Physics

in Four-Part Harmony

by Dean A. Stocker

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Contents

Chapter 1

Our Physics Toolbox

This book approaches physics in a way that is unusual for a physics textbook but is quite natural in our daily lives. If you were an apprentice carpenter, you would expect to quickly start working with an array of different tools. You would also learn to use these tools in a variety of ways, perhaps reading, watching, listening to instruction, and trying them out yourself. Think of yourself now as an apprentice physicist, learning to use the tools of physics.

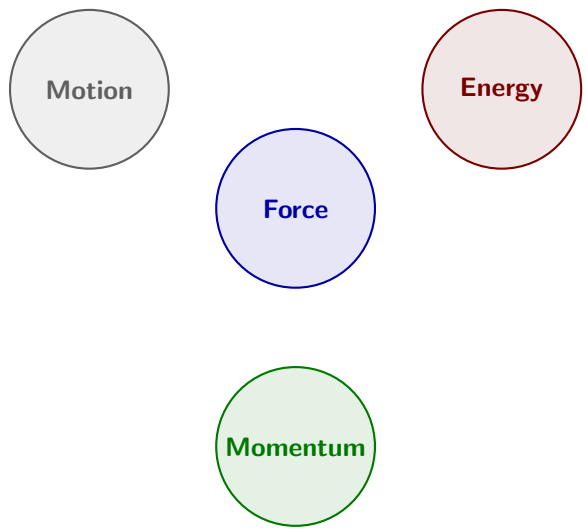


Figure 1.1: Template for a concept map in physics[1]

This chapter introduces two different types of "tool sets." One is a set of conceptual ideas: motion, momentum, energy, and force. It may be helpful to create a concept map to give yourself a framework for how all of these ideas are connected. The template in Figure 1.1 is a good place to start. Make a larger version of it, then add ideas and connections within and between the colored areas. All of these concepts will be built up in parallel over the course of this book.

The other "tool set" that is introduced in this chapter includes three different but interrelated approaches to help us understand each of these concepts: Words, Graphics, and Numbers. Each of these approaches is presented in parallel in different columns on most pages of this book, to emphasize that these three approaches are interrelated and can be used together to build a more complete understanding of a physical situation.

1.1 Motion, Momentum, Energy, and Force

Words

The left columns of this book generally contain verbal representations of physics.

Physics is an attempt to describe the behavior of the physical universe. But the universe is complicated, so it is necessary to make some simplifications.

The approach of this book is to start out by looking at the behavior of very simple systems, ignoring complicating effects, and as we learn to describe the simple systems we will gradually move to more and more complicated systems.

We will start with the area of physics that is called mechanics. There are four main concepts involved in mechanics: motion, momentum, energy, and force.

Pay attention to the color coding. The words, graphics, and math are all color coded to help show which concept is being considered.

Motion is a description of how an object moves over the course of time. This includes the object's position; its velocity, which is another way of saying its speed and its direction of motion; and its acceleration, which is how quickly the object's velocity is changing in time.

Graphics

The center columns of this book generally contain graphical representations of physics.

Photos or drawings

Figure 1.2: A galaxy. Because every physics textbook should include a picture of a galaxy![2]

Motion maps

Figure 1.3: Example of a motion map[1]

Numbers

The right columns of this book generally contain mathematical representations of physics.

A letter or symbol in italics is a variable used to describe some physical quantity. The same letter will always represent the same type of quantity. A lower-case m, for example, will always represent a mass.

A letter or symbol with a half-arrow on top, like $\vec{x}$, represents a vector. With a vector it is important to remember that it has a specific direction, often positive (+) or negative (−). Subscripts are used to differentiate between several of the same type of variable in a given situation. For example, if a problem includes an adult and a child, their positions could be $\overrightarrow{x_{adult}}$ and $\overrightarrow{x_{child}}$.

Boxed equations are true except for any limitations described in the accompanying text.

$$\boxed{2 = 2}$$

Unboxed equations are true for a specific example but are not generalizable to all situations.

Motion is described by position $\vec{x}$, velocity $\vec{v}$, and acceleration $\vec{a}$.

Note that all of these physical quantities are vectors.

Momentum is related to the effort that would be needed to stop a moving object. This physics definition overlaps well with the way the word "momentum" is used in our everyday language. If you own a successful business we say that it has momentum, and your competitors will have a hard time stopping you!

In physics, momentum increases with an object's velocity and its mass.

Momentum bars

Figure 1.4: Example of momentum bars[1]

The symbol for momentum is $\vec{p}$.

Note that momentum is a vector. It is associated with an object's velocity $\vec{v}$ and its mass m, and an object's momentum always points in the same direction as the velocity $\vec{v}$.

Energy is an ability to do work. This physics definition also overlaps well with the way the words "energy" and "work" are used in our everyday language. If you have no energy, you can't do any work!

There are several different forms of energy, associated with an object's motion, an object's position, or an object's temperature. Energy is able to transform from one form to another through various physical processes.

Energy Bar Graphs

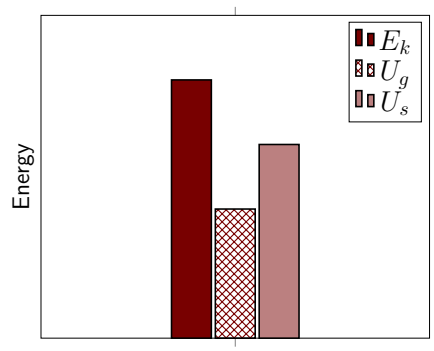


Figure 1.5: Example of an energy bar graph[1]

There are two different symbols for energy: U is used to represent potential energy, which depends upon position; and E is used to represent other types of energy that do not depend upon position. Examples of potential energy are gravitational potential energy U_g and spring (or elastic) potential energy U_s. Examples of other types of energy are kinetic energy E_k and thermal energy E_{th}

Note that energy is a scalar.

Force is a push or a pull. Forces cause changes in motion, momentum, and energy, so forces are truly the heart of mechanics. All the grand theories of physics seek to study the forces that are at work in the universe.

Free-Body Diagrams (FBD's)

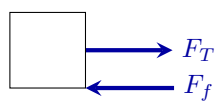


Figure 1.6: Example of a free-body diagram[1]

The symbol for force is $\vec{F}$. Forces cause acceleration. A force acting over time changes momentum. And a force acting over a distance does work, changing energy.

Note that force is a vector.

1.2 A Motionless Rock, in the Horizontal Direction

Words

A 5-kg rock is sitting on a flat place on the ground on a calm day. No wind. No earthquake. You watch for ten seconds. It just sits there in the same place, doing absolutely nothing, for the whole time.

This may seem like a very simple physical situation already, and to make it as simple as possible, we will only consider the horizontal direction.

For this example, it doesn't matter too much which concept we consider first–we're going to end up with a lot of zeros in any case!

In the last section, we ended with forces, so just for fun this time let's start with forces. Remember, a force is a push or a pull. Looking at the photo and considering the description above, what are the forces in the horizontal direction? There is nothing pushing or pulling the rock to the left or to the right! That means there are no forces in the horizontal direction.

Graphics

Figure 1.7: A motionless rock[3]

FBD of Rock - Horizontal Direction
A Free-Body Diagram (FBD for short) is a simple diagram showing the forces that are acting on an object. The object is represented simply by a rectangle. Then arrows are used to represent the forces acting on the object. In this case, there are no forces acting on the rock in the horizontal direction (which is all that we are considering right now), so our FBD ends up being just a rectangle!

Figure 1.8: Free-body diagram of a rock with no forces acting on it[1]

Numbers

The only number given in this example is 5. The number by itself is meaningless; it needs to be attached to some physical quantity, and almost always with a specific unit.

The kilogram, abbreviated [kg], is the unit of mass in the Système International (SI) unit system that has been adopted as the official standard by nearly every country in the world.

The other base SI units are the meter [m] for distance and the second [s] for time. This book will use SI units almost exclusively.

When considering forces and how they affect an object, there are often multiple forces acting at one time. It is important to consider the net force that is acting. The net force is defined as the sum of all of the forces...

$$\boxed{\overrightarrow{F_{net}} \equiv \sum \overrightarrow{F}} \tag{1.1}$$

...where the three horizontal lines mean "is defined as," and the Greek letter Sigma means "the sum of..."

In this particular example, there are no forces at all acting in the horizontal direction, so the net force is zero...

$$\overrightarrow{F_{net}} = 0$$

What can be said about the rock's motion? The rock is not moving, so its position is constant. If it was 2 feet in front of you when you started watching, it was 2 feet in front of you when you stopped (assuming you didn't move).

And no matter where it started, its velocity, which is its change in position over time, is zero. Its acceleration, which is the change in velocity over time, is also zero, because the velocity isn't changing.

Motion map- motionless rock

A motion map uses dots to show position at various points in time, as if you were holding a video camera steady and putting a dot at an object's position in each frame. For this scenario, drawing and numbering one dot per second gives...

Figure 1.9: Motion map showing no motion[1]

If the net force on an object is constant, as it is in this situation, position is given by...

$$\boxed{\vec{x} = \vec{x_0} + \vec{v_0} \cdot t + \frac{1}{2}\vec{a} \cdot t^2} \tag{1.2}$$

...where the subscript "0" means the value at time zero. In this case, $\vec{v_0} = 0$ and $\vec{a} = 0$, so...

$$\vec{x} = \vec{x_0}$$

...which means that $\vec{x}$ at any time t is equal to whatever $\vec{x}$ was at time $t = 0$.

How much effort would be needed to stop this rock? The rock isn't moving, so it would take no effort at all! That means the rock has no momentum.

Momentum bar - motionless rock

A momentum bar is a rectangle with a width proportional to mass and a height proportional to velocity, giving an area proportional to momentum. This momentum bar has a height of zero, so zero area. That means zero momentum.

Figure 1.10: Momentum bar of motionless rock[1]

The momentum of an object is equal to its mass times its velocity:

$$\boxed{\vec{p} = m \cdot \vec{v}} \tag{1.3}$$

As stated above, the velocity $\vec{v} = 0$ in this scenario, so...

$$\vec{p} = m \cdot 0 = 0$$

Remember, energy is an ability to do work. This rock can't do anything! So it has no (useful) energy. Technically, the rock does have thermal energy, because it has a non-zero temperature. And Albert Einstein correctly theorized that mass is also a form of energy. But for our purposes, we will consider only mechanical energy, which consists of kinetic, gravitational potential energy, and spring potential energy.

Energy bar graph - motionless rock

Kinetic energy, gravitational potential energy, and spring potential energy are all are zero, so a bar graph would have no bars.

No motion, so $E_k = 0$. No spring, so $U_s = 0$. And gravitational potential energy is defined based on vertical position y. But we are only considering the horizontal direction, so $U_g = 0$ as well.

1.3 A Motionless Rock, in the Vertical Direction

Words

A 5-kg rock is sitting on a flat place on the ground on a calm day. No wind. No earthquake. You watch for ten seconds. It just sits there in the same place, doing absolutely nothing, for the whole time.

Yes, this is the same physical situation that we have already studied, but now we will consider the vertical direction, which adds another level of complexity.

Let's start by considering the motion of the rock. It is doing nothing more in the vertical direction than it did in the horizontal direction. Its vertical position is unchanging; its vertical velocity is zero; and its vertical acceleration is also zero.

What about the rock's vertical momentum? Again, since this rock isn't moving in the vertical direction, it won't take any effort at all in the vertical direction to make it stop! It has no vertical momentum.

Graphics

Figure 1.11: A motionless rock[3]

Motion map- motionless rock

Figure 1.12: Motion map of no motion. Again![1]

Momentum bar - motionless rock
Zero height means zero vertical momentum.

Figure 1.13: Momentum bar of motionless rock[1]

Numbers

It is helpful to create lists of known and unknown quantities. Only one "known" is specifically listed: 5 kg. Read carefully for others!

- m = 5 kg
- $\vec{v_0} = 0$
- $\vec{a} = 0$

We are not asked to find any specific unknowns; instead, we will use our available tools to find everything we can.

This is exactly the same as the horizontal motion of the rock, but to make it more clear that we are looking only at vertical motion, we can remove the half-arrow and use y in place of x...

$$y = y_0$$

As with motion, we can remove the half-arrow and use a y subscript to indicate that we are only considering the y direction, commonly referred to as "$\hat{y}$."

$$p_y = m \cdot v_y = 0$$

This could also have been done in the $\hat{x}$ direction before, with x subscripts instead of y.

When we consider forces, there is a fundamental difference between the horizontal and vertical direction. The rock isn't moving at all, but there are two different forces that are acting on the rock.

One is a gravitational force, often referred to as "weight." On the surface of the earth, gravitational force always points downward.

The other force is a "contact force" that comes from the ground underneath the rock. If the ground were not there, the rock would be falling, so we know that there is a force from the ground that completely opposes the force of gravity. This contact force is called the "normal" force, where the word "normal" means "perpendicular to the surface." In this case, the normal force is pointing directly upward because the ground is flat.

FBD of Rock - Vertical Direction

Gravitational force points downward, and normal force from the ground points upward. Since we know they cancel each other completely, the lengths of the arrows should be the same.

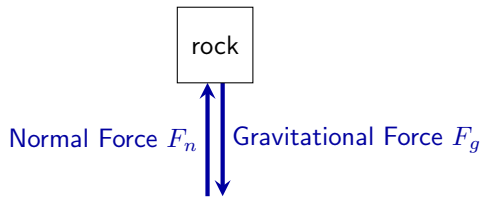


Figure 1.14: Free-body diagram of a rock resting on the ground[1]

Note that we are concerned only with the forces acting on the rock, not about any forces of the rock on something else like the ground.

Since this rock is doing nothing, the net force must be zero.

$$\overrightarrow{F_{net}} \equiv \sum \overrightarrow{F} = \overrightarrow{F_g} + \overrightarrow{F_n} = 0$$

. . . so. . .

$$\overrightarrow{F_g} = -\overrightarrow{F_n}.$$

At the surface of the earth, the gravitational force on an object is. . .

$$\boxed{\overrightarrow{F_g} = -m \cdot g\, \hat{y}} \tag{1.4}$$

. . . where $g = 9.8\ \mathrm{m/s^2}$, the magnitude of the acceleration of gravity at the earth's surface. The magnitude of $\overrightarrow{F_g}$ is given by $m \cdot g$ and the direction is given by $-\hat{y}$ (downward). In this case...

$$\overrightarrow{F_g} = -5\ \mathrm{kg} \cdot 9.8\ \mathrm{m/s^2}\ \hat{y} = -49\ \frac{\mathrm{kg \cdot m}}{\mathrm{s^2}}\ \hat{y} = -49\ \mathrm{N}\ \hat{y}$$

. . . which makes $\overrightarrow{F_n} = +49\mathrm{N}\ \hat{y}$ in this example. The SI unit for force is the Newton [N].

$$1\ \mathrm{N} = 1\ \frac{\mathrm{kg \cdot m}}{\mathrm{s^2}}$$

In the vertical direction, energy depends on position. Breaking an egg requires work, and a rock just lying on the ground can't break an egg. But if you held a rock high above the egg and then dropped it, the rock could break the egg. A rock in an elevated position has the potential to do work–we call that gravitational potential energy. But in this case the rock is on the ground, so it has no gravitational potential energy. Still zero mechanical energy!

Energy bar graph - motionless rock

Again, kinetic energy, gravitational potential energy, and spring potential energy are all are zero, so a bar graph would have no bars.

No motion, so $E_k = 0$. No spring, so $U_s = 0$. And gravitational potential energy is defined based on vertical position:

$$\boxed{U_g = m \cdot g \cdot y}. \tag{1.5}$$

We are free to define the position where $y = 0$, so we can choose our $y = 0$ to make the math as easy as possible. Setting $y = 0$ at ground level gives...

$$U_g = m \cdot g \cdot 0 = 0.$$

1.4 A Rolling Soccer Ball

Words

A 0.4-kg soccer ball is rolling to the right at a constant speed of 9 m/s across a level soccer pitch. Friction force and air resistance are very small, so we will ignore them. We will consider only the horizontal direction.

This situation is different from a motionless rock, because this time the soccer ball is moving. But in terms of horizontal forces, it is exactly the same. Looking at the photo and considering the description above, what are the forces in the horizontal direction? We are specifically told to ignore any friction forces. And there is nothing that is actively pushing or pulling the soccer ball to the left or to the right. That means there are no forces in the horizontal direction.

If there are no forces in the horizontal direction, how does the soccer ball keep moving? It is the ball's momentum that carries it. Until an outside force tries to make the soccer ball stop, it will just continue going in a straight line with the same momentum. This is called "conservation of momentum;" the momentum of any isolated system remains constant.

Graphics

Figure 1.15: A soccer ball rolling to the right[4]

FBD of Soccer Ball - Horizontal
Since we are ignoring friction, there are no forces acting on the soccer ball in the horizontal direction, which is all that we are considering right now.

Figure 1.16: Free-body diagram of a rolling soccer ball, horizontal direction only[1]

Momentum bar - Rolling Soccer Ball

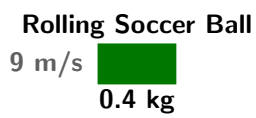


Figure 1.17: Momentum bar of rolling soccer ball[1]

Numbers

Knowns: $m = 0.4$ kg
$v_x = +9\,\mathrm{m/s}$
$a_x = 0$
$F_{net,x} = 0$

Just as "upward" is normally considered the positive vertical direction, "to the right" is normally considered the positive horizontal direction. We know that a_x is zero because the velocity is constant, 9 m/s to the right.

In this example, there are no forces at all acting in the horizontal direction, so the net force is zero...

$$F_{net,x} = 0$$

...where the subscripts "net, x" indicate that we are referring to the net force in the $\hat{x}$ direction.

$$\vec{p} = m \cdot \vec{v}$$

...so...

$$p_x = m \cdot v_x = 0.4\,\mathrm{kg} \cdot +9\,\mathrm{m/s} = +3.6\,\frac{\mathrm{kg \cdot m}}{\mathrm{s}}$$

From this equation we find that the units for momentum are $\left[\frac{\mathrm{kg \cdot m}}{\mathrm{s}}\right]$.

Unlike the motionless rock on the ground, a rolling soccer ball on the ground *does* have energy. It has kinetic energy, or energy of motion. The SI unit for energy is the Joule [J].

Any moving object has kinetic energy that increases with the object's mass and its speed. Since a motionless object has zero kinetic energy and mass and speed are both always positive numbers, kinetic energy can never be negative.

Energy bar graph - rolling soccer ball

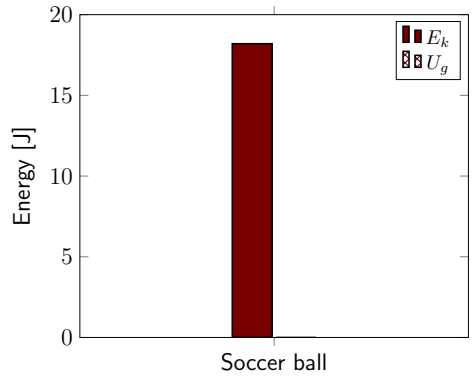


Figure 1.18: Energy bar graph of a rolling soccer ball[1]

As with the rock on the ground, $U_g = 0$ and $U_s = 0$. But this time we have kinetic energy. Kinetic energy depends on an object's mass and its speed v, which is simply the magnitude of its velocity $|\vec{v}|$,...

$$\boxed{E_k = \frac{1}{2} m \cdot v^2} \tag{1.6}$$

...so in this example the kinetic energy from the linear motion is...

$$E_k = \frac{1}{2} 0.4\,\mathrm{kg} \cdot (9\,\mathrm{m/s})^2 = 18.2\,\frac{\mathrm{kg \cdot m^2}}{\mathrm{s^2}} = 18.2\,\mathrm{J}$$

The SI unit for energy is the Joule [J].

$$1\,\mathrm{J} = 1\,\frac{\mathrm{kg \cdot m^2}}{\mathrm{s^2}}$$

Thinking about motion becomes much more interesting when something is moving. Since it is rolling, this soccer ball is definitely moving. What do we mean when we say that an object is moving? That means the object has a non-zero velocity. But when we talk more generally about "motion" in physics, we are considering not only velocity but also position and acceleration. This soccer ball has the simplest motion that we can consider for an object that is actually moving.

Motion map- rolling soccer ball

A motion map tells us about more than just position. It can also tell us about the velocity of the object. Notice that all of the arrows between the points are identical. The arrows represent velocity, so the motion map shows that the velocity is constant.

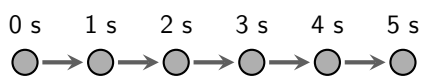


Figure 1.19: Motion map of a soccer ball rolling to the right with a location shown every second for 5 seconds[1]

For this example, Equation 1.2 becomes . . .

$$\vec{x} = \vec{x_0} + \vec{v_0} \cdot t + \frac{1}{2}\vec{a} \cdot t^2 = \vec{x_0} + \vec{v_0} \cdot t + 0$$

. . . where the subscript "0" means the value at time zero. In this case, $\vec{v_0} = 9\,\mathrm{m/s}\,\hat{x}$ and $\vec{a} = 0$, so. . .

$$\vec{x} = \vec{x_0} + (9\,\mathrm{m/s}\,\hat{x}) \cdot t$$

. . . which means that $\vec{x}$ moves 9 meters to the right of its initial position in every second that passes.

1.5 A Falling Rock

Words

A 0.8-kg rock is dropped from a position 2 m above the ground. After 0.5 seconds it is still in the air. Air resistance is very small, so we will ignore it. Describe the behavior of the rock.

We only need to consider the vertical direction, because there is no motion, momentum, or force in the horizontal direction.

The rock starts out not moving, but as soon as it is released gravity starts to pull it downward. In fact gravity was affecting the rock before it was released, but the force applied by the hand kept the rock in place until it was released.

This rock is in "free-fall," which means that the only force that is affecting the rock is gravity.

Since the rock is initially not moving, it has no initial momentum. But after half a second it is moving downward. The momentum is changing because of the force of gravity. The effect of force on momentum is described by Newton's Second Law of Motion, which defines force as something that changes momentum.

The momentum of the rock starts at zero, as it falls downward it gains momentum in the negative direction. This negative momentum gradually increases over the entire time that the rock is falling.

Graphics

Figure 1.20: A rock falling after being dropped[5]

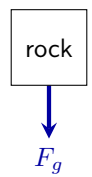


Figure 1.21: FBD of a falling rock[1]

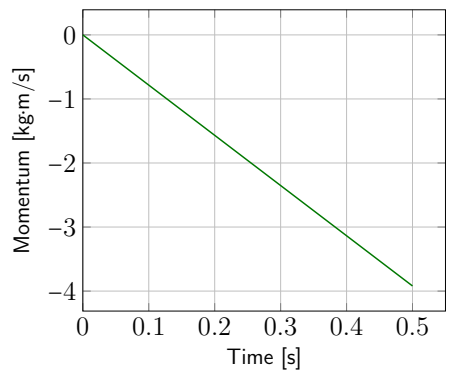


Figure 1.22: Momentum as a function of time[1]

Numbers

Knowns: $m = 0.8$ kg

$y_0 = 2$ m

$v_{0y} = 0$

$g = -9.8\,\mathrm{m/s^2}\,\hat{y}$

Velocity $\vec{v}$ is a vector, but speed v is a scalar that cannot be negative, since it is $|\vec{v}|$. But v_y is the $\hat{y}$ component of $\vec{v}$, so it *can* be negative.

$$\overrightarrow{F_{net}} = \sum \vec{F} = \vec{F_g}$$

$$\overrightarrow{F_{net}} = -m \cdot g\,\hat{y} = \left(-0.8\,\mathrm{kg} \cdot 9.8\,\mathrm{m/s^2}\right)\hat{y} = -7.84\,N\,\hat{y}$$

This is the first situation we have met where the object is not "in equilibrium," meaning that this time the net force on the object is not zero.

Net force is given by Newton's Second Law:

$$\boxed{\overrightarrow{F_{net}} \equiv \frac{\overrightarrow{\Delta p}}{\Delta t}} \tag{1.7}$$

"Δ" means "change in...." $\overrightarrow{\Delta p} = \vec{p_f} - \vec{p_i}$, final momentum, with subscript f, minus initial momentum, with subscript i.

Rearranging, $\overrightarrow{\Delta p} = \overrightarrow{F_{net}} \cdot \Delta t$. The momentum changes linearly from zero to its final value...

$$p_{y,f} = -7.84\,\mathrm{N} \cdot 0.5\,\mathrm{s} = -3.92\,\frac{\mathrm{kg \cdot m}}{\mathrm{s}}$$

Remember, momentum is related to velocity. So if the momentum of the rock is changing, that means its velocity is also changing. Acceleration is a change in velocity over time, so the same force that causes momentum to change also creates an acceleration.

The force of gravity accelerates the rock in the negative (downward) direction, making it fall faster and faster with an acceleration that doesn't depend on its mass.

