of physics related to drinking from a glass become futile, with the result that your shirt is drenched and your friendship with the driver is tested.

If we want to take advantage of using different reference frames, it is important for us to be able to convert between them. The left side of Figure 6.4 shows how a tree sees the world, where we let East be the positive direction. The tree sees Pete driving his red car at 30 m/s East (positive direction), and sees you driving at 20 m/s West (negative direction) in your race cart. Note that the tree sees itself as having no velocity.

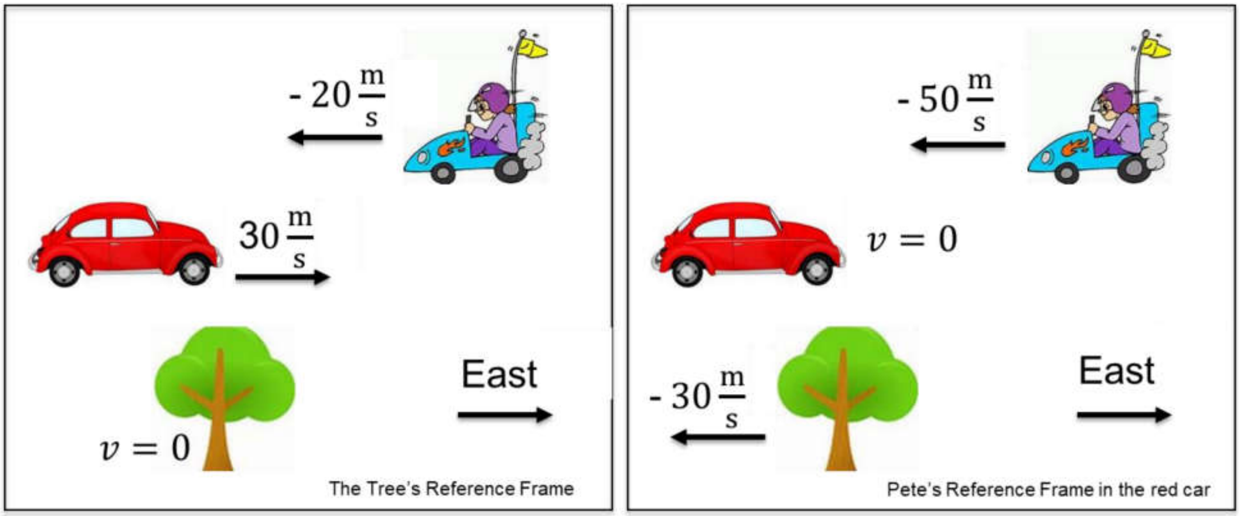


Figure 6.4 Comparison of reference frames

Now, imagine Pete in his red car in the picture on the right side of Figure 6.4. Pete sees himself sitting still (stationary in his reference frame!), but he sees the tree zipping backwards past him, going to the West (negative) at 30 m/s! And Pete also sees you really blasting by him with a velocity of 50 m/s in the negative direction. Note the way that the velocities compare in these two frames. In the tree's reference frame, the tree's velocity is zero, and everything else's velocity is relative to the tree. In Pete's reference frame, his velocity is zero. To switch from the reference frame on the left to the one on the right, in order to change Pete's velocity to zero we had to subtract 30 m/s. So for all other speeds in Pete's reference frame, we need to subtract that same 30 m/s. In other words, to switch from the tree's reference frame to Pete's reference frame, you just need to subtract Pete's velocity from everything in the tree's reference frame.

Exercise 6.1 **Draw the scenario from Figure 6.4 from the reference frame of you in the blue cart. Include the speed and direction of you, Pete, and the tree in your drawing.**

Exercise 6.2 **A claim was made in this section that as long as your reference frame is not accelerating, all of the laws of physics remain true. Test that claim by considering the following car crash: You are traveling North at 15 m/s, and a car next to you with a mass of 400 kg is also traveling North at 15 m/s. Another 400-kg car is ahead of you traveling South, also at 15 m/s. The oncoming car collides with the car that was traveling next to you, bringing both of them to a complete stop as observed by a person standing by the side of the road.**

a. Verify that the momentum of the two cars that were involved in the collision was conserved in the reference frame of the person standing by the road.

b. Verify that the momentum of the two cars that were involved in the collision was conserved in your reference frame as you continue driving North at 15 m/s.