This relationship between force and acceleration is often incorrectly referred to as Newton's Second Law.

When an object is falling, it is undergoing a change in energy. This rock is not moving at the moment it is dropped, so it has no kinetic energy but it does have gravitational potential energy because it is elevated. While it is falling, it is gaining kinetic energy but losing gravitational potential energy.

Energy is a useful concept to use when looking at this situation, because energy is conserved. It can never be created or destroyed; it can only change from one form to another.

If you know how much total energy is in an isolated system at any point in time, you know that same amount of energy is present for all points in time. So if you know, for example, how much gravitational potential energy was lost you can find how much kinetic potential energy was gained.

A motion map can also give information about acceleration. If the arrows are changing, the object is accelerating.

Figure 1.23: Motion map of a falling rock[1]

Due to conservation of energy, the sum of the heights of all of the bars on the energy bar graph at 0 s is equal to the sum of the heights of all of the bars at 0.5 s.

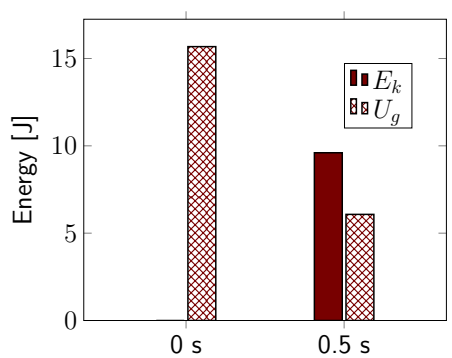


Figure 1.24: Energy bar graph of a falling rock[1]

$$\boxed{\overrightarrow{F_{net}} = m \cdot \vec{a}} \tag{1.8}$$

Rearranging, we find...

$$\vec{a} = \frac{\overrightarrow{F_{net}}}{m} = \frac{-7.84\,\mathrm{N}\,\hat{y}}{0.8\,\mathrm{kg}} = -9.8\,\mathrm{m/s^2}\,\hat{y}$$

In fact, $\vec{a}$ for anything in free-fall is $-9.8\,\mathrm{m/s^2}\,\hat{y}$.

$$y = y_0 + v_{0y} \cdot t + \frac{1}{2} a_y \cdot t^2 = 2\,\mathrm{m} + 0 \cdot t - 4.9\mathrm{m/s^2} \cdot t^2$$

...so after 0.5 s, $y = \left(2 - 4.9 \cdot 0.5^2\right)\,\mathrm{m} = 0.775\,\mathrm{m}$

There is no spring, so $U_s = 0$. Using 0 seconds for the initial time and 0.5 s for the final time...

$$E_{0,total} = E_{f,total}$$

$$E_{0k} + U_{0g} = E_{k,f} + U_{g,f}$$

$$0 + m \cdot g \cdot y_0 = \frac{1}{2} m \cdot v_f^2 + m \cdot g \cdot y_f$$

Everything in the last equation was given in the equation or has already been found, except for the final speed, so we can solve for it:

$$v_f = \sqrt{2 \cdot \frac{m \cdot g \cdot y_0 - m \cdot g \cdot y_f}{m}}$$

Note that mass cancels out of the equation, so the final speed of a falling object is not dependent upon its mass.

1.6 Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

General

- The kilogram [kg] is the SI unit of mass.
- The second [s] is the SI unit of time.

Forces

- The Newton [N] is the SI unit of force. $1\,\mathrm{N} = 1\frac{\mathrm{kg\cdot m}}{\mathrm{s}^2}$
- A force is a push or pull.
- Gravitational force is the same as weight, and is always pointed down toward the earth.
- Normal force is a contact force that points directly out of a surface.
- Forces cause acceleration.
- Forces change momentum.
- An object that is affected only by gravity is said to be in "free-fall."
- An object whose net force is zero is said to be in equilibrium.
- Forces can be shown graphically using a "Free-body diagram," or "FBD," which has a dot representing the object and arrows representing the forces affecting the object. Arrows in a FBD should be drawn in the correct directions, with lengths corresponding to the magnitudes of the forces.

Motion

- The meter [m] is the SI unit of distance.
- An object's motion is described by its position, velocity, and acceleration.
- Velocity is change in position over time.
- Acceleration is change in velocity over time.
- Velocity includes both speed (which is always positive) and direction, so velocity can be negative.
- An object moves at a constant velocity if and only if the net force on the object is zero.
- Motion can be shown graphically using a "Motion map," which has a series of dots representing the position of the object at equally-spaced intervals of time. Arrows are drawn between the dots to indicate the object's velocity, and acceleration appears as changes in the arrows.

Momentum

- $[\mathrm{kg} \cdot \mathrm{m/s}]$ is the SI unit of momentum.
- The momentum of an object is zero if the object is not moving.
- Momentum increases with an object's mass and an object's velocity.
- The momentum of any isolated system is conserved.
- Momentum can be shown graphically using "Momentum bars," which have a height proportional to the object's velocity and a width proportional to the object's mass. The resulting rectangle has an area proportional to the object's momentum.

Energy

- The Joule [J] is the SI unit of energy. $1\,\mathrm{J} = 1\,\frac{\mathrm{kg \cdot m^2}}{\mathrm{s^2}}$
- Energy is an ability to do work.
- Energy is conserved; it cannot be created or destroyed, but it can change form.
- Kinetic energy is energy of motion.
- Kinetic energy can be positive or zero, but never negative.
- Gravitational potential energy is related to an elevated position.
- Spring potential energy and thermal energy are other types of energy that will be dealt with later in this book.
- Mechanical energy consists of kinetic energy, gravitational potential energy, and spring potential energy.
- Energy can be shown graphically using an "Energy bar graph." If the system is isolated, the total height of all energy bars for the system at any point in time is the same as the total height of all energy bars at any other point in time.

Mathematical Models

equation		restrictions on the validity of the equation
$\overrightarrow{F_{net}} \equiv \sum \overrightarrow{F}$	(1.1)	-none-
$\overrightarrow{x} = \overrightarrow{x_0} + \overrightarrow{v_0} \cdot t + \frac{1}{2}\overrightarrow{a} \cdot t^2$	(1.2)	only valid when the net force is constant
$\overrightarrow{p} = m \cdot \overrightarrow{v}$	(1.3)	-none-
$\overrightarrow{F_g} = -m \cdot g\, \hat{y}$	(1.4)	on the surface of the earth, with $+\hat{y}$ defined as "up"
$U_g = m \cdot g \cdot y$	(1.5)	on the surface of the earth, with $+\hat{y}$ defined as "up"
$E_k = \frac{1}{2} m \cdot v^2$	(1.6)	-none-
$\overrightarrow{F_{net}} \equiv \frac{\overrightarrow{\Delta p}}{\Delta t}$	(1.7)	only valid when the net force is constant
$\overrightarrow{F_{net}} = m \cdot \overrightarrow{a}$	(1.8)	-none-

1.7 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**.

Level 1 - Remember

1.1 **[W]** How is energy defined in this book?

1.2 **[W]** How is acceleration defined in this book?

1.3 **[N]** What does a half-arrow over a symbol mean?

1.4 **[N]** What does it mean if an equation has a box around it?

1.5 **[N]** What symbol is used to represent momentum?

1.6 **[G]** What do the arrows represent on a motion map?

1.7 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents. For example, Equation 1.1 should be labeled "net force."

Level 2 - Understand

1.8 **[W]** What information is included in an object's velocity that is not included in its speed?

1.9 **[N]** Give an example of a vector quantity in physics.

1.10 **[N]** Give an example of a scalar quantity in physics.

Level 3 - Apply

1.11 **[G]** Draw a motion map for an object moving to the left at constant speed.

Level 4 - Analyze

1.12 **[G]** Consider the free-body diagram in Figure 1.8, where only the horizontal direction is being considered. Should that figure change if the mass of the rock were doubled? If so, in what way?

1.13 **[G]** Consider the free-body diagram in Figure 1.14, where only the vertical direction is being considered. Should that figure change if the mass of the rock were doubled? If so, in what way?

1.14 **[W & N]** What is the weight of an 80-kg person? Remember to include direction.

1.15 **[N]** One thing that we should have been able to find in Section 1.5 was the time needed for the rock to reach the ground when dropped from a height of 2 m. Find that time for the moment when the rock hits the ground, that is, when $y = 0$.

1.16 **[W & G]** Explain how the free-body diagram for the motionless rock in Figure 1.8 can be the same as that for a moving soccer ball in Figure 1.16.

Level 5 - Evaluate

1.17 **[W]** In the analysis of a motionless rock in the vertical direction in Section 1.3, it is stated that there is no wind. If there were a horizontal wind, would it have affected our analysis of the vertical direction? Explain your answer.

1.18 **[W]** In the analysis of a rolling soccer ball in Section 1.4, we ignored friction to make the analysis simpler, and found that the soccer ball would continue rolling in a straight line indefinitely. Was this a realistic simplification to make? If we had included friction, what would the soccer ball have done?

1.19 **[G & N]** The area of a momentum bar is supposed to represent the momentum of an object. Using that information, what is the momentum of the soccer ball as shown in Figure 1.17? Compare your answer to that found in the column to the right of the figure. Explain why they are the same, or why they are different.

1.20 **[G & N]** Estimate the slope of the line in Figure 1.22. Compare it to the net force found in the column to the right of the figure. Explain why they are the same, or why they are different.

1.21 **[W, G, & N]** Figure 1.24 shows the energies of a rock when it is released from a height of 2 m and when it has fallen for 0.5 s.

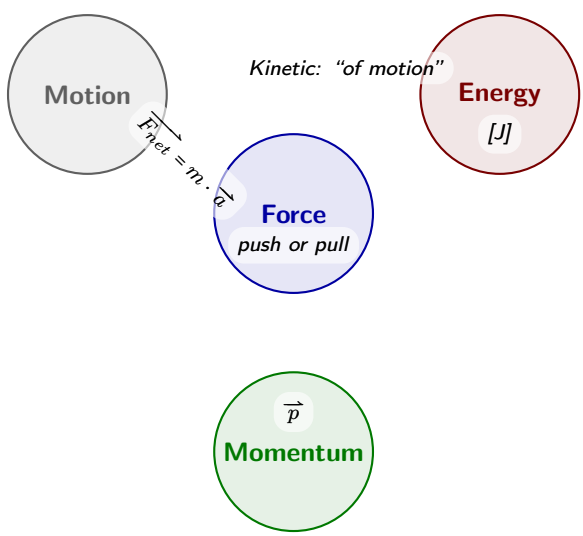
(a) Reproduce this graph, with correct heights for each bar, and add a third set of bars for the moment just before the rock hits the ground. You can label the last set of bars "before hitting."

(b) Find the speed of the rock just before it hits the ground.

(c) Combine the speed of the rock with other information you have about the direction of the rock's motion to find the velocity of the rock just before it hits the ground.

Level 6 - Create

1.22 **[W, G, & N]** At the beginning of this chapter in Figure 1.1 was a template for a concept map. Hand-draw your own large version of it, adding in the main ideas from this chapter. Leave plenty of space to add things from other chapters! Here are some examples to help you start:



Partially completed concept map[1]

1.23 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

1.24 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

Chapter 2

Working in One Dimension

We have now been introduced to most of the concepts and most of the tools that we will use in this entire book to study mechanics. We will continue to look at more and more complicated situations. In this chapter we will continue to restrict ourselves to single objects in one dimension, either horizontal or vertical. We will also continue to consider only constant forces.

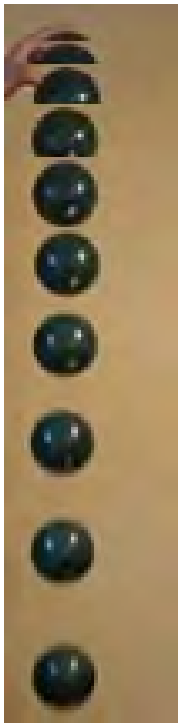


Figure 2.1: Images of a falling ball, taken by a stationary camera at a rate of 20 frames per second[1]

2.1 Units

Words

If you live in the United States or one of a handful of other countries, you may be bothered by the fact that this book is using unfamiliar units. There are good reasons for this.

One reason is that the vast majority of the world has adopted SI units as the official system of measurement.

A second reason is that SI units are based on factors of 10 and universal physical quantities like the speed of light, while most other systems of measurement, including U.S. Customary units, are based upon arbitrary numbers and measurements like the distance between a person's fingertips and elbow.

A third reason is that attempting to use U.S. Customary units to do physics is difficult and confusing even for those who use those units on a daily basis. For example, mass is an important physical property, but in U.S. Customary units people assume that mass is measured in pounds. In fact, pounds are a unit of force, and the correct unit of mass should be slugs, which nobody anywhere in the world uses, even in the U.S.

For all of these reasons, SI units are the standard units used in science throughout the world, and also in this book.

Graphics

Figure 2.2: Rulers often show inches across the top and centimeters (cm) across the bottom.[1]

The U.S. Customary unit for length is the foot, and the SI unit for length is the meter. Conversion factors can be found in the appendices of this book.

Figure 2.3: Weights used in a gym are often labeled in pounds (lb) and kilograms (kg).[1]

On earth, a 10-kg object experiences a gravitational force of 98 N, and 98 N corresponds to 22 pounds of force. On the moon, a 10-kg object experiences a gravitational force of 16.2 N, or 3.65 pounds.

So mass is not dependent on gravity, but weight is. This is why we more often refer to an object's mass than its weight in physics classes. They are related, but they are different.

Numbers

Note: These calculations are wrong! This is why we avoid U.S. Customary Units!

Consider a 50-pound child. What is the child's mass? 50 pounds, right? The acceleration of gravity at the surface of the earth is $32\,\mathrm{ft/s^2}$. Weight is the gravitational force on an object, so a 50-pound child experiences a gravitational force of 50 pounds on earth. Putting these numbers into Equation 1.8 . . .

$$F_y = m \cdot a_y = m \cdot (-32\,\mathrm{ft/s^2})$$

$$-50\,\mathrm{pounds} = (50\,\mathrm{pounds}) \cdot (-32\,\mathrm{ft/s^2})$$

Dividing both sides by 50 pounds gives us . . .

$$1 = 32\,\mathrm{ft/s^2}$$

$1 = 32$!? This cannot be correct. Most people who use U.S. Customary units don't know that the correct unit for mass is the slug. A slug is the mass of an object that weighs 32 pounds at the surface of the earth. That's why our calculation is off by a factor of 32—we used the wrong units.

To do physics using U.S. Customary Units we must introduce units like the slug that are not in regular use anywhere in the world. Rather than taking this step backwards, we step forward into using SI units.

Note: The above calculations are wrong! This is why we avoid U.S. Customary Units!

Failing to pay attention to units can be a costly mistake, as was famously demonstrated in 1999 when NASA lost its $125-million space probe that was supposed to have gone into orbit around Mars. The engineers who designed the rocket system used U.S. Customary Units, but the engineers who designed the guidance system used SI units. When it came time to enter orbit, the probe instead plunged into the Martian atmosphere and was destroyed.

This NASA error was simply a matter of not converting all units into the same system. But there is another type of error that is common for students who are learning physics. That is having answers with the wrong "dimensions" or doing calculations that are dimensionally impossible.

For example, if someone says a baby weighs six pounds, nine ounces, that makes sense. It is dimensionally correct, even though it is not in the SI units that are preferred in physics. If we want SI units, we can convert to find the answer we wanted.

But, if someone says that a baby weighs fifteen inches, that does not make sense. Weight is a force, and inches are a measure of length, so these two things have different "dimensions." In physics, a "dimension" doesn't have to refer to length. It can be any physical quantity: energy, momentum, velocity, *etc.*

Graphing position over time

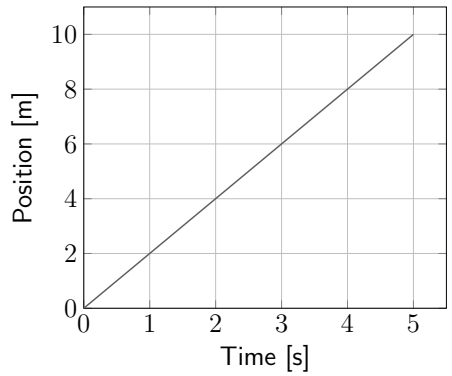


Figure 2.4: Position as a function of time[1]

The slope of a graph often provides useful information. In Figure 2.4, position is changing in time, which means that the object must be moving. The slope of a line is the "rise over the run," how much the vertical value changes divided by how much the horizontal value changes. In this case . . .

$$\text{slope} = \frac{10\,\text{m} - 0\,\text{m}}{5\,\text{s} - 0\,\text{s}} = 2\,\text{m/s}$$

Keeping the units in the slope calculation gives a hint about the meaning of the slope. The unit [m/s] is a velocity, and in fact the slope of a position vs time graph gives velocity.

If you walk at a constant speed of 4 miles per hour, what distance will you travel in 30 minutes? *Figure out the answer to that question before continuing.*

Hopefully you came to an answer of 2 miles without too much of a struggle. Whether you realize it or not, you went through all of these steps, possibly in a different order:

(1) Compare units, finding both hours and minutes for time.

(2) Unit conversion to make time consistent:

$$30\,\text{min} \cdot \frac{1\,\text{hr}}{60\,\text{min}} = 0.5\,\text{hr}$$

(3) Combine given information in such a way that you find an answer with units of distance:

$$\frac{4\,\text{mi}}{1\,\text{hr}} \cdot 0.5\,\text{hr} = 2\,\text{mi}$$

Notice that for unit conversion and general calculations the pattern is the same. Units cancel just like variables and numbers. For unit conversion, multiply by a fraction using the appropriate conversion factor (see the appendices), in such a way that units cancel to give the answer required by the question.

This way of canceling only works with multiplication and division. Two numbers cannot be added or subtracted unless they have the same units.

2.2 Sliding to the Left

Words

A 170-g hockey puck is sliding to the left for four seconds across a smooth sheet of ice at a constant speed of 24 m/s. Frictional force and air resistance are very small, so we will ignore them. We will consider only the horizontal direction.

We have looked at a similar situation with a rolling soccer ball, so some of the analysis will be left for the end-of-chapter exercises. Here we will try a few new approaches. This time there is no picture, and since images are helpful for understanding, we will start with a sketch. The sketch should include as much of the given information as possible.

The net force on an object causes a change in velocity, *i.e.* an acceleration. Since the velocity of this puck is constant, we know that the net force is zero.

A sliding hockey puck can do work on something (for example, if it hits an egg it can break the egg). That means it has energy, in this case kinetic energy. And since the puck is traveling at a constant speed, the kinetic energy would also be constant.

The hockey puck doesn't have any other type of mechanical energy, because there is no spring and it is on the ground.

Graphics

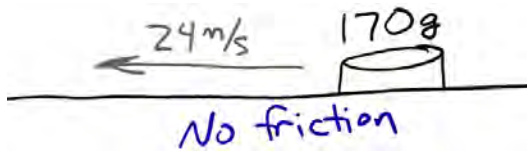


Figure 2.5: Rough sketch of the given physical situation[1]

The sketch does not need to be beautiful, although you can feel free to let your artistic skills shine. But the sketch does need to get across the essential information about the situation.

Figure 2.6: FBD of a puck with no forces[1]

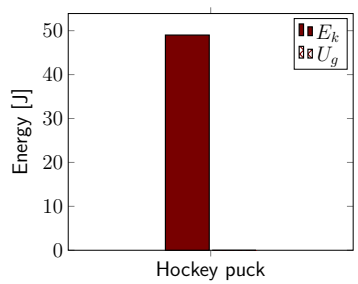


Figure 2.7: Energy bar graph of a sliding hockey puck[1]

Numbers

Knowns: $m = 170\text{ g} = 0.17\text{ kg}$

$t = 4\,\text{s}$

$v_x = -24\,\text{m/s}$

$a_x = 0$

$F_{f,x} = 0$

Mass needs to be converted to teh SI unit kg:

$$170\,\text{g} \cdot \frac{1\,\text{kg}}{1000\,\text{g}} = 0.17\,\text{kg}$$

v_x is negative because conventionally right is positive. a_x is zero because velocity is constant. $F_{f,x}$ is the force of friction in the x direction.

We know from Equation 1.8 that $\overrightarrow{F_{net}} = m \cdot \vec{a}$, and since $a_x = 0$, it follows that $F_{net,x} = 0$.

We know from equation 1.6 that ...

$$E_k = \frac{1}{2} m \cdot v^2$$

... so in this case ...

$$E_k = \frac{1}{2} 0.17\,\text{kg} \cdot (-24\,\text{m/s})^2 = 49\,\text{J}$$

Like the rolling soccer ball that we considered earlier, the hockey puck is moving, so it has momentum. This time, the puck is moving in the opposite direction from the soccer ball we considered earlier, so its momentum is also in the opposite direction, since momentum is a vector. It is important to remember that momentum also includes direction.

The hockey puck is moving to the left with constant speed, so its position is changing throughout the four seconds. We are not given the initial position of the puck, so we can only describe how much the position changes, not whether after 4 seconds it will reach the goal. The change in the position of an object has a special name, "displacement."

From the other columns on this page, we can see that the puck travels 96 m in 4 seconds. The right direction is conventionally considered postive, so left is negative. That means the displacement is 96 m to the left. The total distance it moves during this same time is 96 m. Distance is always positive–you would never say that your home is negative five miles from your workplace, right?

In a situation like this, when the object is moving only in one direction, the displacement is the same as the distance that an object moves, but if the object changes direction at any point in its motion, the total distance it moves will be larger than its displacement. We will encounter this type of situation later in this chapter.

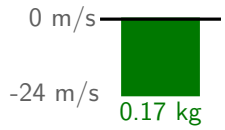


Figure 2.8: Momentum bar of sliding hockey puck. Note that the height is negative![1]

Motion Map

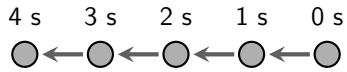


Figure 2.9: A hockey puck sliding to the left for 4 seconds[1]

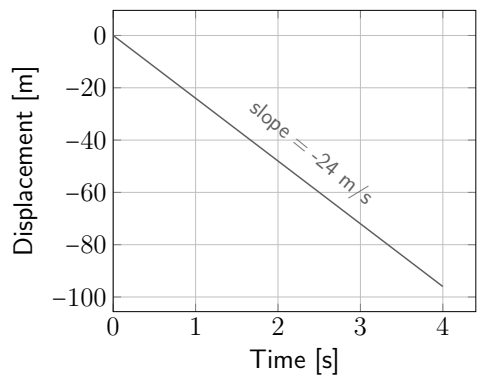


Figure 2.10: Puck displacement as a function of time[1]

We know from Equation 1.3 that

$$\vec{p} = m \cdot \vec{v}$$

. . . so in this case. . .

$$p_x = m \cdot v_x = 0.17\,\mathrm{kg} \cdot -24\,\mathrm{m/s} = -4.08\,\mathrm{kg} \cdot \mathrm{m/s}$$

If the net force on an object is constant, as it is in this situation, position is given by Equation 1.2 . . .

$$\vec{x} = \vec{x_0} + \vec{v_0} \cdot t + \frac{1}{2}\vec{a} \cdot t^2$$

. . . and in this case we are only concerned with the x direction. . .

$$x = x_0 + v_{0x} \cdot t + \frac{1}{2}a_x \cdot t^2$$

$$x = x_0 - 24\,\mathrm{m/s} \cdot t + \frac{1}{2}0 \cdot t^2 = x_0 - 24\,\mathrm{m/s} \cdot t$$

We are not given x_0, but if we subtract it from both sides we have solved the equation for the change in x, also called displacement.

$$x - x_0 = \Delta x = -24\,\mathrm{m/s} \cdot t$$

This allows us to find the displacement for any time t, including the final time 4 s.

$$\Delta x_f = -96\,\mathrm{m}$$

2.3 Falling to the Ground

Words

A 2.4-kg ball is first dropped from a height 1.3 m above the ground. Air resistance is very small, so we will ignore it. We will consider only the vertical direction. Then the same ball is dropped from a height 2.6 m above the ground.

We should be able to find the amount of time that is needed to reach the ground in each situation and also the kinetic energy, velocity, and momentum of the ball just before it hits the ground.

When considering the motion of the ball, we should note that it is in free-fall, which means that it is affected only by the force of gravity, and is accelerating at a rate that doesn't depend on the mass of the ball.

The first "unknown" mentioned in the description above is the time it takes the ball to fall to the ground. At first guess, one might think that the time needed to drop 2.6 m should be twice as much as the time needed to drop 1.3 m, but this is not correct. Notice in the images and motion map provided that with every interval of time the ball moves a greater distance than in the interval before.

From the images, we can see that the ball fell roughly the same distance in the first 0.3 seconds as it did in the next 0.15 seconds. This is because the longer it falls, the faster it is moving.