c. Find the amount of kinetic energy that was changed to thermal energy in the crash in the reference frame of the person standing by the road.

d. Find the amount of kinetic energy that was changed to thermal energy in the crash in your reference frame as you continue driving North at 15 m/s. .

e. **Does the amount of energy that was changed into thermal energy depend upon your reference frame?**

f. **Does the total amount of kinetic energy of each vehicle depend upon your reference frame?**

g. **Did you find anything that defied the laws of physics when you changed reference frames?**

Section 6.2 **Two-dimensional motion maps**

Let's look again at the motion maps in Figure 6.2 and Figure 6.3. The first figure shows the ball falling in your reference frame, and the second shows you holding the ball from the reference frame of somebody who is outside of the train, watching it go by. Now let's try to imagine what the person outside the train would see if you let go of the ball as you were going past them. As was described in the last section, the way to convert from one reference frame to another is to subtract the speed of the object on which the new reference frame will be based from everything in the old reference frame.

If we start in your reference frame on the train, we can watch the ball accelerate downward. Now if we want to switch to the reference frame of the person outside of the train, we need to subtract that person's velocity in your reference frame from the velocity of the falling ball as seen in your reference frame. In a motion map, the arrows that go from one dot to the next represent velocity. In the second motion map, all of the arrows point to the right, indicating that the person who is not on the train sees you moving to the right. That means that you would see that person moving to the left. Now to switch the falling ball from your reference frame to that of the person outside of the train, you have to subtract those left-pointing velocity arrows from the falling ball's downward-pointing arrows. And as it turns out, subtracting arrows that go to the left is the same as adding arrows that go to the right. (A double negative! Read through that a couple of times to convince yourself.) So to see the ball falling from the reference frame of the person outside the train, we need to add the arrows in Figure 6.2 to the arrows in Figure 6.3. Remember that we add vectors by putting the arrows tail-to-tip. So combining those arrows using vector addition gives the motion map shown in Figure 6.5.

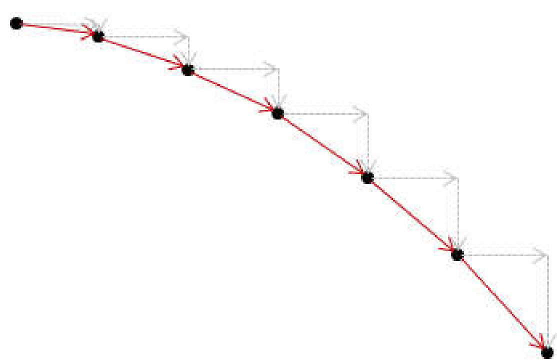
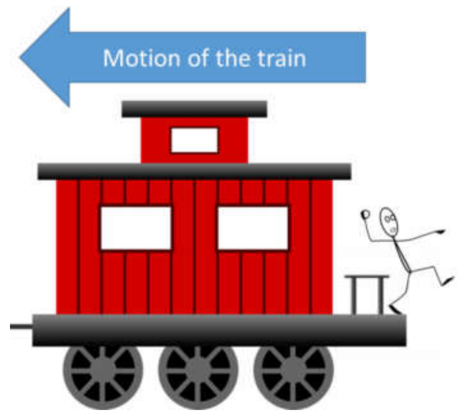


Figure 6.5 A two-dimensional motion map of a ball that is dropped from a moving train, from the frame of reference of someone standing outside of the train. The horizontal dashed gray arrows show the motion of the train as observed by the person standing outside of the train. The vertical dashed gray arrows show the motion of the ball as observed by a person on the train.

So while the person on the train where the ball was dropped sees it falling straight down, the person who is outside of the train sees it falling in a parabolic shape. In both cases, the horizontal velocity is constant—in the train's reference frame the horizontal velocity of the ball is zero, and in the reference frame of an observer who is not on the train, the horizontal velocity of the ball is equal to the velocity of the train.