Graphics

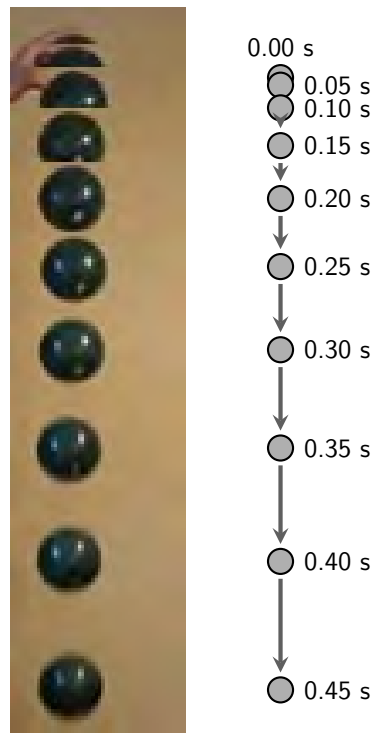


Figure 2.11: Images and corresponding motion map of a falling ball. Images were taken every 0.2 seconds.[1]

Comparing with the calculations on the right, we see that this ball fell slightly less than 1.3 m while it was being recorded by the camera.

Numbers

Knowns	Unknowns
$m = 2.4$ kg	t_f
$y_0 = 1.3$ m or 2.6 m	$E_{k,f}$
$y_f = 0$	$v_{y,f}$
$v_{0y} = 0$	$p_{y,f}$
$a_y = -g$	

We can find the time needed to hit the ground by considering the motion of the ball. Since the acceleration is constant, we can use Equation 1.2:

$$\vec{x} = \vec{x_0} + \vec{v_0} \cdot t + \frac{1}{2}\vec{a} \cdot t^2$$

. . . and since we only need the y direction. . .

$$y = y_0 + v_{0y} \cdot t + \frac{1}{2}a_y \cdot t^2$$

In this equation, we already know everything about the moment just before the ball hits the ground (t_f) except for the time itself.

$$y_f = 0 = y_0 - \frac{1}{2}g \cdot t_f^2$$

Solving for t_f, we find . . .

$$t_f = \sqrt{\frac{2\,y_0}{g}}$$

t_f is 0.515 s when dropped from 1.3 m and 0.728 s when dropped from 2.6 m.

The total energy of an object or system of objects can only change if work is done on the object(s) by external forces. If we know the initial gravitational potential energy and kinetic energy of the ball just before it is released, that is the same as the energy of the ball just before it hits the ground, since no other forces act on it. This is called conservation of energy.

We will not yet start to consider spring or thermal energy. Kinetic energy is energy of motion, and since the ball isn't moving at the moment it is released, it doesn't have any kinetic energy. It does, however, have gravitational potential energy since it is elevated above the ground.

Just before it hits the ground, it will not have any gravitational potential energy, so all of the gravitational potential energy has changed into kinetic energy, and the ball dropped from 2.6 m will have twice as much kinetic energy as the same ball dropped from 1.3 m.

When the ball is dropped from twice as high, it hits the ground at a higher speed, but not quite double. That is because the acceleration is constant the whole time, and acceleration changes velocity over time. Not over distance. Since the time is not double, neither is the final velocity.

The ball started with no momentum, but the force of gravity acting on it while it was in free-fall gave the ball momentum in the downward direction, and as with energy and velocity, when it is dropped from a higher position it gains more momentum before reaching the ground.

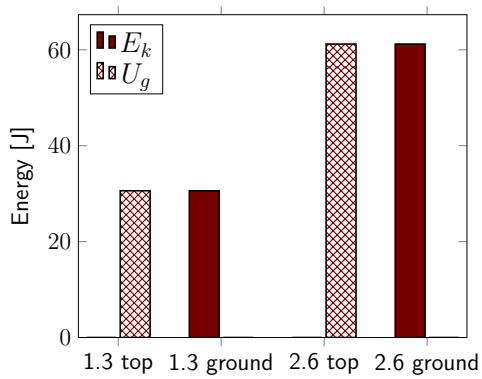


Figure 2.12: Energy bar graph of a ball falling from two different heights[1]

The first set of bars represents the kinetic & gravitational potential energy of a ball at the instant it is dropped from 1.3 m. The second two bars represent the same ball just before it hits the ground. The third and fourth sets of bars are similar, but for a ball dropped from 2.6 m.

The total energy of a system cannot change unless an outside force does work W on it.

$$\boxed{W = E_{tot,f} - E_{tot,i}} \tag{2.1}$$

...where the subscript tot means "total." In this case, there is no work done by an external force, and we only have U_g and E_k, so...

$$U_{g,i} + E_{k,i} = U_{g,f} + E_{k,f}$$

...or...

$$m \cdot g \cdot y_i + \frac{1}{2} m \cdot v_i^2 = m \cdot g \cdot y_f + \frac{1}{2} m \cdot v_f^2$$

...so in this case...

$$m \cdot g \cdot y_0 + 0 = 0 + \frac{1}{2} m \cdot v_f^2$$

The term on the right is $E_{k,f}$, which is equal to $U_{g,i}$. That is 30.6 J when dropped from 1.3 m and 61.2 J when dropped from 2.6 m.

We can find the final velocities by solving the equation above for v_f:

$$v_f = \sqrt{2 \cdot g \cdot y_0}$$

Adding in the direction, $v_{y,f} = -5.05\,\mathrm{m/s}$ for 1.3 m and $v_{y,f} = -7.14\,\mathrm{m/s}$ for 2.6 m

We can use m and $v_{y,f}$ to find $p_{y,f}$:

$$p_{y,f} = m \cdot v_{y,f}$$

That is, -12.1 $\frac{\mathrm{kg \cdot m}}{\mathrm{s}}$ for 1.3 m and -17.1 $\frac{\mathrm{kg \cdot m}}{\mathrm{s}}$ for 2.6 m

2.4 Being Thrown to the Ground

Words

A 2.4-kg ball is thrown straight downward with a speed of 5.05 m/s from a height 1.3 m above the ground. Air resistance is very small, so we will ignore it. We will consider only the vertical direction.

We should be able to find the amount of time that is needed to reach the ground and also the kinetic energy, velocity, and momentum of the ball just before it hits the ground.

The ball is in free-fall, which tells us that it is accelerating downward because of gravity. The ball is being thrown downward, which also indicates that a force was used to throw the ball, but this situation doesn't include the actual throwing. This situation describes the ball after it is released, so there is NO "force of throwing" in this situation. If we looked at how the ball was thrown, then there would be a normal force from a hand or something similar to consider. But in this situation the ball has already been thrown, so there is no hand to consider.

There are many different correct ways to go about finding all of the unknown quantities. To demonstrate this, since we know the net force on the ball, the calculations in the right column can be done for final momentum after finding the final time when the ball reaches the ground. In the previous section of this book, momentum was the last quantity found.

Graphics

If we are not provided with an image of some kind, again we should make a sketch of the situation.

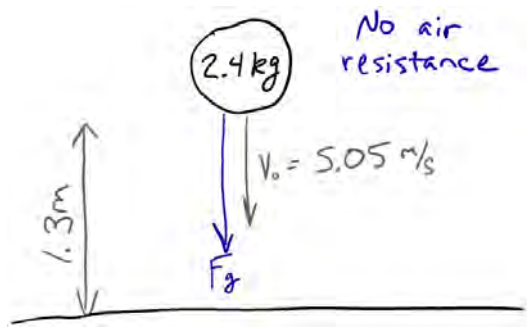


Figure 2.13: Rough sketch of a ball that was thrown downward[1]

Making a sketch like this helps us to see and remember important information, for example in this case the force and the initial velocity are in the same direction.

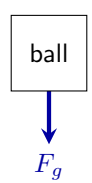


Figure 2.14: FBD of a ball in free-fall[1]

Numbers

Knowns	Unknowns
m = 2.4 kg	t_f
y_0 = 1.3 m	$E_{k,f}$
$y_f = 0$	$v_{y,f}$
v_{0y} = -5.05 m/s	$p_{y,f}$
$a_y = -g$	

We can find t_f by considering the motion of the ball. Since the acceleration is constant...

$$y_f = y_0 + v_{0y} \cdot t_f + \frac{1}{2} a_y \cdot t_f^2$$

...which can be rearranged to...

$$\frac{1}{2} a_y \cdot t_f^2 + v_{0y} \cdot t_f + (y_0 - y_f) = 0$$

t_f and t_f^2 both appear in this equation, so we can use the quadratic formula to get two solutions: 0.213 s and -1.24 s. The negative solution is mathematically correct but cannot physically be the solution, so $t_f = 0.213\,\mathrm{s}$.

Since in this situation $\overrightarrow{F_{net}} = \overrightarrow{F_g}$, we can use Equations 1.4 & 1.7 to find the final momentum.

$$\frac{\Delta p_y}{\Delta t} = \frac{p_{y,f} - p_{y,i}}{\Delta t} = -m \cdot g$$

Rearranging, $p_{y,f} = -m \cdot g \cdot \Delta t + p_{0y}$, or...

$$p_{y,f} = -m \cdot g \cdot \Delta t + m \cdot v_{0y} = -17.1\ \frac{\mathrm{kg} \cdot \mathrm{m}}{\mathrm{s}}$$

The momentum of the ball just before it hits the ground is directed downward, and is actually equal to the final momentum in the last section, for a similar ball that was dropped from a higher initial position. Since the balls have the same mass and the same final momentum, they also have the same final velocity.

We can again use conservation of energy to find the final kinetic energy, but this time we need to remember that the ball also starts with kinetic energy. As it falls, its kinetic energy increases as its gravitational potential energy decreases, so that its total energy stays constant.

Just before hitting the ground, the ball has no gravitational potential energy, because it has changed into kinetic energy. For this situation, since the ball has the same mass and the same final velocity as the ball dropped from the higher position in the last section, the two balls also have the same final kinetic energy.

The final condition of the ball described in this section is the same as that of the ball dropped from a higher position in section 2.3. That is because when the ball dropped from the higher position reaches the same height as the ball described in this section, it has the same velocity as the initial velocity of the ball described in this section. It appears that only the time is different, but the time found in this section is equal to the time needed for the ball in section 2.3 to travel the *last* 1.3 m of its fall.

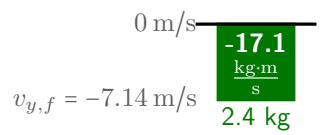


Figure 2.15: Final momentum bar of the falling ball. Since we already know the area and the width of the bar, we can find its height[1]

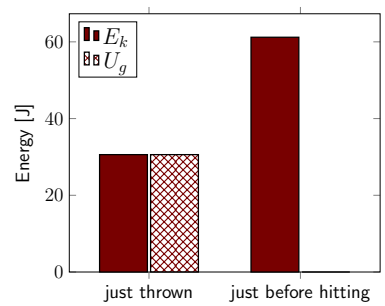


Figure 2.16: Energy bar graph for a ball being thrown to the ground, showing energy when it is just thrown and just before it hits the ground[1]

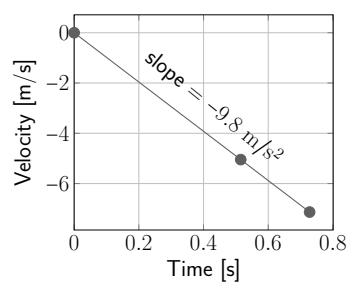


Figure 2.17: Velocity during free-fall[1]

Using Equation 1.3, we can find the final velocity of the ball.

$$p_{y,f} = m \cdot v_{y,f}, \quad \text{so} \ldots$$

$$v_{y,f} = \frac{p_{y,f}}{m} = -7.14\,\mathrm{m/s}$$

We can use work and energy, Equation 2.1, to find $E_{k,f}$, and since no external work W is being done...

$$U_{g,i} + E_{k,i} = U_{g,f} + E_{k,f}$$

...or...

$$m \cdot g \cdot y_0 + \frac{1}{2} m \cdot v_0^2 = m \cdot g \cdot y_f + E_{k,f}$$

...which gives $E_{k,f} = 61.2$ J.

We could also have found the final velocities using the knowledge that acceleration is the change in velocity over time...

$$\boxed{\vec{a} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v_f} - \vec{v_i}}{\Delta t}} \tag{2.2}$$

The y components of this equation can be rearranged to solve for $v_{y,f}$:

$$v_{y,f} = v_{y,i} + a_y \cdot \Delta t$$

2.5 Being Thrown Upward

Words

A 2.4-kg ball is thrown straight *upward* with a speed of 5.05 m/s from a height 1.3 m above the ground. Air resistance is very small, so we will ignore it. We will consider only the vertical direction.

We should be able to find the height at the peak of the ball's flight, the amount of time that is needed to reach the peak, and also the kinetic energy. gravitational potential energy, velocity, and momentum of the ball at the peak.

The ball is in free-fall, which tells us that it is accelerating downward because of gravity. This is *almost* exactly the same as the situation considered in the last section–the only difference is that now the ball has been thrown upward, opposite the force of gravity. Again, there is no force from the hand that threw it (or of whatever else might have thrown it), because the question begins with the ball already having been thrown.

In every situation up until this one, a force has increased both the speed and the magnitude of the momentum of the object we are considering. This time the force does the opposite. That is because the initial velocity is *opposite* the direction of the net force.

The ball still accelerates in the direction of the net force, but the acceleration *slows down* the ball instead of speeding it up!

Graphics

Again we should make a sketch. This is the same as the sketch from section 2.4 except that one arrow changed direction.

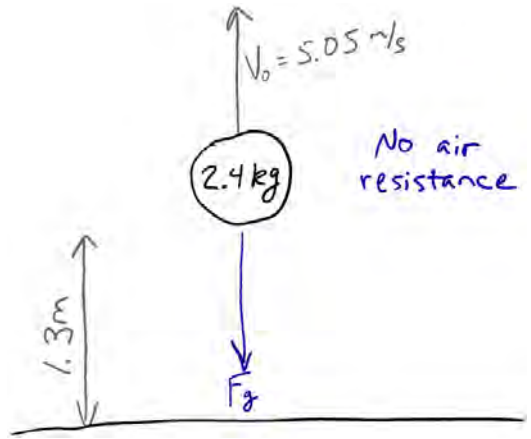


Figure 2.18: Rough sketch of a ball that was thrown *upward*[1]

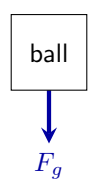


Figure 2.19: FBD of a ball in free-fall[1]

A FBD shows downward force with no indication of upward velocity, exactly as in section 2.4.

Numbers

Knowns	Unknowns
$m = 2.4$ kg	y_f
$y_0 = 1.3$ m	t_f
$v_{0y} = 5.05$ m/s	$E_{k,f}$
$a_y = -g$	$U_{g,f}$
	$v_{y,f}$
	$p_{y,f}$

Note that the only changes to the "knowns" from section 2.4 are that there is no "-" on v_{0y} and y_f is now unknown. $v_{y,f}$ is now unknown; and $U_{g,f}$ has been added as an unknown.

Three of the six unknowns are easy to find, once we realize that at the peak the ball is not moving. That makes $E_{k,f} = 0$, $v_{y,f} = 0$, and $p_{y,f} = 0$.

In regular English, it is normal to refer to "slowing down" as "decelerating," and "speeding up" as "accelerating." That is not the case in physics. We will always use the word "acceleration" to indicate that an object's velocity is changing. Sometimes the magnitude of the velocity (speed) may be increasing; sometimes it may be decreasing; and in some cases the speed may be staying constant but the direction may be changing. To avoid confusion, we will refer to all of these situations as acceleration.

We can again use conservation of energy to find the final gravitational potential energy. At the beginning, the ball has both gravitational potential energy, since it is elevated above the ground, and also kinetic energy, since it is moving. As it goes upward, it slows down, ultimately stopping when it reaches the peak. That means it is losing kinetic energy, and all of the kinetic energy it loses is transforming into gravitational potential energy. So at the peak, all of its initial energy has changed into gravitational potential energy.

When the ball reaches the peak, its height and its gravitational potential energy are at a maximum, while its speed and kinetic energy are at a minimum. The momentum was positive as the ball was going up, since "upward" is usually considered a positive direction. At the top, the momentum is zero, and unless something intervenes, a short time later the ball will be moving downward, so the momentum will be negative.

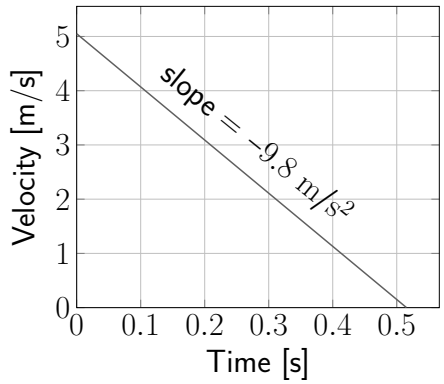


Figure 2.20: Velocity during free-fall[1]

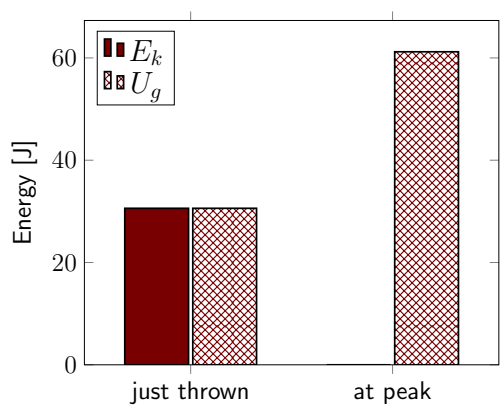


Figure 2.21: Energy bar graph for a ball being thrown upward, showing energy when it is just thrown and at the peak of its flight[1]

We know the acceleration and the initial and final velocity of the ball, so we can rearrange the $\hat{y}$ components of Equation 2.2 to find t_f:

$$\Delta t = t_f - t_0 = \frac{v_{y,f} - v_{0y}}{a_y}$$

...giving $t_f = 0.515\text{s}$, using $t_0 = 0$.

We can use conservation of energy with no external work, Equation 2.1, to find $U_{g,f}$.

$$U_{g,i} + E_{k,i} = U_{g,f} + E_{k,f}$$

...or...

$$m \cdot g \cdot y_0 + \frac{1}{2} m \cdot v_0^2 = U_{g,f} + 0$$

...which gives $U_{g,f} = 61.2$ J.

The final height can be found from the gravitational potential energy using Equation 1.5.

$$U_{g,f} = m \cdot g \cdot y_f$$

This gives $y_f = 2.6$ m.

2.6 Up and Back Down

Words

A 2.4-kg ball is thrown straight *upward* with a speed of 5.05 m/s from a height 1.3 m above the ground. Air resistance is very small, so we will ignore it. We will consider only the vertical direction.

In Section 2.5 we considered this exact same situation, but stopped when the ball reached the peak of its flight. This time we will consider the entire flight of the ball until the moment just before it hits the ground. Let's focus on the initial condition of the ball when it has just been thrown, the peak of its flight, the time when it passes the same height from which it was thrown, and the moment just before it hits the ground.

Again the ball is in free-fall, which tells us that it is accelerating downward because of gravity. That acceleration is valid for the *entire time* that the ball is in the air. Some people are surprised by this, thinking that the acceleration should be zero at the peak of the ball's flight, but if that were true then the ball would just stay there instead of coming back down.

If we look back at the previous sections of this chapter, we have already found all of the pieces of information for the ball as it travels along this entire path. We just need to put it all together.

Graphics

An initial sketch if this situation, would look exactly the same as the sketch from Section 2.5. So instead let's try to draw a motion map for the ball.

Figure 2.22: An attempt at a motion map. The ball's path turns back on itself, making a motion map difficult to read. The numbers give the order of the points, but the time scale is not given.[1]

A motion map is not the best way to show the motion of an object that retraces its path.

Numbers

Knowns	Unknowns
$m = 2.4$ kg	$y_{peak}, y_{down}, y_{bottom}$
$y_0 = 1.3$ m	$t_{peak}, t_{down}, t_{bottom}$
$v_{0y} = 5.05$ m/s	$E_{k,peak}, E_{k,down}, E_{k,bottom}$
$a_y = -g$	$U_{g,peak}, U_{g,down}, U_{g,bottom}$
	$v_{y,peak}, v_{y,down}, v_{y,bottom}$
	$p_{y,peak}, p_{y,down}, p_{y,bottom}$

The subscript "peak" refers to the peak of the ball's flight; "down" refers to the time when it passes the same height from which it was thrown; "bottom" refers to the moment just before it hits the ground.

We have already found the time required to go from each of these positions to the next:

Description	Interval	Where found
$t = 0$ to t_{peak}	0.515 s	Section 2.5
t_{peak} to t_{down}	0.515 s	Section 2.3
t_{down} to t_{bottom}	0.213 s	Section 2.4

Section 2.3 gives the time for t_{peak} to t_{down} if we realize that the time required for a ball to fall 1.3 m does not depend on the height from which it was dropped. So the ball that was dropped from a height of 1.3 m reaches the ground in the same amount of time that a ball dropped from a height of 2.6 m falls 1.3 m.

The ball starts out moving upward, steadily slowing down until it reaches the peak of its flight, so that its velocity is zero at the peak. It doesn't stay at the peak, but turns around and falls downward, steadily speeding up until it hits the ground. When the ball is falling and passes the same point from which it was originally thrown, it will have the same speed that it started with, but the direction will have changed from upward to downward.

If you graph the position as a function of time, the graph forms a parabola shape with its opening pointed downward.

If you graph the velocity as a function of time, the graph forms a straight line, because the acceleration is the same during the entire flight: the acceleration caused by gravity in the downward direction.

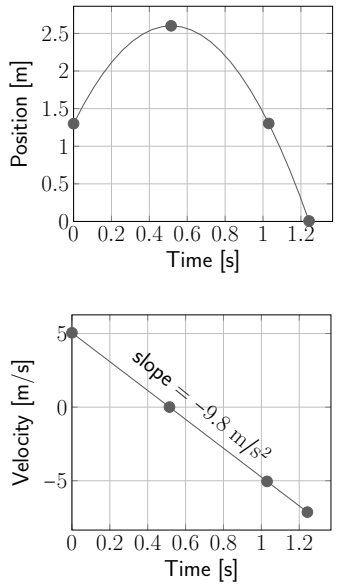


Figure 2.23: Position & velocity for ball in free-fall[1]

Momentum follows the same pattern as velocity. Initially it is upward, and it decreases as the ball goes upward. At the peak, the ball has no momentum, and on the way down its momentum increases, this time in the downward direction.

Initially, the ball has some gravitational potential energy, since it is above ground level, and it also has some kinetic energy, since it is moving. At the peak of its flight the ball is not moving, so it has no kinetic energy. All of the kinetic energy has changed to gravitational potential energy at that point. As the ball falls, the gravitational energy transforms into kinetic energy until by the time it is about to hit the ground it has only kinetic energy.

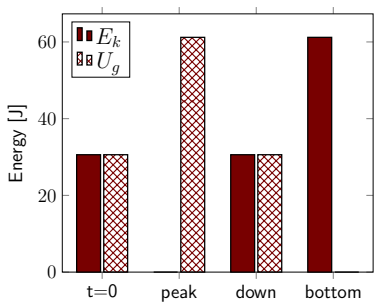


Figure 2.24: Energy bars for ball in free-fall.[1]

Since the acceleration is constant and we are considering only the vertical direction...

$$y = y_0 + v_{0y} \cdot t + \frac{1}{2} a_y \cdot t^2$$

. In this situation...

$$y = 1.3\,\mathrm{m} + 5.05\,\mathrm{m/s} \cdot t - \frac{1}{2} 9.8\,\mathrm{m/s^2} \cdot t^2$$

. And for the velocity we can rearrange Equation 2.2 to find the velocity in the $\hat{y}$ direction at any time t:

$$v_y = v_{0y} + a_y \cdot t$$

In this situation...

$$v_y = 5.05\,\mathrm{m/s} - 9.8\,\mathrm{m/s^2} \cdot t$$

From Equation 1.3, $\vec{p} = m \cdot \vec{v}$, so...

$$p_y = m \cdot v_y = m \cdot (v_{0y} + a_y \cdot t)$$

In this situation...

$$p_y = 2.4\,\mathrm{kg} \cdot 5.05\,\mathrm{m/s} - 2.4\,\mathrm{kg} \cdot 9.8\,\mathrm{m/s^2} \cdot t$$

From Equation 1.5, $U_g = m \cdot g \cdot y$, so...

$$U_g = 2.4\,\mathrm{kg} \cdot 9.8\,\mathrm{m/s^2} \cdot y$$

From Equation 1.6, $E_k = \frac{1}{2} m \cdot v^2$, so...

$$E_k = \frac{1}{2}\, 2.4\,\mathrm{kg} \cdot {v_y}^2$$

2.7 Accelerating in a Car

Words

Up to this point, we have only worked with acceleration due to the force of gravity. But other forces can cause acceleration. Now we will consider a car sitting motionless, accelerating to a given velocity, and maintaining that velocity.

A Ferrari Enzo with a mass of 1500 kg can go from zero to 60 mph in 3 seconds. Assume that it accelerates uniformly for 3 s and then maintains its speed for another 3 s. Describe the net force on the car, the energy of the car, the displacement of the car, and the momentum of the car during these time intervals.

Let's start by just imagining ourselves in the Ferrari. What would it feel like? We would feel the car accelerating as we started, rapidly gaining speed. It is interesting to note that it actually feels like our body being shoved back against the seat as the seat tries to accelerate forward through us. Then it would feel different after the first three seconds because we would just be moving at constant speed. In fact, if you are moving at constant speed it feels almost like you are not moving at all, unless you are watching the scenery go past outside the window.

Since it is force that causes acceleration, there must be a large net force for the first three seconds pushing us forward, but then the net force drops to zero after three seconds.

Graphics

Figure 2.25: A Ferrari Enzo.[6]

Motion Map

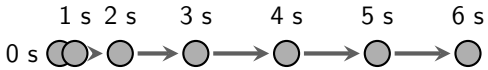


Figure 2.26: A car accelerating for 3 seconds and then moving at constant velocity for 3 seconds[1]

Acceleration on a motion map appears as a change in the arrows from one dot to the next. For the first three seconds, the arrows are getting longer to the right, showing that there is acceleration to the right, and thus a net force to the right. After three seconds the arrows are all the same length and the same direction, showing that there is no acceleration, so zero net force.

Numbers

This situation has two distinct parts that need to be separated mathematically. Acceleration is constant in the first 3 s, and constant in the second 3 s. But since they are different, we will have one set of mathematical models from $t = 0$ to $t = 3\,\mathrm{s}$ and another from $t = 3\,\mathrm{s}$ to $t = 6\,\mathrm{s}$.

Knowns	Unknowns
$m = 1500$ kg	$\overrightarrow{F_{net,0-3}}$; $\overrightarrow{F_{net,3-6}}$
$v_{0x} = 0$	E for the whole time
$v_{x,3-6} = 60\,\mathrm{mph}$	Δx_{0-3} ; Δx_{3-6}
	$\vec{p}$ for the whole time

Let's begin by finding the net force. Using Equation 1.7 and Equation 1.3 we find...

$$\overrightarrow{F_{net}} \equiv \frac{\overrightarrow{\Delta p}}{\Delta t} = \frac{(\vec{p_f} - \vec{p_i})}{\Delta t} = \frac{m \cdot (\vec{v_f} - \vec{v_i})}{\Delta t}$$

Velocity is not changing from 3 to 6 s, so $\overrightarrow{F_{net,3-6}} = 0$. And from 0 to 3 s...

$$\overrightarrow{F_{net}} = \frac{1500\,\mathrm{kg} \cdot (60\,\mathrm{mph}\,\hat{x} - 0)}{3\,\mathrm{s}}$$

We still need to convert the units, so...

$$\overrightarrow{F_{net}} = \left(\frac{1500\,\mathrm{kg} \cdot 60\,\mathrm{mph}}{3\,\mathrm{s}}\right)\left(\frac{1609\,\mathrm{m}}{1\,\mathrm{mile}}\right)\left(\frac{1\,\mathrm{hour}}{3600\,\mathrm{s}}\right)\hat{x}$$

$$\overrightarrow{F_{net,0-3}} = 1.34 \times 10^4\,\mathrm{N}\,\hat{x}$$

What about the energy of the car? If we are on flat ground, the height is not changing, so we don't need to worry about gravitational potential energy. We don't need to consider the vertical direction at all for this situation. We can focus on just the horizontal direction.

The only type of energy that we need to consider is kinetic energy. Initially the car is not moving, so there is no kinetic energy. But over the first three seconds the speed of the car is increasing, so the kinetic energy is also increasing. But after the first three seconds the speed is constant, so for the last three seconds the kinetic energy will remain constant at whatever value it had after the first three seconds.

Energy is conserved in a closed system, but in this system the energy of the car is changing. That is because there is an external force acting on the car. This force is the friction between the tires and the road. To convince yourself of that, ask what would happen if the car were sitting on an oil slick or a sheet of ice. Without friction, the tires would spin uselessly and the car would not move. This force from friction does work on the car, giving it kinetic energy.

The car's momentum will change in the same way that its velocity changes. So initially it has no momentum; then momentum steadily increases for the first three seconds while a constant net force is applied; and finally the momentum remains constant after three seconds when there is no net force acting on the car.

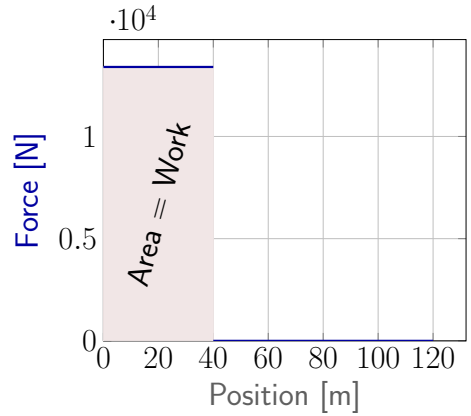


Figure 2.27: A force in the direction of motion does an amount of work equal to the area under the curve in a Force-vs-Position graph.[1]

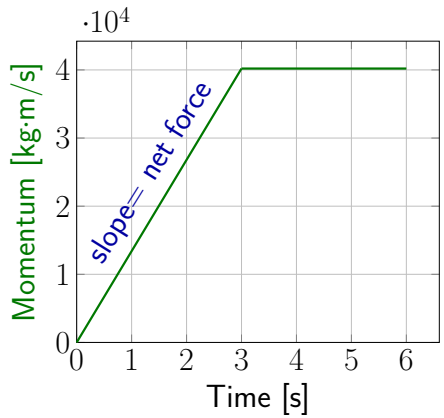


Figure 2.28: Momentum changes while there is a net force, then remains constant when the net force is zero.[1]

The net force acting on the car does work on the car, giving it kinetic energy. The amount of work done by the force depends on the displacement:

$$\boxed{W_{net} = \overrightarrow{F_{net}} \cdot \overrightarrow{\Delta x} = F_{net} \cdot \Delta x \cdot \cos(\theta)} \qquad (2.3)$$

. . . where θ is the angle between $\overrightarrow{F_{net}}$ and $\overrightarrow{\Delta x}$.

This work will often be used in conjunction with Equation 2.1 to find the change in energy of a system. In this example, we are only working with kinetic energy. We know the mass of the car and its velocity at $t = 0$ and $t = 3\,\mathrm{s}$, so that gives us the kinetic energy at each of these times. Using Equation 2.3 and Equation 2.1 we can find the displacement of the car during that time.

$$\Delta E_k = W_{net} = \overrightarrow{F_{net}} \cdot \overrightarrow{\Delta x} = F_{net} \cdot \Delta x \cdot \cos(0)$$

θ is zero because $\overrightarrow{F_{net}}$ and $\overrightarrow{\Delta x}$ are in the same direction. Rearranging to solve for Δx gives. . .

$$\Delta x_{0-3} = \frac{\Delta E_{k,0-3}}{F_{net,0-3}} = \frac{5.39 \times 10^5\ \mathrm{J}}{1.34 \times 10^4\ \mathrm{N}} = 40\ \mathrm{m}$$

We can use the $\hat{x}$ part of Equation 1.2 to find Δx_{3-6}:

$$x = x_0 + v_{0x} \cdot t + \frac{1}{2} a_x \cdot t^2$$

With $a_x = 0$ and using 3 s as our "$t = 0$", . . .

$$\Delta x_{3-6} = x - x_0 = v_{0x} \cdot t + 0 = 60\ \mathrm{mph} \cdot 3\ \mathrm{s} = 80\ \mathrm{m}$$

2.8 Braking in a Car

Words

A Ferrari Enzo with a mass of 1500 kg is traveling at $26.8\ \mathrm{m/s}$ and suddenly applies its brakes, giving it an acceleration of $15\,\mathrm{m/s^2}$ in the direction opposite its motion. Describe the net force on the car, the car's momentum, the time required to stop, and the stopping distance.

If the car had been traveling at twice that speed and then braked with the same acceleration, by how much would the net force, the momentum, the time required to stop, and the stopping distance change?

When we are in a car that is braking, its speed is decreasing. That means the acceleration is opposite the direction of motion, so if your car is moving forward then the acceleration is backward.

Again, since it is force that causes acceleration, the net force must be pointing in the direction opposite the car's motion. The acceleration described above, $15\,\mathrm{m/s^2}$, is larger than the acceleration caused by gravity on the surface of the earth, so this car must be braking very hard. We should expect it to stop quickly.

The car's momentum changes along with velocity, starting large and steadily decreasing until it reaches zero when the car is stopped.

Graphics

Figure 2.29: A Ferrari Enzo.[6]

Free Body Diagram

Figure 2.30: FBD of a car that is traveling to the right and braking, horizontal direction only[1]

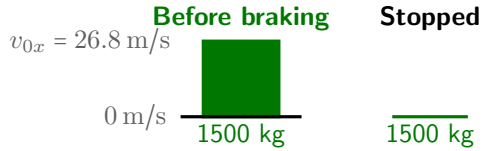


Figure 2.31: Momentum bars before and after braking[1]

Numbers

Knowns	Unknowns
$m = 1500$ kg	$\overrightarrow{F_{net}}$
$v_{0x} = 26.8\ \mathrm{m/s}$	t_f
$a_x = -15\ \mathrm{m/s^2}$	Δx
$v_f = 0$	

Note that the acceleration is negative, while the initial velocity is positive. This is equivalent to the statement that acceleration is opposite the direction of motion. The stopping distance would be the change in position, in other words, the displacement.

Let's begin by finding the net force. Using Equation 1.8 we find...

$$\overrightarrow{F_{net}} = m \cdot \vec{a} = 1500\ \mathrm{kg} \cdot \left(-15\ \mathrm{m/s^2}\ \hat{x}\right)$$

...so $\overrightarrow{F_{net}} = -2.25 \times 10^4\ \mathrm{N}\ \hat{x}$

Since the acceleration is constant until the car stops moving, this force is also constant until the car stops moving.

Given the mass and the initial and final velocity, we can use Equation 1.3 to find the initial and final momentum of the car:

$$\vec{p_0} = 1500\ \mathrm{kg} \cdot 26.8\ \mathrm{m/s}\ \hat{x} = 4.02 \times 10^4\ \mathrm{kg \cdot m/s}\ \hat{x}$$

$$\vec{p_f} = m \cdot \vec{v_f} = 0$$

What would be different if the car had been traveling at twice the speed, and had braked with the same acceleration?

Since force is directly related to acceleration but not related to velocity, the frictional force of braking would be the same.

The final momentum would still be zero, but the initial momentum would double along with the velocity.

Remember that acceleration is a change in velocity over time. Since we are more familiar with speed and distance, let's use that as an example. Speed is a distance over time. If you travel at the same speed but need to go double the distance, the time would have to double, right? It is the same with acceleration and velocity:

- Speed is a distance over time; double the distance means double the time.
- Acceleration is velocity over time; double the velocity means double the time.

The distance traveled before stopping would certainly be longer if we started with double the speed and braked with the same acceleration. In fact the distance increases by much more than double. If you start at double the speed, it takes half of the braking time just to get down to the original speed; in the first half of the braking time you would move a much farther distance than if you had been traveling at the original speed that whole time. And you still wouldn't have stopped!

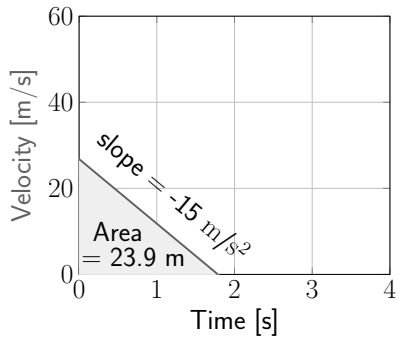


Figure 2.32: Velocity vs time, braking.[1]

The area under the curve of a velocity vs time graph is equal to the displacement. In this case, the maximum velocity is 26.8 m/ s and the maximum time is 1.79 s. The area of a triangle is:

$$A_{triangle} = \frac{1}{2} b \cdot h$$

. . . where b is the triangle's base and h is its height.

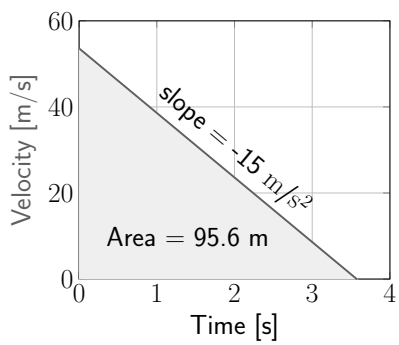


Figure 2.33: Velocity vs time, braking.[1]

The time required to stop can be found using Equation 2.2:

$$\vec{a} = \frac{\overrightarrow{\Delta v}}{\Delta t}$$

If we set $t_i = 0$, this can be rearranged to. . .

$$t = \frac{\vec{v_f} - \vec{v_0}}{\vec{a}} = \frac{0 - 26.8\,\mathrm{m/s}\,\hat{x}}{-15\,\mathrm{m/s^2}\,\hat{x}} = 1.79\,\mathrm{s}$$

The displacement can be found by considering energy. Initially the car has kinetic energy, and the braking force acting over a distance does *negative* work on the car, removing all of its kinetic energy. Using Equation 1.6. . .

$$\Delta E_k = E_{k,f} - E_{k,i} = 0 - \frac{1}{2} m \cdot v_{0x}^2$$

And according to Equation 2.3 and Equation 2.1. . .

$$\Delta E_k = W_{net} = \overrightarrow{F_{net}} \cdot \overrightarrow{\Delta x}$$

Setting these equal to each other and solving for Δx gives. . .

$$\Delta x = \frac{-\frac{1}{2} m \cdot v_{0x}^2}{\overrightarrow{F_{net}}} = \frac{-5.39 \times 10^5\,\mathrm{J}}{-2.25 \times 10^4\,\mathrm{N}} = 23.9\mathrm{m}$$

2.9 Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

General

- Units in an equation cancel just like variables or numbers.
- Numbers with different dimensions can be multiplied or divided, but not added or subtracted.
- Physical quantities should be converted to SI units before being used in calculations.
- When thinking about a physical scenario, it is often helpful to make a sketch.
- Conventionally, right is the positive "x" direction $\hat{x}$.
- Conventionally, up is the positive "y" direction $\hat{y}$.
- Sometimes a calculator can give answers that are correct mathematically but cannot be physically correct.

Forces

- In U.S. Customary units, pound is the unit of force; slug is the unit of mass.
- If a force acts in the direction opposite to the velocity of an object it will slow the object down.
- Depending on the situation, a force of friction can either increase or decrease an object's speed.
- A force acting in the direction of an object's displacement does positive work on the object.
- A force acting opposite the direction of an object's displacement does *negative* work on the object.

Motion

- The change in the position of an object is called its displacement.
- Displacement can be positive or negative.
- The total distance an object moves, or the length of the path it follows, is always positive.
- The slope of the line on a postion-vs-time graph is the velocity.
- The slope of the line on a velocity-vs-time graph is the acceleration.
- Acceleration refers to a change in velocity whether the speed is increasing, decreasing, or staying the same.
- The acceleration of an object in free-fall is constant throughout the time the object is in the air, even if it is not moving at some point in time during the flight.
- The area under the curve of a velocity-vs-time graph is the displacement of the object.

Momentum

- Momentum is negative if velocity is negative.
- If velocity is negative, momentum bars have negative height, and thus negative area.
- The slope of the line on a momentum-vs-time graph is the net force on the object.

Energy

- The area under the curve of a force-vs-distance graph is the amount of work done by the force.
- Work changes an object's kinetic energy.

Mathematical Models

equation		restrictions on the validity of the equation
$W = E_{tot,f} - E_{tot,i}$	(2.1)	-none-
$\vec{a} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v_f} - \vec{v_i}}{\Delta t}$	(2.2)	only valid when the net force is constant
$W_{net} = \overrightarrow{F_{net}} \cdot \overrightarrow{\Delta x} = F_{net} \cdot \Delta x \cdot \cos(\theta)$	(2.3)	only valid when the net force is constant

2.10 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

Level 1 - Remember

2.1 **[W]** List two reasons for using SI units instead of U.S. Customary Units in physics.

2.2 **[N]** Give the appropriate SI units and U.S. Customary units for each of the following:

	SI (Système International) Unit	**U.S. Customary Unit**
length		
force		
mass		
speed		

2.3 **[G]** How does one find work from a graph of force vs. position?

2.4 **[G]** How does one find force from a graph of momentum vs. time?

2.5 **[G]** How does one find acceleration from a graph of velocity vs. time?

2.6 **[G]** How does one find displacement from a graph of velocity vs. time?

2.7 **[G]** How does one find velocity from a graph of position vs. time?

2.8 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

Level 2 - Understand

2.9 **[W]** The word "acceleration" is used in physics to mean a change in velocity. What is happening to an object's speed when it is accelerating?

Level 3 - Apply

2.10 **[G]** Draw a free-body diagram for the ball in Section 2.3.

2.11 **[G & N]** What is the total displacement of the ball described in Section 2.6, from the time it is thrown to the moment just before it hits the ground? Remember that displacement is a vector.

2.12 **[G & N]** What is the total distance (path length) traveled for the ball described in Section 2.6, from the time it is thrown to the moment just before it hits the ground?

2.13 **[W]** Describe the momentum of a ball that is dropped from the time it leaves a person's hand until the time just before it hits the ground.

2.14 **[G]** Section 2.7 describes a physical situation of a car accelerating and then traveling at constant speed.

(a) Draw a velocity vs. time graph for the car in this situation. Be sure to use SI units.

(b) Use the graph to find the displacement of the car in the first 3 s.

(c) Use the graph to find the displacement of the car in the second 3 s.

(d) Use the graph to find the acceleration of the car in the first 3 s.

(e) Use the graph to find the acceleration of the car in the second 3 s.

Level 4 - Analyze

2.15 **[G]** Figure 2.4 shows a graph of position vs. time for an object that is moving at 2 m/s. Draw a similar graph for an object that is moving at 5 m/s for 10 s.

2.16 **[W, G, & N]** In Section 2.2, there was a 170-g hockey puck sliding across a sheet of ice. If the mass of the hockey puck had been 340 g and everything else in the situation stayed the same, which of the following would change?

(a) The sketch at the beginning of Section 2.2 (and make a new sketch if any change is needed)

(b) The free body diagram in Section 2.2 (and make a new FBD if any change is needed)

(c) The calculation of kinetic energy in Section 2.2 (and find the new value for the kinetic energy if any change is needed)

(d) The motion map in Section 2.2 (and make a new motion map if any change is needed)

(e) The calculation of momentum in Section 2.2 (and find the new value for the momentum if any change is needed)

(f) The calculation of displacement in Section 2.2 (and find the new value for the displacement if any change is needed)

2.17 **[W, G, & N]** In Section 2.2, there was a hockey puck sliding across a sheet of ice at 24 m/s. If the speed of the hockey puck had been 12 m/s and everything else in the situation stayed the same, which of the following would change?

(a) The sketch at the beginning of Section 2.2 (and make a new sketch if any change is needed)

(b) The free body diagram in Section 2.2 (and make a new FBD if any change is needed)

(c) The calculation of kinetic energy in Section 2.2 (and find the new value for the kinetic energy if any change is needed)

(d) The motion map in Section 2.2 (and make a new motion map if any change is needed)

(e) The calculation of momentum in Section 2.2 (and find the new value for the momentum if any change is needed)

(f) The calculation of displacement in Section 2.2 (and find the new value for the displacement if any change is needed)

Level 5 - Evaluate

2.18 **[N]**Compare the change in gravitational potential energy for an object of mass m falling a distance d to the work done by the force of gravity on an object of mass m falling a distance d. Explain your reasoning.

2.19 **[N]**Compare the work done bringing an object of mass m from a complete stop to a speed v over a distance d to the work done bringing that same object from the same speed v to a stop over a distance $d/4$. Explain your reasoning.

Level 6 - Create

2.20 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Add the main ideas from this chapter to the concept map that you began for the question at the end of Chapter 1.

2.21 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

2.22 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

Chapter 3

Two Objects

Up to this point we have only been looking at a single object. That object has always interacted with something else: A soccer ball rolling across the *ground*; a hockey puck sliding on *ice*; a ball falling under the influence of the *earth*'s gravity; but we have never stopped to consider the second object and how it interacts with the first.

There are two new things we will consider in this chapter: reference frames (also called frames of reference or points of view) and force pairs.

Whenever we look at a physical situation, we need to be mindful of our reference frame. Up to this point, the surface of the earth has been used as a reference, and is assumed to be fixed in place as we consider the physics of each situation. But sometimes it will be easier to look at a situation from another reference frame that is moving with respect to the surface of the earth. For example, if you are a passenger in a car that is traveling at high speed, it is still quite easy to pick up an object that is sitting next to you, even though that object is traveling at high speed compared to the ground outside of the car. As long as the car is moving at a constant velocity, the physics inside the car is just the same as if the car were sitting still.

Figure 3.1: Here we see two objects that have interacted with each other. To understand the collision, we need to take both objects into account, and we will have to decide whether to look at the collision from the reference frame of a person in the truck, a person in the car, or a person standing by the side of the road.[7]

The idea of force pairs comes about because forces are always interactions between objects. If any object exerts a force on something else, that something exerts the same amount of force back on the first object.

3.1 Reference Frames

Words

Imagine standing on a bridge over the highway in the picture on the right. You would look at these vehicles and say that the two cars are going North at 25 m/s and the truck is going South at 25 m/s. You are comparing the velocities of the vehicles to your own velocity, which is zero if the surface of the earth is stationary.

Of course the surface of the earth is not stationary because the earth is spinning and orbiting the sun! But in many situations we can ignore the motion of the earth itself. Since you are not moving with respect to the surface of the earth, we say that you are looking at the vehicles from the earth's reference frame.

What are the velocities of the vehicles in each others' reference frames?

To consider other reference frames, we just need to imagine ourselves in a different place. What would this situation look like if we were inside the black car? Since the white car that is ahead of us is moving at the same speed, in the same direction as us, it will actually appear not to be moving at all. If it is 20 meters ahead of us, it will stay 20 meters ahead of us as long as neither of us accelerates.

Graphics

Figure 3.2: Two cars traveling North and a truck traveling South, all at 25 m/s[8]

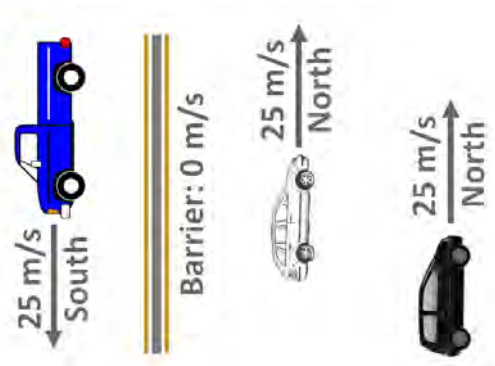


Figure 3.3: The earth's reference frame[9]

Numbers

We will need to introduce new notation for velocity in different reference frames: "$\overrightarrow{v_{1\leftarrow 2}}$" will mean "the velocity of object 1 as seen by object 2." Using this notation, making North the positive direction, and referring to the cars just as "white" and "black"...

Knowns	**Unknowns**
$\overrightarrow{v_{truck\leftarrow earth}} = -25\ \mathrm{m/s}$	$\overrightarrow{v_{white\leftarrow truck}}$
$\overrightarrow{v_{white\leftarrow earth}} = +25\ \mathrm{m/s}$	$\overrightarrow{v_{black\leftarrow truck}}$
$\overrightarrow{v_{black\leftarrow earth}} = +25\ \mathrm{m/s}$	$\overrightarrow{v_{truck\leftarrow white}}$
	$\overrightarrow{v_{black\leftarrow white}}$
	$\overrightarrow{v_{truck\leftarrow black}}$
	$\overrightarrow{v_{white\leftarrow black}}$

To change from one reference frame to another, simply subtract the velocity of the object whose frame we are entering:

$$\boxed{\overrightarrow{v_{2\leftarrow 3}} = \overrightarrow{v_{2\leftarrow 1}} - \overrightarrow{v_{3\leftarrow 1}}} \tag{3.1}$$

To go from the earth's reference frame to the black car's reference frame, we can make "1" the earth, "2" the white car, and "3" the black car:

$$\overrightarrow{v_{white\leftarrow black}} = \overrightarrow{v_{white\leftarrow earth}} - \overrightarrow{v_{black\leftarrow earth}}$$

$$\overrightarrow{v_{white\leftarrow black}} = +25\ \mathrm{m/s} - (+25\ \mathrm{m/s}) = 0$$

Staying in the black car's reference frame and looking at the truck, it appears to be moving very fast. From the bridge, the truck appeared to be moving 25 m/s, but since we are moving in the opposite direction of the truck, it appears to be moving much faster.