Exercise 6.3 **Draw motion maps for a rock that is not dropped on a train, but thrown from a platform on a train. Use right as the "positive" direction, and have the person on the train throw the rock horizontally to the right while the train is moving to the left.**



a. **Draw a motion map for the rock in the reference frame of the person throwing it.**

b. **Draw a motion map for the rock in the reference frame of a person standing next to the track, if the train was moving 10 m/s to the left and the person on the train (in their frame of reference) threw it at 10 m/s to the right.**

c. **Draw a motion map for the rock in the reference frame of a person standing next to the track, if the train was moving 20 m/s to the left and the person on the train (in their frame of reference) threw it at 10 m/s to the right. Would the rock hit the train in this situation? Why or why not?**

Section 6.3 **Motion in Two Dimensions**

In the two-dimensional motion map in Figure 6.5, notice that there is an acceleration, caused by gravity. It is changing the vertical part of the velocity. But look at the horizontal part—even though there is an acceleration of gravity, the part of the velocity that is horizontal stays constant, as if there were no acceleration at all. ***This is the key to understanding two-dimensional motion: the horizontal (usually called "x") components of the motion are completely independent of the vertical (usually called "y") components of the motion.***

If you need to cross a river in a boat, the speed of the water flowing in the river could be very small or very large, but no matter how quickly the water is flowing it will not help you to get across. It's going the wrong way for that. Although this is less obvious, a rapidly flowing river doesn't actually slow down your ability to get to the other side of the river if you row straight across the river. But if it is flowing quickly it will mean that you will reach the other side far downstream from where you started. This is shown in Figure 6.6.

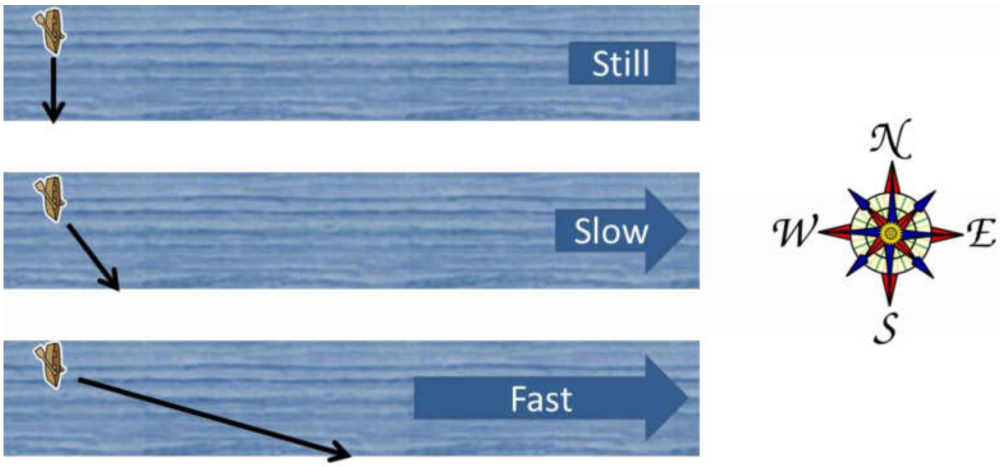


Figure 6.6 Three rowboats pointing directly South across a rivers that are flowing East. The rate that the boat crosses the river depends only on the velocity of the boat going South. But the relative speed of the East-flowing river determines how far the boat will travel downstream before reaching the other side.

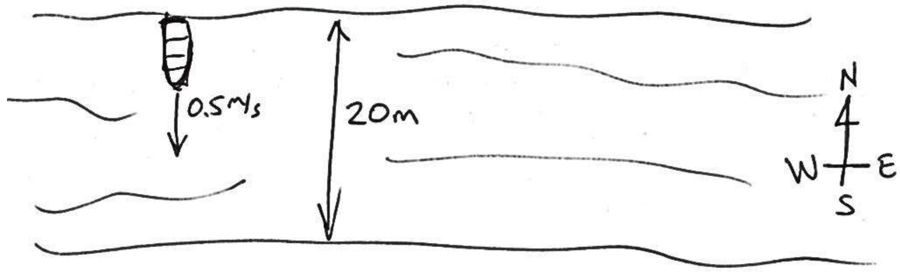
The distance the boat will travel downstream depends upon the velocity of the river and also the time it takes to cross. The "kinematic equations" of motion can be used to solve this problem. Since the "x" and "y" directions are independent of each other, the equations can be written completely separately for x and y:

Table 6.1 Kinematics equations in two dimensions. Remember that these equations are only valid when acceleration is constant!