Interestingly, the earth itself and everything hooked to it appears to be moving in the reference frame of the black car. If you are sitting in the black car, you see houses, trees, and the cement barrier going past your window, even though they are motionless in the earth's reference frame.

Often, it will be easiest to think about situations in the earth's reference frame, but all of the laws of physics are still true in any reference frame that is not accelerating. This type of reference frame is often called an "inertial reference frame."

If you are riding in a train car and sipping from a cup of coffee, you understand the physics of taking a small careful sip, and you are able to do this as naturally on a train as you would if you were standing on solid ground. The laws of physics work just the same in the moving train as outside on solid ground. That is, unless the train is suddenly lurching out of the station, or applying the brakes, or going around a sharp curve. At those times, the inside of the train is a non-inertial reference frame, so the normal laws of physics do not apply inside the train car, and you spill your coffee.

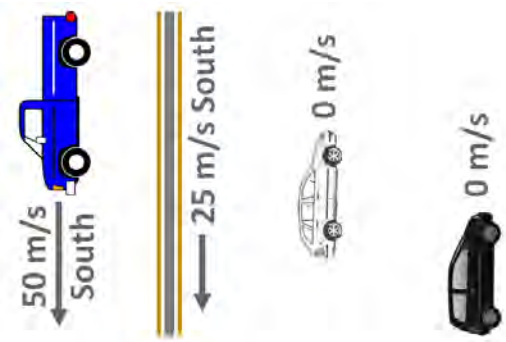


Figure 3.4: The black car's reference frame[9]

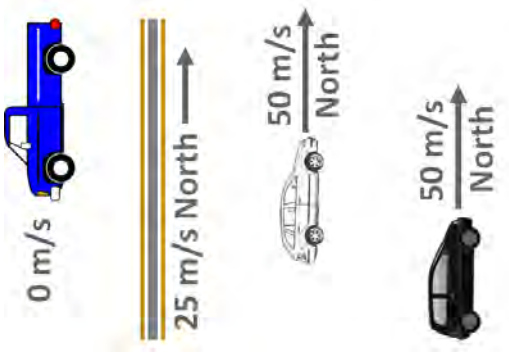


Figure 3.5: The truck's reference frame[9]

Similarly,

$$\overrightarrow{v_{truck\leftarrow black}} = \overrightarrow{v_{truck\leftarrow earth}} - \overrightarrow{v_{black\leftarrow earth}}$$

$$\overrightarrow{v_{truck\leftarrow black}} = -25\,\mathrm{m/s} - (+25\,\mathrm{m/s}) = -50\,\mathrm{m/s}$$

What happens if we look at the *earth* from the black car's reference frame? Using Equation 3.1 where "3" is again the black car but this time the earth is both "1" and "2,"

$$\overrightarrow{v_{earth\leftarrow black}} = \overrightarrow{v_{earth\leftarrow earth}} - \overrightarrow{v_{black\leftarrow earth}}$$

. . . and recognizing that the velocity of the earth in the earth's reference frame is zero. . .

$$\overrightarrow{v_{earth\leftarrow black}} = 0 - (+25\,\mathrm{m/s}) = -25\,\mathrm{m/s}$$

The earth and everything solidly connected to it has a velocity of -25 m/s in the black car's reference frame!

Notice that $\overrightarrow{v_{earth\leftarrow black}}$ is the opposite of $\overrightarrow{v_{black\leftarrow earth}}$. This will be true for any pair of objects:

$$\boxed{\overrightarrow{v_{1\leftarrow 2}} = -\overrightarrow{v_{2\leftarrow 1}}} \tag{3.2}$$

3.2 An Ant Pushing a Rock

Words

The image on the right shows an ant pushing a rock. Let's imagine that at first neither the ant nor the rock is moving, but then the ant begins pushing and the rock and the ant both start moving to the right, accelerating together at 0.5 m/s^2. The rock looks much larger than the ant, so we will say that the ant has a mass of 0.005 kg and the rock has a mass of 0.015 kg.

The ant is applying a force to the rock to make it accelerate. What are the other forces involved, and how do they compare with the force that the ant is applying to the rock? For simplicity, we will assume that the rock slides freely on the ground, and we will consider only the horizontal direction.

The possible forces in the horizontal direction are at each interface. So between the ant and the rock, between the ant and the ground, and between the rock and the ground.

At each interface there are two forces involved– for example, the ant pushes on the rock, and the rock pushes back on the ant. To some people, it may seem obvious that since the ant is the thing doing the pushing, it is applying the larger force. To others, it may seem obvious that since the rock is larger than the ant, it is applying the larger force. But in fact neither of these is correct!

If a drawing is not given, it is good to make a sketch to help with understanding the situation clearly.

Graphics

Figure 3.6: An ant pushing a rock.[10]

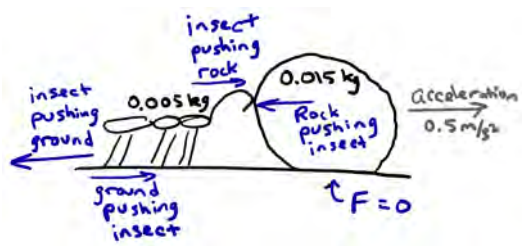


Figure 3.7: A sketch of the ant, the rock, and the forces affecting them. [1]

Numbers

Knowns	Unknowns
m_{ant} = 0.005 kg	$\overrightarrow{F_{ant\to rock}}$
m_{rock} = 0.015 kg	$\overrightarrow{F_{ground\to ant}}$
$\vec{a}$ = +0.5 m/s^2 $\hat{x}$	$\overrightarrow{F_{rock\to ant}}$
$\overrightarrow{F_{ground\to rock}}$ = 0	

Here, $\overrightarrow{F_{ant\to rock}}$ is the force applied by the ant on the rock, and similarly for the other subscripts.

Note that only one acceleration $\vec{a}$ is listed as an unknown, instead of listing $\overrightarrow{a_{ant}}$ and $\overrightarrow{a_{rock}}$ separately. Since the ant and the rock are in contact with each other the whole time, their motion will be identical, so they have the same acceleration. No subscript is needed.

Since we have the acceleration and masses of the objects and are looking for force, we will need to use Equation 1.8:

$$\overrightarrow{F_{net}} = m \cdot \vec{a}$$

First, let's consider the rock because there is only one force that is acting on it in the horizontal direction: the force from the ant. That force is just enough to accelerate the rock at 0.5 m/s^2: 0.0075 N.

Figure 3.8: FBD of rock, horizontal direction only[1]

The ant has two forces acting on it, and we don't yet know either one, so it will be easier instead to consider the ant and the rock to be a single system, and look at the forces affecting that system. From Figure 3.7 we can see that the only external horizontal force affecting the system is the frictional force of the ground pushing the ant. That force is just enough to accelerate the system at 0.5 m/s^2: 0.01 N.

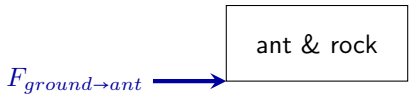


Figure 3.9: FBD of ant & rock system, horizontal direction only[1]

Now we can consider just the ant. The net force from the ground and the rock has to be just enough to accelerate the ant at 0.5 m/s^2. We already know the size of the force from the ground on the ant, so now we can find the force from the rock on the ant. Interestingly, it is exactly the same magnitude as the force of the ant on the rock, but in the opposite direction!

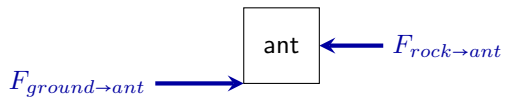


Figure 3.10: FBD of ant, horizontal direction only[1]

This is a general rule that will be true for all forces. For every force, there is an equal and opposite force on another object. This is called Newton's Third Law.

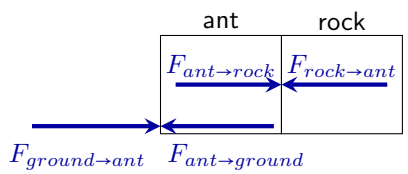


Figure 3.11: Combined FBD[1]

When determining how forces affect an object, for example when finding an object's acceleration, its change in momentum, or its change in energy, only the external forces, those acting on the object from outside, should be considered.

It is possible to combine the FBD's of the ant and the rock. For each object, only the external forces, the arrows touching the object from the outside, affect the object's motion and momentum.

We have the mass and the acceleration of the rock, so we can find the net force on it, and the only force on the rock is from the ant, so the net force is just $\overrightarrow{F_{ant\to rock}}$.

$$\overrightarrow{F_{ant\to rock}} = m_{rock} \cdot \vec{a} = +0.0075\ \mathrm{N}\ \hat{x}$$

We have the mass and the acceleration of the combined ant & rock system, so we can find the net force on it as well, and the only force on the system is from the ground on the ant.

$$\overrightarrow{F_{ground\to ant}} = m_{ant\&rock} \cdot \vec{a} = +0.01\ \mathrm{N}\ \hat{x}$$

We have the mass and the acceleration of the ant, so we can find the net force on it as well, and since we already know the force from the ground on the ant we can find the force of the rock on the ant.

$$\overrightarrow{F_{net}} = \overrightarrow{F_{ground\to ant}} + \overrightarrow{F_{rock\to ant}}$$

Rearranging gives...

$$\begin{aligned}\overrightarrow{F_{rock\to ant}} &= \overrightarrow{F_{net}} - \overrightarrow{F_{ground\to ant}} \\ &= \left(0.005\ \mathrm{kg} \cdot 0.5\ \mathrm{m/s^2} - 0.01\ \mathrm{N}\right)\ \hat{x} \\ &= -0.0075\ \mathrm{N}\ \hat{x}\end{aligned}$$

The force of the ant on the rock is equal and opposite to the force of the rock on the ant. This will be true for any physical situation.

$$\boxed{\overrightarrow{F_{1\to 2}} = -\overrightarrow{F_{2\to 1}}} \tag{3.3}$$

3.3 Kicking Horizontally

Words

The image on the right shows a soccer player about to kick a ball. The 450-gram ball was initially moving to the left at 5 m/s and after the kick it moves to the right at 15 m/s.

What can be said about what happened during the kick from just this information?

There is not really much information to work with here. The mass of the ball shouldn't change because of the kick. The only change described is the velocity of the ball.

Velocity was to the left, and after the kick velocity is to the right at a higher speed. So there has been a change in momentum, not only in magnitude but also in direction.

Imagine the kick in slow motion. The ball is moving left, then it comes into contact with a foot. The foot stops the motion to the left, removing all of the ball's initial momentum, and then gives the ball momentum to the right.

This happens quickly, so in a short time the foot created a large change in the ball's momentum. A change in momentum is often called an *impulse*, especially when the interaction occurrs over a very small amount of time.

Graphics

Figure 3.12: A soccer player kicking a ball.[11]

Momentum Bars

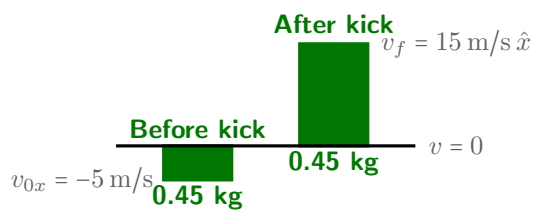


Figure 3.13: Momentum of the ball before and after the kick.[1]

Since momentum is always conserved in an isolated system, we know that something outside of the ball must have affected it.

Numbers

Knowns	Unknowns
$m_1 = 0.45$ kg	???
$v_{0x} = -5\,\mathrm{m/s}$	
$v_f = +15\,\mathrm{m/s}$	

In this situation we are asked to find anything that we can, so there are no specific unknowns to look for.

Given mass and velocity, perhaps momentum would be a good place to start.

Since we have initial and final information about the ball, we can calculate its change in momentum. From Equation 1.3,

$$\vec{p} = m \cdot \vec{v}$$

. . . so. . .

$$\vec{\Delta p} = \vec{p_f} - \vec{p_i} = m \cdot \vec{v_f} - m \cdot \vec{v_i} = m \cdot \vec{\Delta v}$$

$$\vec{\Delta p} = 0.45\,\mathrm{kg} \cdot (+15\,\mathrm{m/s} - (-5\,\mathrm{m/s}))\,\hat{x}$$

$$\vec{\Delta p} = 9\,\mathrm{kg} \cdot \mathrm{m/s}\,\hat{x}$$

$\vec{\Delta p}$ is often called *impulse*.

Changing the momentum of the ball would have taken a force, so the foot applied a force to the ball, to the right. And since forces come in equal-and-opposite pairs, the ball must also have applied that same force to the foot, but in the opposite direction.

This all happened very quickly, so the forces involved had to be very large but acting over a very small amount of time.

We also know something about the kinetic energy of the ball: After the kick it was moving more quickly than it was before the kick. So its kinetic energy increased. That means work must have been done on the ball.

The foot must have done work on the ball, and if the foot continued moving to the right the entire time, it was doing work for the entire time that it was in contact with the ball.

At the instant the ball came into contact with the foot, it was moving to the left, so it should be doing work on the foot, except that the foot was moving in the opposite direction. But what actually happens during the collision is some of the energy gets stored as elastic, or spring, potential energy and some of the energy is transformed into heat, warming up the ball. So while total energy is conserved, some energy is usually lost to heat in a collision. That means that *mechanical energy* is not conserved in the collision.

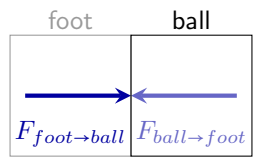


Figure 3.14: Combined FBD of the foot and ball during the kick, focusing on forces on the ball[1]

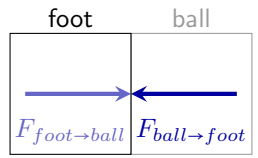


Figure 3.15: Combined FBD of the foot and ball, focusing on forces on the foot.[1]

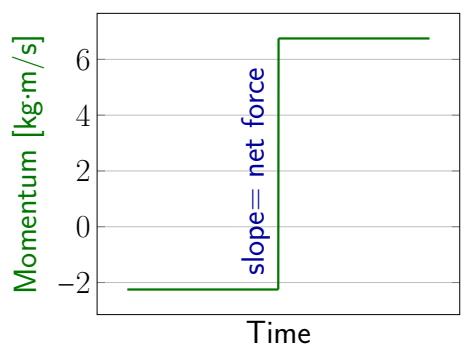


Figure 3.16: Momentum is constant before the kick, changes rapidly during the kick, and then is constant again after the kick.[1]

The definition of force in Equation 1.7 tells us that

$$\overrightarrow{F_{net}} = \frac{\overrightarrow{\Delta p}}{\Delta t}$$

...but we don't know Δt. We do, however, know that Δt is very small, probably a few milliseconds, so the forces involved must be very large. Using Equation 3.3,

$$\overrightarrow{F_{foot\to ball}} = -\overrightarrow{F_{ball\to foot}}$$

...so there is also a very large force applied from the ball to the foot.

We don't know the amount of time that the force is applied, but we do know that the time that the ball was in contact with the foot is exactly equal to the time that the foot was in contact with the ball. We also know that the force on the foot from the ball was exactly equal to the force on the ball from the foot. Since we know that the times are the same and the forces are equal an opposite...

$$\overrightarrow{F_{foot\to ball}} \cdot \Delta t = -\overrightarrow{F_{ball\to foot}} \cdot \Delta t$$

...or...

$$\overrightarrow{\Delta p_{ball}} = -\overrightarrow{\Delta p_{foot}}$$

$$\overrightarrow{p_{ball,f}} - \overrightarrow{p_{ball,i}} = -\left(\overrightarrow{p_{foot,f}} - \overrightarrow{p_{foot,f}}\right)$$

...which can be rearranged to show that...

$$\boxed{\overrightarrow{p_{tot,i}} = \overrightarrow{p_{tot,f}}} \tag{3.4}$$

This is true for any system of objects that is not affected by any external net force.

3.4 Elastic Collision

Words

One hard steel ball with a mass of 0.7 kg is sitting motionless when it is hit by an identical steel ball that is moving to the right at 3 m/s. A collision between steel balls is normally an *elastic* collision, which means that kinetic energy is conserved. Assume that there are no external forces affecting the balls.

What is the final velocity of each ball?

In any collision, the first thing to consider is momentum, since momentum is always conserved for any isolated system. And the forces between the objects during a collision is usually so large that all other forces can be neglected during the collision. So for practical purposes, momentum is conserved for any collision, whether there are external forces or not.

We are also told that kinetic energy is conserved in this collision, so the total kinetic energy before the collision is equal to the total kinetic energy after the collision.

Graphics

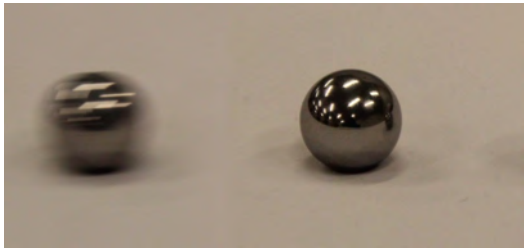


Figure 3.17: A moving ball about to hit a stationary ball[1]

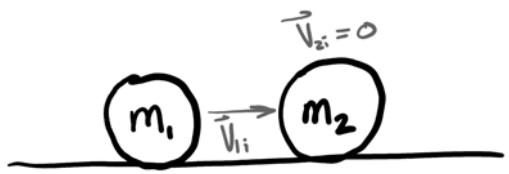


Figure 3.18: Sketch of balls before the collision[1]

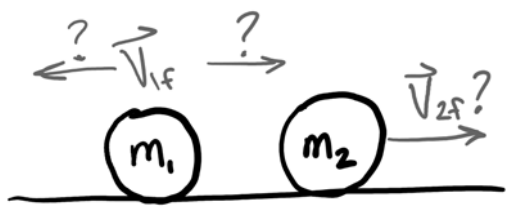


Figure 3.19: Sketch of balls after the collision[1]

Numbers

Knowns	Unknowns
$m_1 = m_2 = 0.7$ kg	$\overrightarrow{v_{1,f}}$
$\overrightarrow{v_{1,i}} = +3\ \mathrm{m/s}\ \hat{x}$	$\overrightarrow{v_{2,f}}$
$\overrightarrow{v_{2,i}} = 0$	
$E_{k,i} = E_{k,f}$	

Equation 3.4 tells us that:

$$\overrightarrow{p_{1,i}} + \overrightarrow{p_{2,i}} = \overrightarrow{p_{1,f}} + \overrightarrow{p_{2,f}}$$

$$m_1 \cdot \overrightarrow{v_{1,i}} + m_2 \cdot \overrightarrow{v_{2,i}} = m_1 \cdot \overrightarrow{v_{1,f}} + m_2 \cdot \overrightarrow{v_{2,f}}$$

...but we don't know $\overrightarrow{v_{1,f}}$ or $\overrightarrow{v_{2,f}}$. We need a second equation.

Given that kinetic energy is conserved (it is an elastic collision)...

$$E_{k,i} = E_{k,f}$$

...or...

$$\frac{1}{2}m_1 \cdot v_{1,i}^2 + \frac{1}{2}m_2 \cdot v_{2,i}^2 = \frac{1}{2}m_1 \cdot v_{1,f}^2 + \frac{1}{2}m_2 \cdot v_{2,f}^2$$

In this case, the ball on the right starts with no momentum, and the ball on the left has momentum. So the total momentum of the system is to the right. Whatever happens during the collision, the total momentum will still have to be to the right after the collision. That could mean both balls will end up going to the right, one ball stops and the other goes to the right, or one ball goes to the left and the other ball goes *faster* to the right.

Since there are so many options, in order to determine what actually happens in a collision like this, you have to crank through the numbers.

After going through the numbers, we find that there are three possibilities. They are illustrated in the center column with momentum bars.

Possibility #1: Ball 1 stops completely and Ball 2 has a final velocity that is the same as the initial velocity of Ball 1.

Possibility #2: The velocities of both balls are the same as their initial velocities. This describes the situation if the first ball would have missed the second ball completely, so it is not the solution to what happens after a collision.

Possibility #3: Ball 1 stops completely and Ball 2 stays still. This cannot be correct, because this solution does not conserve momentum.

So the first possibility has to be the solution: Ball 1's final velocity is zero and Ball 2's final velocity is 3 m/s to the right.

Momentum before collision

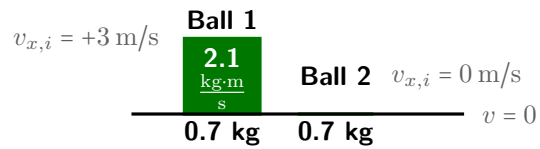


Figure 3.20: Total momentum before collision was $+2.1\,\mathrm{kg{\cdot}m/s}$[1]

Possible momenta after collision

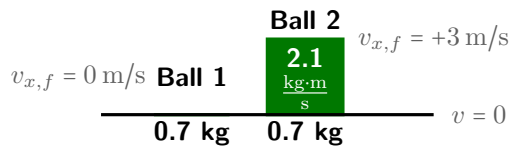


Figure 3.21: Possibility #1: momentum is conserved, so if $v_{1,f,x} = 0$ then $v_{2,f,x} = 3\,\mathrm{m/s}$[1]

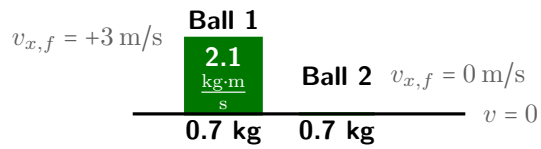


Figure 3.22: Possibility #2: if $v_{2,f,x} = 0$ then $v_{1,f,x} = 3\,\mathrm{m/s}$. This is identical to the situation before the collision[1]

Possibility #3: There is no way to draw momentum bars that conserve the initial momentum if the final velocities of both balls are zero.

Now we have two equations and two unknowns, so we should be able to solve for both velocities. We can simplify the equations, since $m_1 = m_2$ and $\overrightarrow{v_{2,i}} = \vec{0}$, giving...

$$\overrightarrow{v_{1,i}} = \overrightarrow{v_{1,f}} + \overrightarrow{v_{2,f}}$$

and

$$v_{1,i}^2 = v_{1,f}^2 + v_{2,f}^2$$

Since there is only motion in the x direction, we can just consider the $\hat{x}$ part of the vectors:

$$v_{1,i,x} = v_{1,f,x} + v_{2,f,x}$$

$$v_{1,i,x}^2 = v_{1,f,x}^2 + v_{2,f,x}^2$$

Squaring both sides of the first equation...

$$v_{1,i,x}^2 = v_{1,f,x}^2 + 2 \cdot v_{1,f,x} \cdot v_{2,f,x} + v_{2,f,x}^2$$

Now we have two equations for $v_{1,i,x}^2$. Setting them equal to each other...

$$v_{1,f,x}^2 + v_{2,f,x}^2 = v_{1,f,x}^2 + 2 \cdot v_{1,f,x} \cdot v_{2,f,x} + v_{2,f,x}^2$$

...which can only be true if...

$$2 \cdot v_{1,f,x} \cdot v_{2,f,x} = 0$$

There are three possibilities:

Possibility #1	$v_{1,f,x} = 0$	$v_{2,f,x} \neq 0$
Possibility #2	$v_{1,f,x} \neq 0$	$v_{2,f,x} = 0$
Possibility #3	$v_{1,f,x} = 0$	$v_{2,f,x} = 0$

Which ones make physical sense?

3.5 Two Moving Balls

Words

Two identical steel balls have an elastic collision with each other. Both have a mass of 0.7 kg. One is initially moving to the right at 1.5 m/s and the other is initially moving to the left at 1.5 m/s.

What is the final velocity of each ball?

In any collision, the first thing to consider is momentum. The total momentum will be conserved. In this case Ball 1 has momentum to the right and Ball 2 has the exact same amount of momentum but to the left. So the total momentum of the system is zero. So after the collision, the balls will also have equal and opposite momentum. And since they both have the same mass, they will also have equal and opposite velocity, so the same speed but opposite directions.

Since this is an elastic collision, we know that kinetic energy is conserved. Initially, the speeds and the masses of the two balls are the same. Since the final speeds of the two balls are equal, the only way for the final kinetic energy to be the same as the initial kinetic energy is if the final speeds are the same as the initial speeds. Ball 1 ends up moving to the left at 1.5 m/s and Ball 2 ends up moving to the right at 1.5 m/s.

Graphics

Figure 3.23: Two steel spheres that are about to collide.[1]

Figure 3.24: Sketch of balls before the collision[1]

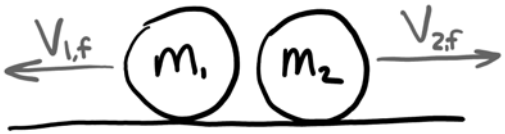


Figure 3.25: Sketch of balls after the collision[1]

Numbers

Knowns	Unknowns
$m_1 = m_2 = 0.7$ kg	$\overrightarrow{v_{1,f}}$
$\overrightarrow{v_{1,i}} = +1.5\,\mathrm{m/s}\,\hat{x}$	$\overrightarrow{v_{2,f}}$
$\overrightarrow{v_{2,i}} = -1.5\,\mathrm{m/s}\,\hat{x}$	
$E_{k,i} = E_{k,f}$	

As in Section 3.4,

$$\overrightarrow{p_{1,i}} + \overrightarrow{p_{2,i}} = \overrightarrow{p_{1,f}} + \overrightarrow{p_{2,f}}$$

and

$$\frac{1}{2}m_1 \cdot v_{1,i}^2 + \frac{1}{2}m_2 \cdot v_{2,i}^2 = \frac{1}{2}m_1 \cdot v_{1,f}^2 + \frac{1}{2}m_2 \cdot v_{2,f}^2$$

But in this case $\overrightarrow{p_{1,i}} + \overrightarrow{p_{2,i}} = 0$, so...

$$\overrightarrow{p_{2,f}} = -\overrightarrow{p_{1,f}}$$

Since the masses are equal, $\overrightarrow{v_{2,f}} = -\overrightarrow{v_{1,f}}$. Since the speeds $v_{1,i} = v_{2,i}$ and $v_{1,f} = v_{2,f}$, the kinetic energy equation simplifies to:

$$v_{1,i} = v_{2,i} = v_{1,f} = v_{2,f}$$

So we are left only with determining the final directions of motion.

Let's try looking at this question from a different reference frame, that of Ball 1. In its own reference frame, Ball 1 is never moving. But initially Ball 2 will appear to be moving at 3 m/s to the left because the two balls are moving toward each other at 1.5 m/s in the earth's reference frame. After the collision, again Ball 1 is not moving in its own reference frame, but Ball 2 now appears to be moving to the right at 3 m/s because the two balls are moving away from each other at 1.5 m/s in the earth's reference frame.

This is exactly the same as the physical scenario in Section 3.4. In the reference frame of Ball 1, initially Ball 2 would be moving to the left, toward Ball 1, at 3 m/s. And after the collision Ball 2 is moving to the right at 3 m/s in the reference frame of Ball 1, which is now the same as the reference frame of the earth since Ball 1 is no longer moving in the earth's reference frame.

So these two sections have presented exactly the same physical scenario, seen from two different reference frames. But the scenario in this section was easier to understand conceptually and mathematically. It would be nice if we could always find a reference frame for collision problems that would make it easier.

And in fact, there is such a reference frame for any collision. It is called the "Center of Mass" reference frame.

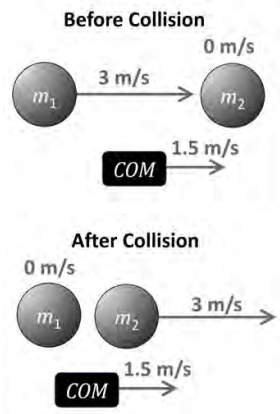


Figure 3.26: The collision from Section 3.4 in earth's reference frame, with an added "COM"[1]

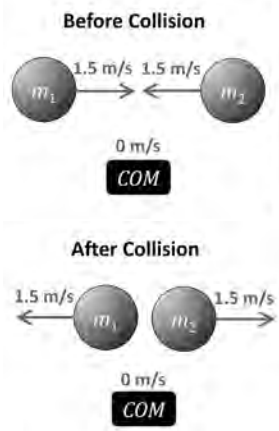


Figure 3.27: The same collision in the COM reference frame[1]

There is a reference frame that changes the collision from Section 3.4 into the collision from this section, which makes it easier to solve.

The diagrams to the left show the collision from Section 3.4 in a "COM" (Center of Mass) reference frame that is moving to the right at 1.5 m/s. Compare the diagrams in Figure 3.27 for the collision from Section 3.4 to the sketch in Figure 3.25 for the physical scenario in this section. They are identical!

3.6 Center of Mass

Words

For some situations, it is easy to find the center of mass–it's just in the center!

This is the case for something like a bocci ball, which is a uniform, solid sphere. It is also the case for something like a tennis ball, which is a uniform, *hollow* sphere. Even though there is nothing actually at the center of the ball, that point is still the center of mass.

And even for something with a more unusual shape, like a dumbbell, as long as the object is symmetrical, the center of mass will be at its center.

For objects with asymmetrical shapes or for a system of different objects, the center of mass is somewhere in the middle, shifted toward the side with more mass. In the photo of the tennis ball, bocce ball, and dumbbell, the tennis ball has a much smaller mass than the other two, so the center of mass of this group of three objects is somewhere inbetween the dumbbell and the bocci ball. Where exactly that center of mass is located depends upon the relative masses of the objects.

On the right is a stack of books with varying widths and masses. The center of mass of the whole stack of books would be somewhere near the middle of the stack, where the ruler shows approximately 13.6 cm.

Graphics

Figure 3.28: A tennis ball (top left), bocci ball (top right), and dumbbell[1]

Figure 3.29: A stack of books of varying widths and masses[1]

Numbers

The center of mass of an object or system of objects is a position, so it is a vector quantity.

$$\boxed{\overrightarrow{x_{com}} = \frac{m_1 \cdot \overrightarrow{x_1} + m_2 \cdot \overrightarrow{x_2} + \cdots}{m_1 + m_2 + \cdots}} \tag{3.5}$$

...where $m_1$and $\overrightarrow{x_1}$ refer to the mass and position of object 1, m_2 and $\overrightarrow{x_2}$ are for object 2, and every object in the system is included in the sums.

The easiest way to use this equation is to break the system up into pieces that have an obvious center of mass, and use those as objects 1, 2, *etc.*

On the left is a stack of four books. The center of mass of each individual book is at the center of the book. If we know the mass and position of each book we can find the center of mass of the stack:

Book	Mass	Position
1	0.202 kg	10.7 cm
2	0.242 kg	12.2 cm
3	0.302 kg	13.9 cm
4	0.324 kg	16.2 cm

Combining this data using Equation 3.5 gives:

$$x_{com} = \frac{14.6\,\mathrm{kg} \cdot \mathrm{cm}}{1.07\,\mathrm{kg}} = 13.6\,\mathrm{cm}$$

Center of mass is a useful concept in several different situations, for example when trying to make a stable structure or when an object is rotating. It is also useful when looking at collisions.

When two objects collide, they generally exert very large forces on each other, so any external forces can be neglected during the collision. Since the external forces are negligible, the momentum of the system of objects involved in the collision does not change. And since the mass isn't changing, this means that the collective velocity of the system of objects does not change in a collision.

So if we know the initial velocity of the center of mass of the system of objects, that will be the same as the final velocity of the center of mass of the system of objects.

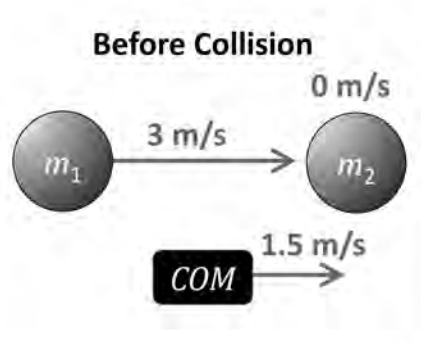


Figure 3.30: Before the collision from Section 3.4 in earth's reference frame, with the velocity of the Center of Mass "COM" reference frame included[1]

The velocity of the center of mass of an object or system of objects is calculated in the same way that the position of the center of mass is calculated:

$$\boxed{\overrightarrow{v_{com}} = \frac{m_1 \cdot \overrightarrow{v_1} + m_2 \cdot \overrightarrow{v_2} + \cdots}{m_1 + m_2 + \cdots}} \tag{3.6}$$

...where $\overrightarrow{v_1}$ is the the velocity of object 1, $\overrightarrow{v_2}$ is the velocity of object 2, and every object in the system is included in the sums.

It is now possible to see why the velocity of the "COM" reference frame in Section 3.4 was $+1.5\,\mathrm{m/s}\,\hat{x}$ when it was revisited in Section 3.5.

$$\overrightarrow{v_{com}} = \frac{m_1 \cdot \overrightarrow{v_{1,i}} + m_2 \cdot \overrightarrow{v_{2,i}}}{m_1 + m_2}$$

$$\overrightarrow{v_{com}} = \frac{(0.7\,\mathrm{kg}) \cdot (+3\,\mathrm{m/s}\,\hat{x}) + (0.7\,\mathrm{kg}) \cdot 0}{0.7\,\mathrm{kg} + 0.7\,\mathrm{kg}}$$

$$\overrightarrow{v_{com}} = +1.5\,\mathrm{m/s}\,\hat{x}$$

3.7 Unequal Masses

Words

In some countries, rubber bullets can be used by government forces in riot situations, or even by civilians in self-defense. For this physical scenario, consider a 30 gram rubber bullet initially moving at 100 m/s and then colliding elastically with a motionless 60 kg person and bouncing back in the direction from which it came. The total time that the bullet is in contact with the person is 0.001 s.

What can you say about: Final velocities? Kinetic energy? Force? Momentum?

Something about kinetic energy is given to us in the statement of the question. The collision is elastic, which means that kinetic energy is conserved. The total kinetic energy before the collision has to be equal to the total kinetic energy after the collision. Since the person is not moving before the collision, the person's initial kinetic energy is zero. If the person is moving after the collision, that means that there would be less kinetic energy for the bullet, so the bullet would have to be moving at a slower speed after the collision than it had before the collision.

We can also say something about momentum. It is always conserved in any collision, so if the bullet's momentum changes then the person's momentum also has to change. In this case we know that the bullet's momentum changes because it is initially moving to the right (as drawn in the sketch) and

Graphics

Figure 3.31: A rubber bullet[12]

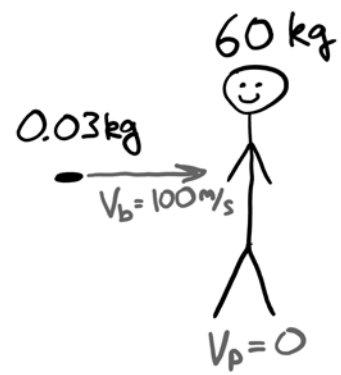


Figure 3.32: Sketch of the rubber bullet and person before the collision, in the earth's reference frame[1]

Numbers

Knowns	Unknowns
m_b = 0.03 kg	$\overrightarrow{v_{b,f}}$
m_p = 60 kg	$\overrightarrow{v_{p,f}}$
$\overrightarrow{v_{b,i}} = +100\,\mathrm{m/s}\,\hat{x}$	E_k's
$\overrightarrow{v_{p,i}} = 0$	$\vec{F}$'s
$E_{k,i} = E_{k,f}$	$\vec{p}$'s
t=0.001 s	

We also know that $\overrightarrow{v_{b,f}}$ is in the $-\hat{x}$ direction.

Since this is a collision, we will first change to the center of mass frame. From Equation 3.6...

$$\overrightarrow{v_{com}} = \frac{(0.03\,\mathrm{kg})\cdot(+100\,\mathrm{m/s}\,\hat{x}) + (60\,\mathrm{kg})\cdot 0}{0.03\,\mathrm{kg} + 60\,\mathrm{kg}}$$

$$\overrightarrow{v_{com}} = +0.05\,\mathrm{m/s}\,\hat{x}$$

According to Equation 3.1, to change to the center of mass reference frame, we have to subtract $\overrightarrow{v_{com}}$ from all velocities in the earth's reference frame that we were using before:

$$\overrightarrow{v_{b,i}} = +100\,\mathrm{m/s}\,\hat{x} - 0.05\,\mathrm{m/s}\,\hat{x} = +99.95\,\mathrm{m/s}\,\hat{x}$$

$$\overrightarrow{v_{p,i}} = 0 - 0.05\,\mathrm{m/s}\,\hat{x} = -0.05\,\mathrm{m/s}\,\hat{x}$$

Note that in the center of mass frame, $m_b \cdot \overrightarrow{v_{b,i}} = -m_p \cdot \overrightarrow{v_{p,i}}$ the total momentum is zero! That will

ends up moving to the left. So it had a change in momentum in the "left" direction. That means the person must have a change in momentum in the "right" direction.

So the final velocity of the bullet is to the left and the final velocity of the person is to the right.

And now that we have confirmed that the person has a final momentum to the right, we know that the person must have kinetic energy after the collision, so the final speed of the bullet is slower than its initial speed.

As for the force, we know that there was a force between the bullet and the person, because they both changed momentum in the collision, and force is what causes a change in momentum over time. In this case, the time of the collision is very small, so the force during the collision must be very large. The bullet created a force to the right on the person, and since forces come in equal-but-opposite pairs, we know that the person also created the same amount of force on the bullet, but to the left.

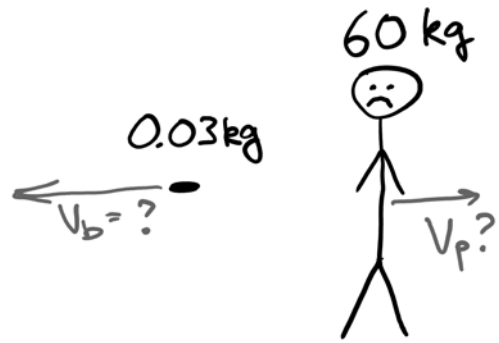


Figure 3.33: Sketch of rubber bullet and person after the collision, in the earth's reference frame[1]

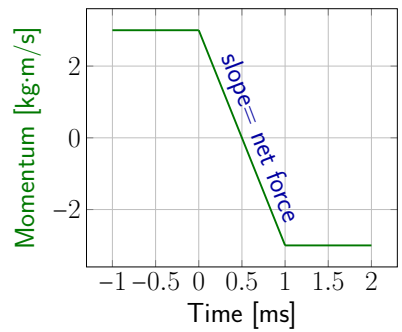


Figure 3.34: Momentum of the bullet before, during, and after the collision. Note that the time axis is in milliseconds.[1]

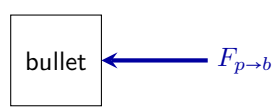


Figure 3.35: FBD of the bullet during the collision.[1]

always be true, for both initial and final momenta. This greatly simplifies the math.

We also know that kinetic energy is conserved, and there is only one way to conserve kinetic energy while keeping the total momentum zero before and after the collision: both objects simply reverse direction in the center of mass frame: $\overrightarrow{v_{b,f}} = -\overrightarrow{v_{b,i}}$ and $\overrightarrow{v_{p,f}} = -\overrightarrow{v_{p,i}}$.

We then change back to the earth reference frame by adding the center of mass velocity to each of these other velocities (which is equivalent to using Equations 3.1 & 3.2).

$$\overrightarrow{v_{b,f}} = -99.95\,\mathrm{m/s}\,\hat{x} + 0.05\,\mathrm{m/s}\,\hat{x} = -99.9\,\mathrm{m/s}\,\hat{x}$$

$$\overrightarrow{v_{p,f}} = +0.05\,\mathrm{m/s}\,\hat{x} + 0.05\,\mathrm{m/s}\,\hat{x} = +0.1\,\mathrm{m/s}\,\hat{x}$$

The change in momentum of the person is...

$$\overrightarrow{p_{p,f}} - \overrightarrow{p_{p,i}} = 60\,\mathrm{kg}\cdot 0.1\,\mathrm{m/s}\,\hat{x} - 0 = +6\,\mathrm{kg}\cdot\mathrm{m/s}\,\hat{x}$$

Because momentum conserved, the change in momentum of the bullet is $-6\,\mathrm{kg}\cdot\,\mathrm{m/s}\,\hat{x}$.

Using Equation 1.7, the average force on the person during the collision is

$$\overrightarrow{F_{b\to p}} = \frac{\overrightarrow{\Delta p}}{\Delta t} = \frac{+6\,\mathrm{kg}\cdot\mathrm{m/s}}{0.001\,\mathrm{s}} = +6000\,\mathrm{N}\,\hat{x}$$

And using Equation 3.3, $\overrightarrow{F_{p\to b}} = -6000\,\mathrm{N}\,\hat{x}$.

3.8 Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

General

- Physical situations can be analyzed from different reference frames.
- The laws of physics are true in any inertial (not accelerating) reference frame.
- It is often useful to look at collisions in the center of mass reference frame.
- It is often useful to consider the physics of a system of objects, not just the physics related to a single object.

Forces

- For every force, there is an equal and opposite force on another object.
- A free body diagram of an object includes only forces acting *on* the object, never forces *caused by* the object.

Motion

- Objects, even stationary objects, change velocity when we change to a different reference frame.
- The velocity of the center of mass of a system of objects does not change during a collision between the objects in the system.
- In the center of mass reference frame, an elastic collision between two objects causes the objects to reverse direction but keep the same speeds

Momentum

- Change in momentum is often called "impulse."
- The total momentum of a system in its center of mass reference frame is always zero.

Energy

- Elastic collisions are collisions in which kinetic energy is conserved.

Mathematical Models

equation		restrictions on the validity of the equation
$\overrightarrow{v_{2\leftarrow 3}} = \overrightarrow{v_{2\leftarrow 1}} - \overrightarrow{v_{3\leftarrow 1}}$	(3.1)	-none-
$\overrightarrow{v_{1\leftarrow 2}} = -\overrightarrow{v_{2\leftarrow 1}}$	(3.2)	-none-
$\overrightarrow{F_{1\rightarrow 2}} = -\overrightarrow{F_{2\rightarrow 1}}$	(3.3)	-none-
$\overrightarrow{p_{tot,i}} = \overrightarrow{p_{tot,f}}$	(3.4)	only valid when there is no external net force
$\overrightarrow{x_{com}} = \frac{m_1 \cdot \overrightarrow{x_1} + m_2 \cdot \overrightarrow{x_2} + \cdots}{m_1 + m_2 + \cdots}$	(3.5)	-none-
$\overrightarrow{v_{com}} = \frac{m_1 \cdot \overrightarrow{v_1} + m_2 \cdot \overrightarrow{v_2} + \cdots}{m_1 + m_2 + \cdots}$	(3.6)	-none-

3.9 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

Level 1 - Remember

3.1 **[W]** What does"impulse" mean in physics?

3.2 **[W]** What is the name of the reference frame in which the total momentum of a system of objects is always zero?

3.3 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

Level 2 - Understand

3.4 **[W]** Which of the following describe inertial reference frames where the normal laws of physics are valid?

(a) A typical physics classroom

(b) A physics classroom during a violent earthquake

(c) The back of a truck that is traveling at constant speed on a long, straight road

(d) The back of a truck that is traveling at constant speed on a long, straight, uphill road

(e) The back of a truck that is accelerating away from a stoplight

(f) A spaceship floating far off in space, away from any sources of gravity

(g) A spaceship firing its rockets far off in space, away from any sources of gravity

Level 3 - Apply

3.5 **[N]** There are six "Unknowns" listed at the beginning of Section 3.1, but numerical values are only found for two of them. Find the rest of the numerical values.

3.6 **[G]** Section 3.1 includes illustrations in the reference frames of the earth, the black car, and the truck. Draw a similar illustration in the white car's reference frame.

3.7 **[W & G]** Draw an energy bar graph for the soccer ball in Section 3.3 for the time just before and just after the kick. Is energy conserved in this situation? If not, where did the extra energy come from, or where did it go?

3.8 **[G]** Draw momentum bars representing the collision described in Section 3.7, in the earth's reference frame. Use them to show that momentum was conserved in the collision.

3.9 **[G]** Draw momentum bars representing the collision described in Section 3.7, in the center of mass reference frame. Use them to show that the total momentum of the system was zero before and after the collision.

3.10 **[G]** Draw an energy bar graph representing the collision described in Section 3.7, in the center of mass reference frame. Use it to show that mechanical energy was conserved during the collision.

Level 4 - Analyze

3.11 **[G]** Figure 3.35 shows a free body diagram for the bullet during the collision. Draw a free body diagram for the person during the collision, including only the forces in the horizontal direction.

Level 5 - Evaluate

3.12 **[W]** In the situation given in Section 3.2 it is stated that the ant and rock start motionless. Now imagine what would happen if the forces applied stayed the same but the ant and rock started with an initial velocity in one direction or the other.

(a) Would the initial velocity affect the acceleration? Explain why or why not.

(b) Would the initial velocity affect the force applied by the ant to the rock or the rock to the ant? Explain why or why not.

3.13 **[W]** Two people are moving toward each other across an open field. One is running and the other is walking. Rank the following in terms of the amount of time needed for the two people to come together:

(a) Considering this situation in the earth's reference frame

(b) Considering this situation in the reference frame of the person who is running

(c) Considering this situation in the reference frame of the person who is walking

Ignore the effects of "special relativity," if you know what that is!

Level 6 - Create

3.14 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Add the main ideas from this chapter to the concept map that you began for the question at the end of Chapter 1.

3.15 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

3.16 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

Chapter 4

Thermal Energy and Friction

In all of the collisions that we looked at in the last chapter, the objects bounced off of each other. This is a common characteristic of elastic collisions. But most collisions are not elastic. In a typical car crash, the cars don't bounce off of each other and they become deformed. These are common characteristics of an inelastic collision. If the objects stick together after the collision, it is called a "completely inelastic" collision.

An inelastic collision is one in which kinetic energy is not conserved. Some of the kinetic energy is transformed into thermal energy in the collision.

Kinetic energy can also be transformed into thermal energy by frictional force when two objects slide against each other.

Figure 4.1: In Chapter 3 we considered several different collisions, but all of them were elastic collisions. A car crash is an inelastic collision.[7]

4.1 Railway Couplers

Words

When railway cars are assembled into a train, they are connected using railway couplers. This is an example of an inelastic collision, because the two train cars stick together after the collision that couples them.

Consider a 90,000 kg boxcar sitting motionless on a track in a railyard. A 500,000 kg engine backs into it at 0.2 m/s. The collision between the cars takes 0.08 seconds, after which they are locked together. We should be able to describe their velocity after the collision, the change in kinetic energy during the collision, the amount of force they applied to each other during the collision, and the acceleration of the engine and the boxcar during the collision.

In any collision, you can't go wrong by starting to think about it in terms of momentum. Initially the boxcar was not moving, so it had no momentum. The engine did have momentum. After the collision, the two train cars were stuck together, so they have the same velocity. Since momentum is conserved, there must be momentum after the collision. So after the collision both of the train cars have momentum in the same direction as the initial momentum. The engine transfers just enough of its momentum to the boxcar to bring them both to the same final velocity.

Graphics

Figure 4.2: Two train cars connected by railway couplers.[13]

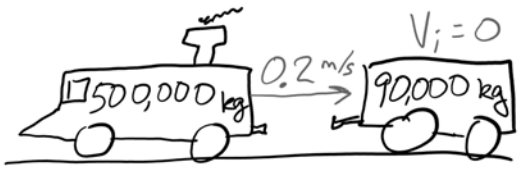


Figure 4.3: Sketch of the train cars before coupling.[1]

Numbers

Knowns	Unknowns
$m_{boxcar} = 90{,}000$ kg	$\overrightarrow{F_{boxcar \to engine}}$
$m_{engine} = 500{,}000$ kg	$\overrightarrow{F_{engine \to boxcar}}$
$\overrightarrow{v_{i,boxcar}} = 0$	$\overrightarrow{v_f}$
$\overrightarrow{v_{i,engine}} = 0.2\,\mathrm{m/s}\,\hat{x}$	ΔE_k
$t_{collision} = 0.08\,\mathrm{s}$	$\overrightarrow{a_{boxcar}}$
	$\overrightarrow{a_{engine}}$

Since a direction is not given, it is easiest to set the problem up so that the initial velocity is in the positive direction. There is only one $\overrightarrow{v_f}$ because the train cars are stuck together after the collision.

We can start by considering momentum in the center of mass reference frame, where the total momentum is always zero. The velocity of the center of mass reference frame is given by Equation 3.6:

$$\overrightarrow{v_{com}} = \frac{m_{boxcar} \cdot \overrightarrow{v_{i,boxcar}} + m_{engine} \cdot \overrightarrow{v_{i,engine}}}{m_{boxcar} + m_{engine}}$$

$$\overrightarrow{v_{com}} = \frac{0 + 500{,}000\,\mathrm{kg} \cdot (0.2\,\mathrm{m/s})}{(90{,}000 + 500{,}000)\,\mathrm{kg}}\,\hat{x} = 0.169\,\mathrm{m/s}\,\hat{x}$$

Since the total momentum in the center of mass frame is zero and the two objects are stuck together, they must both have zero velocity in the center of mass rest frame. So $\overrightarrow{v_f} = \overrightarrow{v_{com}}$

Kinetic energy depends upon the reference frame. If you stand at the side of a road and watch a bus go past, in your reference frame the bus has kinetic energy and you do not. But in the bus's reference frame you have kinetic energy and it does not. But other forms of energy do not depend upon the reference frame. So we are free to consider transformations involving thermal and potential energy in any reference frame that makes it easy to understand what is happening.

For this collision, we can use the center of mass reference frame. In this reference frame, both train cars are initially moving, but after they smash into each other they both have zero velocity. So initially they had kinetic energy, but then they lost it. Energy is always conserved, so where did it go? It was converted to thermal energy as the couplings moved, rubbed together, and latched into place. Any inelastic collision results in the conversion of some kinetic energy into thermal energy.