$v_x = v_{0x} + a_x t$	$v_y = v_{0y} + a_y t$
$\Delta x = \left(\frac{v_x + v_{0x}}{2}\right) t$	$\Delta y = \left(\frac{v_y + v_{0y}}{2}\right) t$
$\Delta x = v_{0x} t + \frac{1}{2} a_x t^2$	$\Delta y = v_{0y} t + \frac{1}{2} a_y t^2$
${v_x}^2 = {v_{0x}}^2 + 2a_x \Delta x$	${v_y}^2 = {v_{0y}}^2 + 2a_y \Delta y$

Notice that "x" position, velocity, and acceleration are all only in the "x" side of this table of equations, and the "y" position, velocity, and acceleration are all only in the "y" side of this table of equations. The only thing that appears on both sides is the time "t." The time can be a useful link between the two.

Consider the rowboat crossing the rivers in Figure 6.6. If the rowboat can go forward at 0.5 m/s, the river is 20 m wide, and it is completely still like in the top part of the figure, how long will it take for the rowboat to reach the other side, and how far will it have drifted downstream by the time it reaches the opposite bank? We can solve the problem like this:



We should only need to use the motion lens.
We can define North as the "+y" direction.

The rowboat moves at −0.5 m/s in the y direction (down in the picture, and up was the positive direction).

The boat has to go a displacement −20m in the y direction to reach the opposite shore.
There is no acceleration in the y direction. We can use

$$\Delta y = V_{oy}t + \frac{1}{2}\cancel{a_y}^{0}t^2$$

$$\Delta y = V_{oy}t$$

$$t = \frac{\Delta y}{V_{oy}} = \frac{-20\,m}{-0.5\,m/s} = \boxed{40\,s}$$

It will take 40 s to cross the river.
In the "x" direction, which would be East, there is no velocity and no acceleration, so

$$\Delta x = \cancel{V_{ox}}^{0}t + \frac{1}{2}\cancel{a_x}^{0}t^2 = 0$$

It will not drift downstream at all!

And what if the river is flowing, like in the middle part of Figure 6.6? We would solve it this way:

The "y" direction is exactly the same, so it will still take 40s to cross the river. But the "x" direction is different.

Now there is a constant velocity of 0.6 m/s in the positive x (East) direction. So,

$$\Delta x = V_{0x} t + \frac{1}{2} a_x t^2 = (0.6\,m/s)(40\,s)$$

$$\boxed{\Delta x = 24\,m}$$

The rowboat will drift 24 m downstream before hitting the opposite shore.

Exercise 6.4 **Refer to Figure 6.6. If the rowboat can go forward at 0.8 m/s, the river is 24 m wide, the rowboat is pointed directly South, and the river is flowing East at 4 m/s…**

a. How much time will it take the rowboat to reach the opposite bank?

b. How far will the rowboat have drifted downstream by the time it reaches the opposite bank?

Section 6.4 **Ballistic Motion**

There is a class of physics problems that fall into the category of "ballistic motion." Ballistic motion is the motion of an object under the influence of gravity and no other forces. This is the situation with many objects that are launched into the air, so we will look at a few of these problems. We have actually already done some ballistic motion, for example the motion maps in Figure 6.5 and Exercise 6.3. Let's look again at that exercise. If the person throws the rock at 10 m/s in a horizontal direction, the rock has a mass of 2 kg, and it leaves his hand at an initial height that is 3 m from the ground, let's see what we can learn about the path the rock takes, and any other information we can learn along the way as well.

To start, we already learned from the motion map what the path will look like, and we can add the numbers given to the picture.