Because of conservation of momentum, the less massive boxcar has a larger change in velocity than the more massive engine. And since this change in velocity occurs in the same amount of time, we also know that the acceleration of the boxcar is larger than the acceleration of the engine during the collision.

The force that the boxcar exerts on the engine would have the same magnitude as the force that the engine exerts on the boxcar, but these two forces would be in opposite directions.

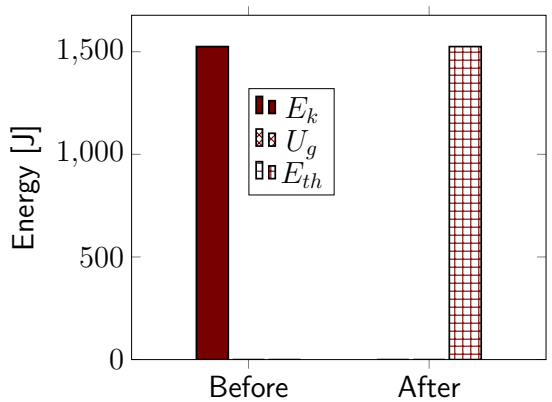


Figure 4.4: Energy bar graph for the collision in the center of mass reference frame. Initial thermal energy is taken to be zero. Gravitational energy is zero because both train cars are on the ground before and after the collision.[1]

Note that the total heights of the columns on the left is equal to the total heights of the columns on the right.

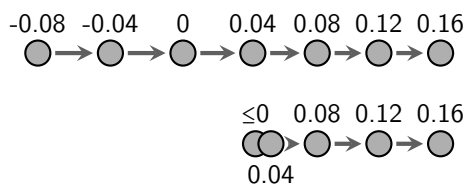


Figure 4.5: Motion map of the engine (top) and the boxcar (bottom). The labels are in seconds, and the collision is from 0 to 0.08 s.[1]

Note that the change in the length of the arrows is much larger for the boxcar than for the engine.

Since we started in the center of mass reference frame, let's consider energy in this frame as well. We will need to transform the initial velocities into the center of mass reference frame using Equation 3.1.

$$\overrightarrow{v_{i,boxcar\leftarrow com}} = \overrightarrow{v_{i,boxcar\leftarrow earth}} - \overrightarrow{v_{com\leftarrow earth}}$$

$$\overrightarrow{v_{i,boxcar\leftarrow com}} = 0 - 0.169\,\mathrm{m/s}\,\hat{x}$$

. . . and similarly. . .

$$\overrightarrow{v_{i,engine\leftarrow com}} = (0.2 - 0.169)\,\mathrm{m/s}\,\hat{x} = 0.031\,\mathrm{m/s}\,\hat{x}$$

Now we can find the change in kinetic energy in the center of mass reference frame.

$$\Delta E_{k,com} = E_{k,f,com} - E_{k,i,com}$$

$$\Delta E_{k,com} = 0 - (1285\,\mathrm{J} + 480\,\mathrm{J})$$

$$\Delta E_{k,com} = -1525\,\mathrm{J}$$

Conservation of energy tells us that this energy doesn't just disappear. It has to go somewhere. In this case, it changes to thermal energy.

$$\Delta E_{th} = -\Delta E_k = 1525\,\mathrm{J}$$

If we assume that the force is constant during the collision, then the acceleration would be. . .

$$\vec{a} = \frac{\overrightarrow{\Delta v}}{\Delta t}$$

Whether the calculation is done in the center of mass reference frame or the earth's reference frame, the result is the same: $\overrightarrow{a_{boxcar}} = +2.11\,\mathrm{m/s^2}\,\hat{x}$ and $\overrightarrow{a_{engine}} = -0.388\,\mathrm{m/s^2}\,\hat{x}$

4.2 Curling

Words

The conversion of kinetic energy to thermal energy is not limited to collisions. It happens any time that two surfaces rub together. Take, for example, the olympic sport of curling. One person pushes a heavy "stone" on a sheet of ice and then releases it, and the rest of the curling team sweeps the area in front of the stone as it slides, trying to control the friction to get the stone to stop in a target area.

When it leaves the curler's hand, the stone has momentum and kinetic energy, but the force of friction opposes the momentum, doing negative work on the stone and slowing it until it eventually comes to rest. After the stone has stopped moving, all of the kinetic energy is gone, changed into thermal energy.

Let's consider an 18 kg curling stone that was initially released with a momentum of $60\ \mathrm{kg \cdot m/s}$, and stops after traveling 45 m across the ice. We should be able to describe the velocity of the stone, the force of friction on the stone, the energy conversions that take place as it slides across the ice, and the amount of time that the stone is in motion after it has been released.

Graphics

Figure 4.6: Top: A curler preparing to push a stone. Bottom: Stones in the target area.[14]

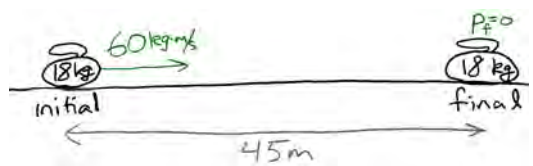


Figure 4.7: A sketch of the curling stone before and after sliding across the ice.[1]

Numbers

Many of our equations are only valid if the force is constant. Sometimes, like for this situation, the assumption is valid; other times, like in a collision, it is not. Even when the force is not constant, versions of these equations are useful for finding average values. Most notably:

$$\boxed{\overrightarrow{F_{net,avg}} = \frac{\overrightarrow{\Delta p}}{\Delta t}} \tag{4.1}$$

and

$$\boxed{\overrightarrow{a_{avg}} = \frac{\overrightarrow{\Delta v}}{\Delta t}} \tag{4.2}$$

. . . where $\overrightarrow{F_{net,avg}}$ and $\overrightarrow{a_{avg}}$ are the average values over time. One of our other equations is true for the average force over a distance:

$$\boxed{W_{net} = \overrightarrow{F_{net,avg}} \cdot \overrightarrow{\Delta x} = F_{net,avg} \cdot \Delta x \cdot \cos(\theta)} \tag{4.3}$$

Knowns	Unknowns
$m = 18\ \mathrm{kg}$	$\vec{v}$
$\vec{p_i} = 60\ \mathrm{kg \cdot m/s}\ \hat{x}$	$\vec{F_f}$
$\vec{p_f} = 0$	Energy transformations
$\overrightarrow{\Delta x} = 45\ \mathrm{m}\ \hat{x}$	Δt

Since specific directions aren't given, we can choose a direction that is easy to draw or easy to think about. Since "to the right" is conventionally taken to be the positive direction, for simplicity Figure 4.7 is set up so that the initial momentum is to the right. Given that, we know that the initial velocity is also to the right, or positive. And since the stone stops at the end, its final velocity is zero.

The force of friction always opposes the relative motion of two objects, so the force on the stone is to the left, or negative.

The amount of time that the stone spends sliding across the ice depends on the force of friction and the initial momentum. The larger the force, the less time it will take to stop; and the larger the initial momentum, the more time it will take to stop.

Note that the question doesn't ask about just the initial or final velocity. It says we should be able to describe the velocity. So what about during the time during which it is sliding? The velocity will always be to the right, but dropping continuously while the stone slides across the ice, as is shown in Figure 4.9. We could speak of an average velocity of the stone during this time, which would be to the right, and if the force is constant, the average velocity would be half of the initial velocity.

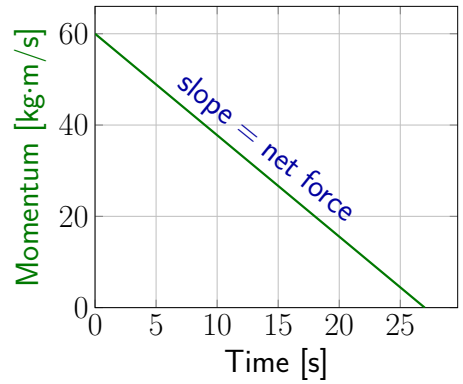


Figure 4.8: Momentum is decreasing over time, indicating a negative net force.[1]

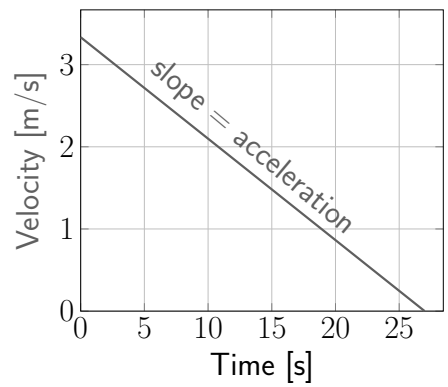


Figure 4.9: Velocity is decreasing over time, indicating a negative acceleration.[1]

It is relatively easy to find the initial velocity, and from it the initial kinetic energy:

$$\vec{p_i} = m \cdot \vec{v_i}$$

$$\vec{v_i} = \frac{\vec{p_i}}{m} = \frac{60\,\mathrm{kg \cdot m/s}}{18\,\mathrm{kg}} = 3.33\,\mathrm{m/s}\,\hat{x}$$

$$E_{k,i} = \frac{1}{2} m \cdot v_i^2 = \frac{1}{2}(18\,\mathrm{kg})(3.33\,\mathrm{m/s})^2 = 100\,\mathrm{J}$$

All of the initial kinetic energy is converted to thermal energy, so $\Delta E_k = -100\,\mathrm{J}$ and $\Delta E_{th} = 100\,\mathrm{J}$. Now we can use Equations 2.3 and 4.3 to find the average force of friction.

$$\Delta E_k = W_{net} = F_f \cdot \Delta x \cdot \cos(180°)$$

$$F_f = \frac{\Delta E_k}{-\Delta x} = \frac{-100\,\mathrm{J}}{-45\,\mathrm{m}} = 2.22\,\mathrm{N}$$

$180°$ was used as the angle because we assume that the force is opposite the direction of the displacement. Since we got a positive value for the force, we know that our assumption was correct. If the number turned out to be negative then we would know that our assumption was not correct. The velocity varies continuously over the time the stone is moving. We have already found the initial and final velocity. Another useful piece of information is the average velocity, which is given by...

$$\boxed{\vec{v_{avg}} = \frac{\vec{\Delta x}}{\Delta t}} \tag{4.4}$$

In situations where the net force (and therefore the acceleration) is constant, average velocity is also given by...

$$\boxed{\vec{v_{avg}} = \frac{\vec{v_i} + \vec{v_f}}{2}} \tag{4.5}$$

4.3 Car crash, initial impact

Words

Let's consider the car crash in Figure 3.1. We can imagine what happened. Perhaps the 2500 kg truck was stopped, and the 1500 kg car hit it from behind at 25 m/s. The truck would have been shoved forward by the car, and the two stuck together. The time in which the car and truck were smashing into each other would have been very short, let's say 30 milliseconds. The drivers of both vehicles probably were applying the brakes, and the vehicles skidded together to a stop after traveling together for 4 m. For now let's focus on what happened during the early part of the crash when they were smashing into each other, before the vehicles skidded together.

We should be able to find the average force that each vehicle applied to the other during the early part of the crash, the velocity of the car-truck wreckage just after they had finished smashing into each other but before they started skidding together, and the kinetic energy of the vehicles just before and just after they smashed into each other.

Before the car hit the truck, the truck was not moving but the car was; a short time after contact the car and the truck were connected together and moved as if they were a single object with the combined mass of both vehicles. Because of this, there is only one velocity after the collision, and it is the velocity of both the car and the truck.

Graphics

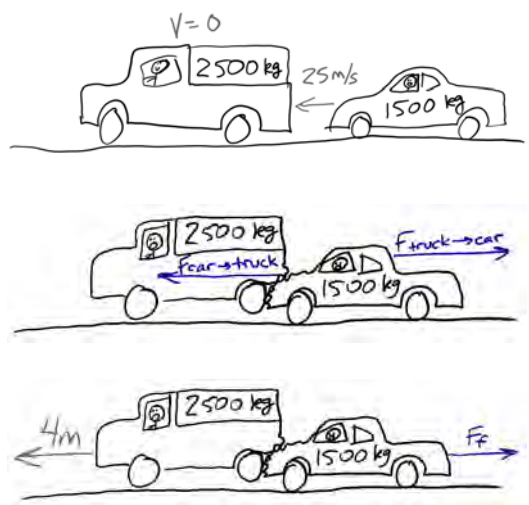


Figure 4.10: Sketches of what could have happened in the crash. In this section we will consider only the top two sketches.[1]

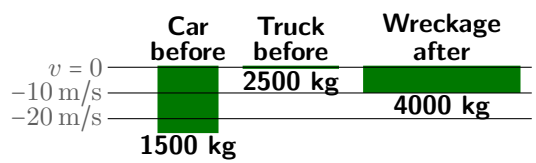


Figure 4.11: Momentum bars just before and just after the collision. Area of the third bar has to be equal to sum of the areas of the first two bars.[1]

Numbers

Knowns	Unknowns
$m_{car} = 1500$ kg	$\overrightarrow{F_{car\to truck}}$
$m_{truck} = 2500$ kg	$\overrightarrow{F_{truck\to car}}$
$\overrightarrow{v_{i,car}} = -25\,\mathrm{m/s}\,\hat{x}$	$\overrightarrow{v_f}$
$\overrightarrow{v_{i,truck}} = 0$	$E_{k,i}$
$t_{collision} = 0.03\,\mathrm{s}$	$E_{k,f}$

The place to start in any collision is momentum, and using the center of mass reference frame usually simplifies the math. The velocity of the center of mass reference frame is given by Equation 3.6:

$$\overrightarrow{v_{com}} = \frac{m_{car}\cdot\overrightarrow{v_{i,car}} + m_{truck}\cdot\overrightarrow{v_{i,truck}}}{m_{car}+m_{truck}}$$

$$\overrightarrow{v_{com}} = \frac{1500\,\mathrm{kg}\cdot(-25\,\mathrm{m/s})+0}{(1500+2500)\,\mathrm{kg}}\,\hat{x} = -9.375\,\mathrm{m/s}\,\hat{x}$$

Since the total momentum in the center of mass frame is zero and the two objects are stuck together, they must both have zero velocity in the center of mass rest frame. So...

$$\overrightarrow{v_f} = \overrightarrow{v_{com}} = -9.375\,\mathrm{m/s}\,\hat{x}$$

This is the velocity of the two vehicles after they have smashed together and before they start sliding together down the road.

In the crash, the car transferred some of its momentum to the truck, applying a force to the truck. So the truck applied an equal-but-opposite force to the car. Since momentum is conserved in an isolated system, the change in the truck's momentum is the same as the change in the car's momentum, but in the opposite direction.

This tells us something about the acceleration. They have the same change in momentum, but different masses. The more massive truck has a smaller change in velocity than the car, so the acceleration of the truck is less than the acceleration of the car.

For this collision, we can use the center of mass reference frame. In this reference frame, both vehicles are initially moving, but after they smash into each other they both have zero velocity. So initially they had kinetic energy, but then they lost it. Energy is always conserved, so where did it go? It was converted to thermal energy as the car and truck were deforming. Remember, any inelastic collision results in the conversion of some kinetic energy into thermal energy.

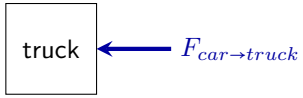


Figure 4.12: Free body diagram (FBD) of the truck during the collision, horizontal direction only. The force of friction is small compared to the force caused by the collision, so it is being neglected.[1]

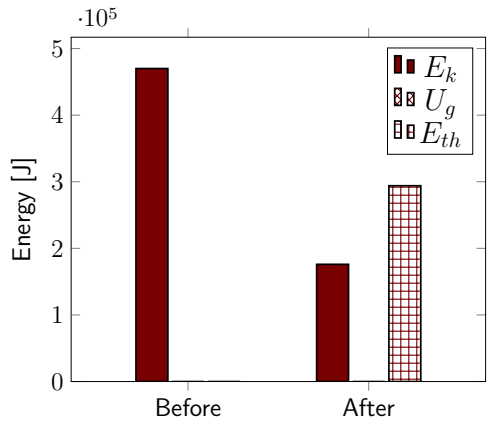


Figure 4.13: Energy bar graph for the collision. Initial thermal energy is taken to be zero. Gravitational energy is zero because both vehicles are on the ground before and after the collision.[1]

Note that the total heights of the columns on the left is equal to the total heights of the columns on the right.

We can use Equation 1.7 to find the force on the truck during the collision:

$$\overrightarrow{F_{car\to truck}} = \frac{\overrightarrow{\Delta p_{truck}}}{\Delta t} = \frac{m_{truck}\cdot(\vec{v_f} - \overrightarrow{v_{i,truck}})}{t_{collision}}$$

$$\overrightarrow{F_{car\to truck}} = \frac{2500\,\mathrm{kg}\cdot(-9.375-0)\,\mathrm{m/s}\,\hat{x}}{0.03\,\mathrm{s}}$$

$$\overrightarrow{F_{car\to truck}} = -7.8\times10^5\,\mathrm{N}\,\hat{x}$$

From Equation 3.3, we know that

$$\overrightarrow{F_{truck\to car}} = -\overrightarrow{F_{car\to truck}} = +7.8\times10^5\,\mathrm{N}\,\hat{x}$$

We will consider kinetic energy in the earth's reference frame, since we already know all of the velocities in this frame.

$$E_{k,i} = \frac{1}{2}m_{car}\cdot v_{i,car}^2 + \frac{1}{2}m_{truck}\cdot v_{i,truck}^2$$

$$E_{k,i} = \frac{1}{2}(1500\,\mathrm{kg})\cdot(-25\,\mathrm{m/s})^2 + 0 = 4.7\times10^5\,\mathrm{J}$$

$$E_{k,f} = \frac{1}{2}m_{total}\cdot v_f^2$$

$$E_{k,f} = \frac{1}{2}(4000\,\mathrm{kg})\cdot(-9.375\,\mathrm{m/s})^2 = 1.76\times10^5\,\mathrm{J}$$

Energy is conserved whenever no work is being done by an external force. Gravity and springs do not play a role in this situation, so lost kinetic energy changes into thermal energy.

$$E_{k,i} + E_{th,i} = E_{k,f} + E_{th,f}$$

$$E_{k,i} - Ek,f = E_{th,f} - E_{th,i}$$

$$\Delta E_{th} = (4.7 - 1.76)\times10^5\,\mathrm{J} = 2.94\times10^5\,\mathrm{J}$$

4.4 Car crash, sliding

Words

Now we consider the same collision as in the last section, but focus on what happened when the vehicles skidded together across the ground. During that time, the vehicles were braking and they skidded to a stop after traveling for 4 m.

We should be able to describe the total braking force that was required, which is a frictional force that we will assume to be constant, acting between the tires and the road; the amount of time that they were skidding; and the energy transformations.

It is usually best to start any analysis with whatever part seems easiest. In this case, it is easiest to think about the energy involved. We know the vehicles are moving when they start to skid down the road, so they have kinetic energy. The road is flat, so we don't have to worry about any changes in gravitational potential energy. And there are not any springs that are trying to stop them or make them go faster, so we also don't need to worry about spring potential energy. So we started with only kinetic energy. And after the vehicles stop moving, they no longer have kinetic energy. So where did all of the energy go? The only option left is thermal since we have ruled out potential energies. All of the kinetic energy is transformed into thermal energy.

Graphics

Figure 4.14: Sketch of the car and truck skidding together after the initial impact.[1]

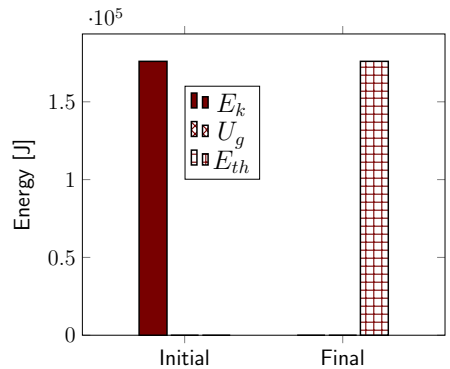


Figure 4.15: Energy bar graph for the vehicles skidding on the road. Initial thermal energy is taken to be zero. Gravitational energy is zero because both vehicles are on the ground before and after the collision.[1]

Numbers

Knowns	Unknowns
$m_{wreckage} = 4000$ kg	$\vec{F_f}$
$\vec{v_i} = -9.375\,\mathrm{m/s}\,\hat{x}$	$t_{skidding}$
$\vec{v_f} = 0$	ΔE_{th}
$\vec{\Delta x} = -4\,\mathrm{m}\,\hat{x}$	ΔE_k

Note that the initial velocity for the skidding part of the collision is the final velocity of impact part of the collision.

We already know from the last section that $E_{k,i}$ for the skidding portion of the collision is 1.76×10^5J. And since $\vec{v_f} = 0$, $E_{k,f} = 0$. Using Equation 2.1, and considering friction to be an internal change of energy and not work being done by an external force),

$$W = 0 = E_{tot,f} - E_{tot,i}$$

$$E_{k,i} + E_{th,i} = E_{k,f} + E_{th,f}$$

$$\Delta E_k = -\Delta E_{th}$$

$$\Delta E_{th} = 1.76 \times 10^5\,\mathrm{J}$$

As the tires skid across the ground, the frictional force does negative work on the vehicles, slowing them down. Whenever two surfaces slide against each other with friction, the friction fights the relative motion of the two surfaces, converting kinetic energy into thermal energy.

This only happens when the surfaces move relative to each other. If a car is sitting on the side of a hill with the brakes locked, there is still friction between the tires and the road, but there is no relative motion, so no work is being done and kinetic energy is not being transformed by friction into thermal energy.

We can find the time needed to stop the vehicles by considering momentum. The car-truck wreckage has momentum after the initial impact, and then the force of friction reduces the momentum to zero over a certain amount of time.

The larger the force of friction, the less time needed to reduce the momentum to zero.

We could also consider the question of the force between the car and the truck during the time they were skidding. It would depend on how much of the frictional force is coming from the truck's tires and how much from the car's tires. If the car were not braking at all then there would be a large force applied on the car from the truck during the time they were skidding. If they were both braking then the force between them would be smaller.

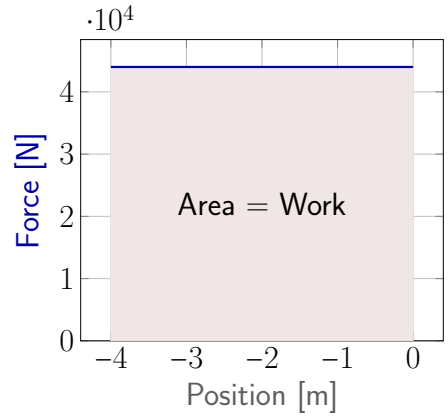


Figure 4.16: A force opposite the direction of motion does negative work equal to the area under the curve in a Force-vs-Position graph.[1]

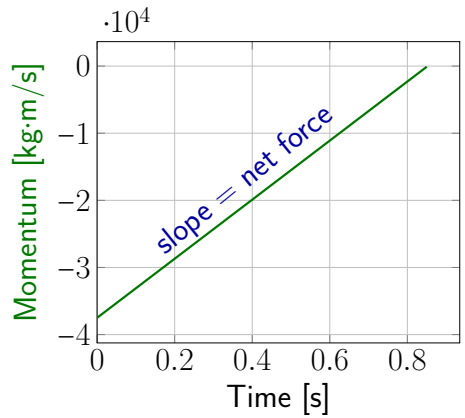


Figure 4.17: Momentum starts with a large negative value, and goes to zero over time because of the force of friction.[1]

Now shifting to look at the change in energy as being done by the force of friction, Equation 2.1 gives.

$$W_{net} = \Delta E_k = -1.76 \times 10^5 \text{J} = \overrightarrow{F_{net}} \cdot \overrightarrow{\Delta x}$$

Using only the "x" components since we are working in one dimension,

$$F_{net,x} = \frac{\Delta E_k}{\Delta x} = \frac{-1.76 \times 10^5 \text{ J}}{-4 \text{ m}} = 4.4 \times 10^4 \text{ N}$$

The negative signs for change in energy and displacement cancel out, giving a force in the positive direction, opposite the direction of motion. Since friction is the only force in the horizontal direction, $F_{net,x} = F_f$.

We can use Newton's Second Law, Equation 1.7, to find the time needed to stop.

$$\overrightarrow{F_{net}} = \frac{\overrightarrow{\Delta p}}{\Delta t} = \frac{\overrightarrow{p_f} - \overrightarrow{p_i}}{t_{skidding}}$$

$$t_{skidding} = \frac{m_{wreckage} \cdot v_{f,x} - m_{wreckage} \cdot v_{i,x}}{F_f}$$

$$t_{skidding} = \frac{4000 \text{ kg} \cdot 0 - 4000 \text{ kg} \cdot (-9.375 \text{ m/s})}{4.4 \times 10^4 \text{ N}}$$

$$t_{skidding} = 0.85\text{s}$$

4.5 Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

General

- When you make an assumption about the direction of a vector quantity and then you calculate a positive magnitude, your assumption about direction was correct.
- When you make an assumption about the direction of a vector quantity and then you calculate a negative magnitude, your assumption about direction was not correct.