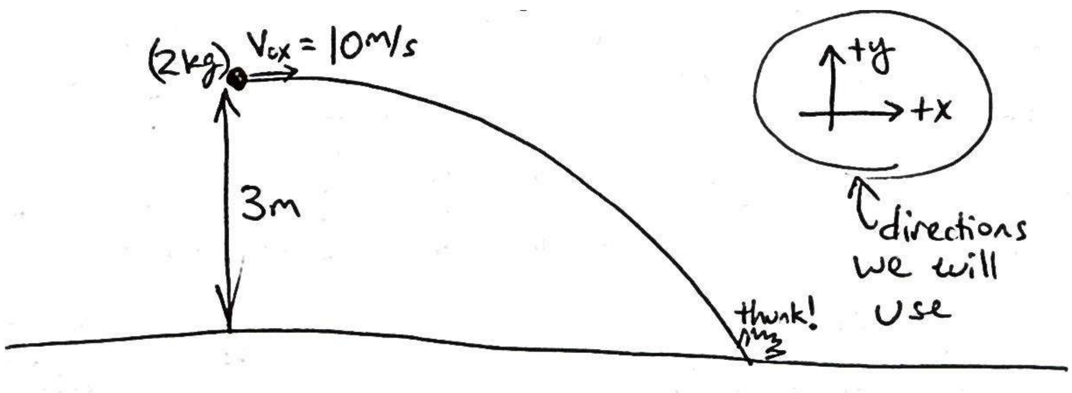


Figure 6.7 Sample problem for ballistic motion

Next, we know the initial energy of the rock,

$$E_i = mgy_i + \frac{1}{2}mv_i^2$$

where y_i is 3 m and v_i is 10 m/s in the positive x direction.

$$E_i = (2\,kg)(10\,m/s^2)(3\,m) + \frac{1}{2}(2\,kg)(10\,m/s)^2$$

$$\boxed{E_i = 160\,J}$$

And if we assume that there is no air resistance, that tells us the final speed of the rock just before it hits the ground:

$$E_f = mg\cancelto{0}{y_f} + \frac{1}{2}mv_f^2$$

$$160\,J = \left(\frac{1}{2}\right)(2\,kg)\,v_f^2$$

$$v_f^2 = 160\left(\frac{m}{s}\right)^2$$

$$\boxed{v_f = 12.6\,m/s}$$

So now we know the final speed, but we do NOT know the direction in which the rock is moving. That 12.6 m/s is partly in the x direction and partly in the y direction. Now let's try the motion lens. First in the x direction. We know v_{0x} = 10 m/s. The only acceleration in the air is gravity in the y direction, so a_x = 0. That means

$$\Delta x = v_{0x}t + \frac{1}{2}\cancelto{0}{a_x}t^2 = v_{0x}t$$

$$\Delta x = (10\,m/s)\,t$$

so x is constantly changing. And if we knew the time ***t*** when the rock hit the ground, we would know how far it traveled horizontally before hitting the ground. Maybe we can get that from the y direction. The equation will look similar to the one we used in the x direction.

$$\Delta y = v_{0y} t + \frac{1}{2} a_y t^2$$

The initial velocity was only in the x direction, so $v_{0y} = 0$. Acceleration in the y direction was gravity, pointing down, so $a_y = -10\ m/s^2$. So,

$$\Delta y = v_{0y} t + \frac{1}{2} a_y t^2 = -\frac{1}{2}(10\ m/s^2) t^2$$

$$\Delta y = (-5\ m/s^2) t^2$$

We know that the rock will hit the ground when $y = 0$, and the initial y was +3 m, so the rock hits the ground when

$$\Delta y = y_f - y_i = 0 - 3m = -3m$$

So now we know

$$\Delta y = (-5\ m/s^2) t^2$$

and

$$\Delta y = -3m.$$

So,

$$-3m = (-5\ m/s^2) t^2$$

$$t^2 = \left(\frac{3}{5}\right) s^2$$

$$t = \sqrt{\frac{3}{5}}\ s = 0.77s$$

That is the amount of time that the rock was in the air after being thrown. We can use that to find how far the rock went horizontally before it hit the ground:

$$\Delta x = (10\ m/s) t = 7.7m$$

We also know a bit more about the velocity just before the rock hit the ground.

$$V_x = V_{ox} + a_x t \quad (a_x \to 0)$$

$$\text{So} \quad V_x = V_{ox} = 10\ \text{m/s.}, \quad \text{so} \quad V_{fx} = 10\ \text{m/s}$$

$$\text{And,} \quad V_y = V_{oy} + a_y t \quad (V_{oy} \to 0) = (-10\ \text{m/s}^2)(0.77\ \text{s})$$

$$V_{fy} = -7.7\ \text{m/s}$$

Wait a minute! Do we have a mistake somewhere? The first thing we found was that the final speed $\boldsymbol{v_f}$ = 12.6 m/s. But then we found the final velocity in the x direction was $\boldsymbol{v_{fx}}$ = 10 m/s and the final velocity in the y direction was $\boldsymbol{v_{fy}}$ = -7.7 m/s. But how can that be? Shouldn't $\boldsymbol{v_f} = \boldsymbol{v_{fx}} + \boldsymbol{v_{fy}}$? Actually, it does, sort of! Remember, ***v*** is supposed to be a vector, so in fact if we want to add the x and y components of the velocity you have to use vector addition:

$$\overrightarrow{v_f} = \overrightarrow{v_{fx}} + \overrightarrow{v_{fy}}$$

And remember, with vector addition we need to add tail-to-tip. So, the vector addition of the x and y components of the velocity looks like this:

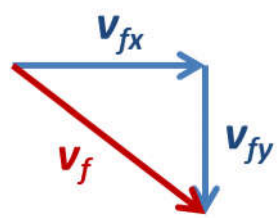


Figure 6.8 Using vector addition to combine final x and y components of velocity into a final velocity vector

And if you have ever learned the Pythagorean Theorem, you may recognize that this is a right triangle, so the sum of the squares of the two legs ($\boldsymbol{v_{fx}}$ and $\boldsymbol{v_{fy}}$) should equal the square of the hypotenuse ($\boldsymbol{v_f}$). And in fact this is the case. $10^2 + 7.7^2 = 12.6^2$! I didn't use -7.7, because the arrow in the picture already shows it going downward, in the negative direction. But even if I did use the negative value it wouldn't matter, because $10^2 + (-7.7)^2 = 12.6^2$ as well. So we don't have a mistake after all!

Exercise 6.5 **The sample problem we just did had a 2-kg rock being thrown horizontally from 3 m above the ground and we found the time it took to hit the ground (0.77 s), the distance it moved horizontally in that time (7.7 m), and the final velocity of the rock (10 m/s horizontally to the right and 7.7 m/s vertically downward). Now let's change the problem slightly.**

This time, imagine the rock is launched upward from the ground back along the direction it just came from. So it starts on the ground with an initial velocity of 10 m/s horizontally to the left and 7.7 m/s vertically upward.

a. Make a diagram of the path the rock will take.

b. What is the initial energy of the rock?

c. What will the velocity of the rock (x and y direction) be after 0.77 seconds?

d. How high above the ground will the rock be after 0.77 seconds?

e. What will the energy of the rock be after 0.77 seconds?

f. If the rock continues in its flight until it hits the ground again, what will its velocity in the x and y direction be just before it hits the ground?

g. What will the displacement of the rock be from its original position just before it hits the ground?

h. What will the energy of the rock be just before it hits the ground?

i. Do your answers make sense when you compare them to the sample problem? Explain.

Figure 6.9 Two archers are aiming at targets the same distance away. The arrow of the archer on the left is pointed upward more steeply than the arrow of the archer in the center. They are both proficient at archery so both of their arrows will actually hit the targets.

Exercise 6.6 See Figure 6.9. The two archers are aiming at targets that are 40 m away. The targets are both at the same height above the ground, roughly the same height from which the arrows are being fired. The archer on the left is using a recurve (Olympic style) bow, and the archer in the middle is using a compound (hunting style) bow. If the arrows traveled in a straight line with no gravitational force, the one fired from the recurve bow would be too high, missing the target by 12 m. The arrow fired from the compound bow would also be too high, but only by 2 m.

a. Make a drawing showing both archers firing from the same location, hitting the same target, but along different paths based on the angle that the archers are aiming their bows. Include dotted lines showing the paths the arrows would take if gravity had no effect.

b. **By how much does the acceleration of gravity change the vertical position of the arrow fired from the recurve bow by the time it reaches the target?**

c. **How much time would the arrow from the recurve bow have to spend in the air so that gravity would have the effect described in part b?**

d. **Find the initial horizontal speed of the arrow fired from the recurve bow.**

e. **Find the initial vertical speed of the arrow fired from the recurve bow. (Use your dotted line!)**

f. **Use the Pythagorean Theorem to find the initial speed of the arrow fired from the recurve bow.**

g. **By how much does the acceleration of gravity change the vertical position of the arrow fired from the compound bow by the time it reaches the target?**

h. **How much time would the arrow from the compound bow have to spend in the air so that gravity would have the effect described in part g?**

i. **Find the initial horizontal speed of the arrow fired from the compound bow.**

j. **Find the initial vertical speed of the arrow fired from the compound bow.**

k. **Use the Pythagorean Theorem to find the initial speed of the arrow fired from the compound bow.**