Forces

- During a collision, the forces caused by the collision are usually so much larger than any other forces that all other forces can be neglected.
- During a collision, forces change rapidly, so usually average forces during a collision are calculated.
- The force of friction opposes the relative motion of two objects.
- The force of friction converts kinetic energy into thermal energy when two surfaces slide against each other.

Motion

- In a collision between two objects with different masses, the more massive object experiences less acceleration than the less massive object.

Momentum

- Considering momentum is a good way to begin any investigation of a collision.

Energy

- An inelastic collision is one in which kinetic energy is not conserved.
- Completely inelastic collisions are collisions in which the two objects together after the collision.
- Kinetic energy depends upon your reference frame.
- Energy transformations involving potential energy or thermal energy do not depend upon your reference frame.
- In an inelastic collision, some kinetic energy is transformed into thermal energy.
- If the force is opposite the direction of motion in a force-vs-distance graph, the area "under" the curve represents negative work. In this case either the distance or the force is shown as a negative number on the graph.

Mathematical Models

equation	restrictions on the validity of the equation
$\overrightarrow{F_{net,avg}} = \frac{\overrightarrow{\Delta p}}{\Delta t}$ (4.1)	-none-
$\overrightarrow{a_{avg}} = \frac{\overrightarrow{\Delta v}}{\Delta t}$ (4.2)	-none-
$W_{net} = \overrightarrow{F_{net,avg}} \cdot \overrightarrow{\Delta x} = F_{net,avg} \cdot \Delta x \cdot \cos(\theta)$ (4.3)	-none-
$\overrightarrow{v_{avg}} = \frac{\overrightarrow{\Delta x}}{\Delta t}$ (4.4)	-none-
$\overrightarrow{v_{avg}} = \frac{\overrightarrow{v_i} + \overrightarrow{v_f}}{2}$ (4.5)	only valid when the net force is constant

4.6 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

Level 1 - Remember

4.1 **[W]** What type of energy conversion always happens during an inelastic collision?

4.2 **[W]** Differentiate between what is meant by elastic, inelastic, and completely inelastic collisions.

4.3 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

Level 2 - Understand

4.4 **[N]** Confirm that Equation 4.4 and Equation 4.5 give the same result for the physical scenario described in Section 4.2. Show your work.

4.5 **[W & N]** In Section 4.3 we found the horizontal force on the car and the truck during the impact.

(a) Is the magnitude of the force the same for both the car and the truck? Explain why or why not.

(b) Find the accelerations of the car and the truck during the impact.

(c) Is the magnitude of the acceleration the same for both the car and the truck? Explain why or why not.

4.6 **[W]** Explain why the initial velocity for the physical scenario that is considered in Section 4.4 is the same as the final velocity for the physical scenario that is considered in Section 4.3.

4.7 **[W & N]** Find the acceleration of the car and the truck from Section 4.4 during the skidding. Is the acceleration of the car the same as the acceleration of the truck? Why or why not?

Level 3 - Apply

4.8 **[N]** In Section 4.2 one of the unknowns was time, but it was never explicitly found, though it can be seen in some of the graphs. Exactly how much time does the stone spend sliding across the ice?

4.9 **[G]** Section 4.3 includes a free body diagram for the truck during the collision, but ignoring the force of friction.

(a) Re-draw that free body diagram for the forces on the truck in the horizontal direction only, including the force of friction. Assume that the driver of the truck was applying the brakes during the time of the impact.

(b) Does including the force of friction result in a larger or smaller magnitude of the net force on the truck in the horizontal direction?

(c) Draw another free body diagram for the forces on the *car* in the horizontal direction only, including the force of friction. Assume that the driver of the car was applying the brakes during the time of the impact.

(d) Does including the force of friction result in a larger or smaller magnitude of the net force on the car in the horizontal direction?

Level 4 - Analyze

4.10 **[W & N]** In Section 4.2 it is mentioned that some of the team members use brooms to change the force of friction as the stone slides across the ice. If they were able to reduce the force of friction by 10%, what effect would that have on the displacement and the time? Would they increase or decrease? Would they also change by 10%, or more, or less? Explain your answers.

4.11 **[W & G]** Section 4.3 includes a figure showing momentum bars of the car, truck, and wreckage just before and just after impact in the earth's reference frame.

(a) Verify that the area of the third momentum bar in the figure is equal to the total area of the first two momentum bars.

(b) Re-draw the momentum bars for this collision in the center of mass reference frame.

(c) Is the area of the momentum bar of the wreckage in the center of mass reference frame equal to the total area of the momentum bars for the car and the truck just before the collision? Explain why or why not.

(d) Did the choice of reference frame affect conservation of momentum?

4.12 **[N]** In Section 4.3, energy conservation was calculated in the earth's reference frame and an assertion was made that energy transformations do not depend on the reference frame.

(a) Transform the initial and final velocities of the car and the truck into the center of mass reference frame.

(b) Find the initial and final kinetic energy of this system. Compare with the initial and final kinetic energies in Section 4.3. Are they the same as the values found in the earth's reference frame?

(c) Use conservation of energy to find the amount of thermal energy created in this collision when viewed in the center of mass reference frame. Is it the same as the amount of thermal energy created in the earth's reference frame?

(d) Did the choice of reference frame affect the amount of energy that was transformed into thermal energy in this physical scenario?

Level 5 - Evaluate

4.13 **[N]** An assertion is made in this chapter that during a collision the forces caused by the impact are usually so much larger than any other forces that all other forces can be neglected. Compare the force of the impact in Section 4.3 to the force of friction in Section 4.4 for the same collision. How many times larger is the force of impact than the force of friction with the road? Probably both cars were braking even during the impact. Does ignoring the force of friction cause a significant (which we will define for this question as larger than 10%) error in the calculation of the force in the impact?

4.14 **[W]** We have considered the collision in Sections 4.3 and 4.4 from the earth's reference frame and also from the center of mass reference frame. Would we have gotten valid results if we had done the same analysis in the car's reference frame? Why or why not?

4.15 **[W, G, & N]** At the end of Section 4.4 it is noted that the force between the car and the truck during the time that they are skidding depends on the relative amount of frictional force applied by the tires of each vehicle.

(a) Find the amount of force applied by the truck on the car while they are skidding if the car is not braking at all and all $4.4 \times 10^4\ \mathrm{N}$ of frictional force comes from the truck's tires.

(b) Find the amount of frictional force that would need to be applied by the car's tires and by the truck's tires so that their total frictional force would still be $4.4 \times 10^4\ \mathrm{N}$ and there would be no net force applied by the truck on the car while they are skidding.

Level 6 - Create

4.16 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Add the main ideas from this chapter to the concept map that you began for the question at the end of Chapter 1.

4.17 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

4.18 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

Chapter 5

Working in Two Dimensions

So far, we have only considered objects and forces in one dimension, either horizontally or vertically. But our universe is not one-dimensional. Usually we think of it as being three-dimensional, and that is correct. We live in three spatial dimensions. It's also possible to consider time as another "dimension," in which case we can think of living in a four-dimensional "space-time." And in some theoretical models of physics called "string theories" there have been attempts made to describe the universe as being 10-, 11-, or even 26-dimensional! For now, let's just work on expanding our understanding from one dimension to two.

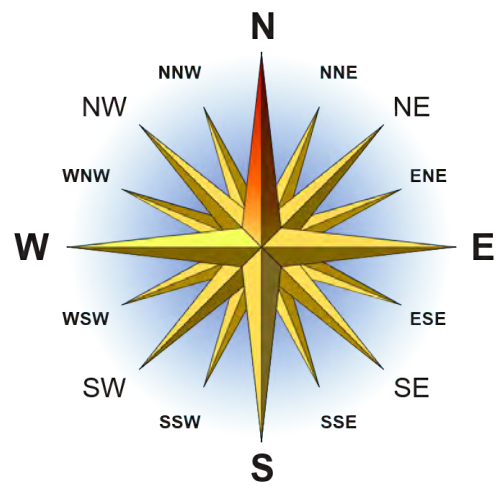


Figure 5.1: A compass rose, showing the traditional direction of North as "up." [15]

At first we will consider a plane that is lying flat on the ground, so North and South, East and West; or left and right, forward and backward. Then we will consider vertical planes, so up and down, left and right. Finally, we will consider planes that are tilted. In all cases, it is important to remember that for our analyses, the two dimensions have to be at right angles to each other. That allows us to consider each direction separately from the other.

For example, if your aunt's home is West of yours, you have to get there by traveling West[1]and not North or South. In fact, if you do go North, you will then have to undo that motion by going South. North and South have to be considered as completely separate from East and West.

[1]You could also get there by going really far East, since the earth is spherical–but we will imagine everything as being flat for now. Curved space gets complicated.

5.1 Floating on the Water

Words

Imagine sitting on an inner tube, just floating in the water with a group of friends. If the river isn't moving and all of you are just lazily floating in the river, you would say that none of you is moving. And if a person were standing on a nearby shore, that person would also say that you aren't moving.

These ideas should sound very familiar, because it is very similar to our discussion of reference frames in an earlier chapter. The only difference now is that some of the objects we are considering are actually riding on one of the objects. But all of the ideas are the same.

But what if the river were moving, and carrying you and your friends along? If the river is flowing North at 2 m/s, and you and your friends are just floating along with the river, you will also go North at 2 m/s. The person standing on the shore would be able to watch you going North along with the river.

Graphics

Figure 5.2: A group of people floating in inner tubes on a river. If they don't try to push themselves through the water, they will all move together, along with the river.[16]

Numbers

Knowns	Unknowns
$\overrightarrow{v_{river\leftarrow shore}} = 0 \text{ or } 2\,\mathrm{m/s}\,\hat{\mathrm{N}}$	$\overrightarrow{v_{you\leftarrow shore}}$
$\overrightarrow{v_{you\leftarrow river}} = 0$	$\overrightarrow{v_{friend\leftarrow you}}$
$\overrightarrow{v_{friend\leftarrow river}} = 0$	$\overrightarrow{v_{shore\leftarrow you}}$

Equation 3.1 works here, though we have never tried to use it where one object is riding on another. Taking objects 1, 2, and 3 to be the river, you, and the shore, respectively...

$$\overrightarrow{v_{you\leftarrow shore}} = \overrightarrow{v_{you\leftarrow river}} - \overrightarrow{v_{shore\leftarrow river}}$$

We aren't given $\overrightarrow{v_{shore\leftarrow river}}$, but we are given $\overrightarrow{v_{river\leftarrow shore}}$, and Equation 3.2 says that...

$$\overrightarrow{v_{shore\leftarrow river}} = -\overrightarrow{v_{river\leftarrow shore}}$$

...So...

$$\overrightarrow{v_{you\leftarrow shore}} = \overrightarrow{v_{you\leftarrow river}} + \overrightarrow{v_{river\leftarrow shore}}$$

If $\overrightarrow{v_{river\leftarrow shore}} = 0$ then $\overrightarrow{v_{you\leftarrow shore}} = 0$, and if $\overrightarrow{v_{river\leftarrow shore}} = 2\,\mathrm{m/s}\,\hat{\mathrm{N}}$ then $\overrightarrow{v_{you\leftarrow shore}} = 2\,\mathrm{m/s}\,\hat{\mathrm{N}}$.

Now what if you and your friend were deep in conversation, not paying any attention to anything except each other? If you didn't happen to notice that the shore and the shoreline was moving past you, it would feel to you exactly like you and your friend were sitting still, and meanwhile the shore is slipping by at 2 m/s to the South.

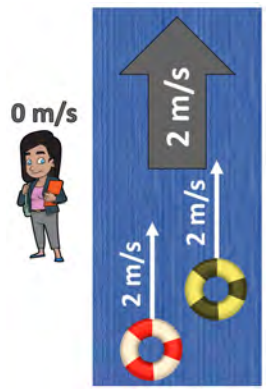


Figure 5.3: From the frame of reference of a person standing on the shore, you and your friend are moving along with the river.[17]

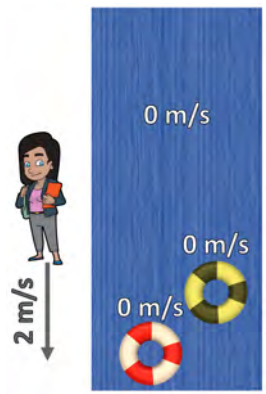


Figure 5.4: From your frame of reference, you and your friend are not moving, and a person standing on the shore is moving South.[17]

Interestingly, $\overrightarrow{v_{friend \leftarrow you}}$ does not depend at all on the velocity of the river. We can again use Equation 3.1, this time taking objects 1, 2, and 3 to be the river, your friend, and you, respectively.

$$\overrightarrow{v_{friend \leftarrow you}} = \overrightarrow{v_{friend \leftarrow river}} - \overrightarrow{v_{you \leftarrow river}}$$

$\overrightarrow{v_{river \leftarrow shore}}$ does not appear in this expression at all. Regardless of the velocity of the river relative to the shore, the velocity of your friend relative to you will always be zero in this scenario.

Equation 3.1 can also tell us the velocity of the shore in your frame of reference as you float down the river...

$$\begin{aligned}\overrightarrow{v_{shore \leftarrow you}} &= \overrightarrow{v_{shore \leftarrow river}} - \overrightarrow{v_{you \leftarrow river}} \\ &= -2\,\mathrm{m/s}\,\hat{\mathrm{N}}\end{aligned}$$

5.2 Boating in a River

Words

If you are in a boat on a still body of water, and you start rowing South, you will go South. If you row East, you will go East. And whatever speed you achieve in the water, that is the speed that somebody standing on a dock would say that you are moving. But, things change if you are in a moving river.

If you are in a river that is slowly flowing East, and you point your boat south and start rowing, you will indeed go South. Compared to the water surrounding you, you will be going directly South. But to somebody who is standing on a dock, they would say that you are going both South and East, because as you row South the current carries you East.

The actual direction that you move, as seen by someone on the dock, depends on how fast you are rowing relative to the speed of the water in the river.

It is tempting to say that the first boat in Figure 5.2 is moving South, which is correct, and the others are moving SouthEast, which is not correct. As shown in figure 5.1, SouthEast is a specific direction, halfway between South and East. So unless we are sure that the boat is traveling in exactly that direction, we should say that it is traveling South *and* East.

Graphics

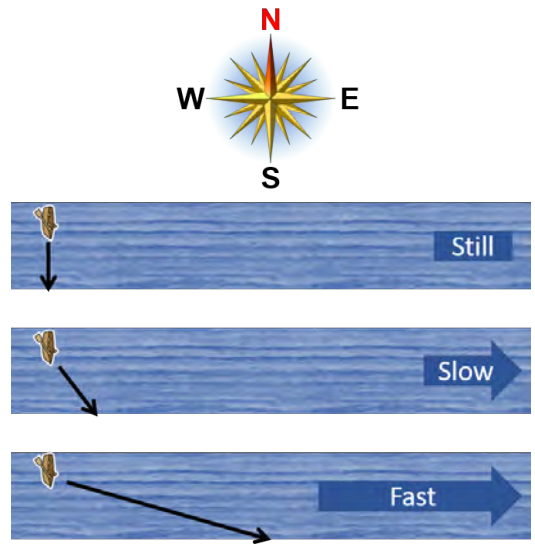


Figure 5.2: The path taken by a boat in a river, as seen by someone on a dock, depends on both the velocity of the boat in the river and the velocity of the river.[18]

The path taken by the boat can be found graphically by adding the velocity vectors "tip to tail," that is, draw the first vector, then draw the second so that its tail starts at the tip of the previous one. Continue with any other vectors that need to be added together. The resultant vector starts at the tail of the first vector and ends at the tip of the last one.

Numbers

Knowns	Unknowns
$\overrightarrow{v_{boat\leftarrow river}}$	$\overrightarrow{v_{boat\leftarrow dock}}$
$\overrightarrow{v_{river\leftarrow dock}}$	

Equation 3.1 works here, though we have never tried to use it where one object is riding on another. Taking objects 1, 2, and 3 to be the river, the boat, and the dock, respectively...

$$\overrightarrow{v_{boat\leftarrow dock}} = \overrightarrow{v_{boat\leftarrow river}} - \overrightarrow{v_{dock\leftarrow river}}$$

We aren't given $\overrightarrow{v_{dock\leftarrow river}}$, but we are given $\overrightarrow{v_{river\leftarrow dock}}$, and Equation 3.2 says that...

$$\overrightarrow{v_{dock\leftarrow river}} = -\overrightarrow{v_{river\leftarrow dock}}$$

...So...

$$\overrightarrow{v_{boat\leftarrow dock}} = \overrightarrow{v_{boat\leftarrow river}} + \overrightarrow{v_{river\leftarrow dock}}$$

In order to add the velocity vectors numerically, you have to break up each vector into its component parts, in this case the East/West component (E subscript below) and the North/South component (N subscript).

$$v_{boat\leftarrow dock,E} = v_{boat\leftarrow river,E} + v_{river\leftarrow dock,E}$$

$$v_{boat\leftarrow dock,N} = v_{boat\leftarrow river,N} + v_{river\leftarrow dock,N}$$

Let's consider a situation where you can row the boat at a speed of 5 m/s, and the river is flowing at a uniform speed of 4 m/s to the East.

If you were to row directly East, along with the river, you would move along very quickly. Since you are rowing the same direction as the current, the speed of the river would add to your rowing speed, and someone on the dock would see you going past at 9 m/s. This is the highest possible total speed when combining 5 m/s and 4 m/s.

If you were to row directly West, you would be fighting the river and moving very slowly. At 5 m/s, you can row slightly faster than the flow of the river. So you would be able to go West but only at 1 m/s. This is the lowest possible speed when combining 5 m/s and 4 m/s. If the river were flowing faster than you could row, you would not be able to go West at all, but would slowly go down the river even when paddling upstream as quickly as you could.

If you were to row directly South, you would go both South and East, and your speed would be somewhere between 1 m/s and 9 m/s. For this example, since you are rowing faster than the flow of the river, your direction would be somewhere between South and SouthEast.

Remember, vectors have magnitude and direction, so as long as you keep the magnitude and direction of a vector the same you are free to shift it up, down, left, or right on the page to get the tips & tails to line up correctly. This same technique will work with any type of vectors, for example forces, momentum, displacement, and acceleration.

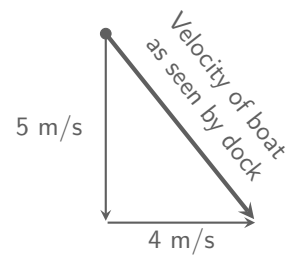


Figure 5.3: Adding a vector of 5 m/s South to a vector 4 m/s East gives a resultant vector at an angle between South and East.[1]

If the velocity vectors are drawn to scale, the resultant vector can be measured with a ruler to get a good estimate of the speed of the boat as seen by the dock. A good estimate of the direction can be found by using a protractor.

From the diagram in Figure 5.3, measurement with a ruler and protractor gives a speed of approximately 6.5 m/s, at an angle approximately $40°$ East of South.

In this example,

$$\overrightarrow{v_{boat \leftarrow dock}} = -5\,\mathrm{m/s}\,\hat{N} + 4\,\mathrm{m/s}\,\hat{E}$$

The North/South direction has been assigned a negative value in the "North" direction. In other words, South! It is usually easier to define North as positive and East as positive when doing calculations, and then if the final answer is negative report it as a positive value in the South (or West) direction.

This vector can also be described in terms of magnitude and direction. Since the North/South and East/West directions are at right angles to each other, the magnitude of the resultant vector has to be found using the Pythagorean theorem:

$$\boxed{A^2 = A_x^2 + A_y^2} \tag{5.1}$$

...where A is the magnitude of any vector $\overrightarrow{A}$, A_x is the $\hat{x}$ component of the vector, and A_y is the $\hat{y}$ component of the vector. In our case, we are using East and North in place of x and y.

The magnitude of the velocity, in other words the speed, of the boat as seen from the dock would be...

$$v_{boat \leftarrow dock} = \sqrt{(-5)^2 + 4^2}\,\mathrm{m/s} = 6.4\,\mathrm{m/s}$$

5.3 Straight Across a River

Words

Let's consider attempting to cross a river directly in a boat. If there is any current flowing in the river, aiming your boat directly across the river will mean that you will reach the opposite bank at a point downstream. Is there a way to aim a boat in such a way that it goes directly across the river?

Suppose we have a river that is 50 meters wide from North to South, and it is uniformly flowing East at 4 m/s. If a boat is rowed at a speed of 5 m/s relative to the river, in what direction should it be pointed so that it crosses directly from the North bank of the river to the South bank, as seen from a dock? How much time would be needed to cross the river in this way?

If we want to end up directly South from our starting point, then we need to make sure that our velocity as seen from a dock is directly South. We will be in the river, which is flowing East, so that means that the way we row our boat will have to exactly cancel out the Eastward flow of the river. That means our boat will have to be going 4 m/s West compared to the river, since the river is going 4 m/s East.

Graphics

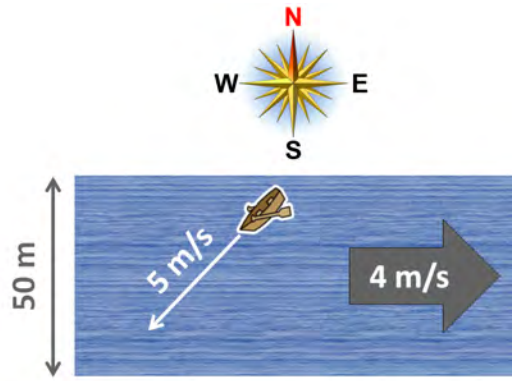


Figure 5.4: In order to go directly South across the river as seen from a dock, the boat will need to be pointed slightly upstream.[18]

If we start by drawing the vector we know, 4 m/s East, we can rotate the 5 m/s until the resultant vector is vertical.

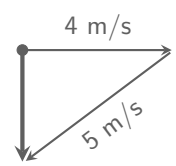


Figure 5.5: Finding the angle to make the resultant vector vertical.[1]

Numbers

Knowns	Unknowns
$\overrightarrow{v_{river \leftarrow dock}} = 4\,\mathrm{m/s}\,\hat{E}$	θ_{boat}
$v_{boat \leftarrow river} = 5\,\mathrm{m/s}$	t
$\overrightarrow{\Delta x} = -50\,\mathrm{m}\,\hat{N}$	

The angle of any vector is related to its $\hat{x}$ and $\hat{y}$ components by sine, cosine, and tangent.

$$\boxed{\sin\theta = \frac{A_y}{A}} \tag{5.2}$$

$$\boxed{\cos\theta = \frac{A_x}{A}} \tag{5.3}$$

$$\boxed{\tan\theta = \frac{A_y}{A_x}} \tag{5.4}$$

...where θ is the angle opposite the $\hat{y}$ component and adjacent to the $\hat{x}$ component of $\overrightarrow{A}$.

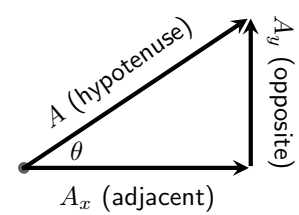


Figure 5.6: Components of a vector are related by "SOHCAHTOA."
[1]

The time needed to cross the river will be longer than if there were no current in the river, because some of the boat's velocity has to be used to fight the flow of the river. The faster the river is flowing, the longer it will take to cross.

Using a protractor to measure between the 4 m/s vector and the 5 m/s vector, we find an angle of approximately $35°$, so the direction is approximately $35°$ South of West.

It is important to clearly specify the reference direction and which way the angle is measured from that direction, because compass bearings are measured differently from angles in mathematics and physics.

Figure 5.7: Compass bearings start with zero at North and proceed clockwise. Angles in physics usually start at East and proceed counterclockwise.[19]

The speed of the boat as seen from the dock can be found by measuring the vertical vector in Figure 5.5. It is approximately 3 m/s. This can be used to determine the amount of time that is needed to cross the river.

"SOHCAHTOA" is a mnemonic that many people use to remember Equations 5.2, 5.3, and 5.4:

Sine=**O**pposite/**H**ypotenuse,
Cosine=**A**djacent/**H**ypotenuse,
Tangent=**O**pposite/**A**djacent.

In this case, using $\hat{x}$ and $\hat{y}$ notation, $\overrightarrow{v_{boat \leftarrow river}}$ has a magnitude $v = 5\,\mathrm{m/s}$ and a horizontal component $v_x = -4\,\mathrm{m/s}$. So we can use Equation 5.3, solving for θ:

$$\theta = \arccos\left(\frac{-4\,\mathrm{m/s}}{5\,\mathrm{m/s}}\right) = 143°$$

The normal way to define angles is clockwise from the positive x axis, so $143°$ is in a direction North and West. Unfortunately, in many cases the inverse trigonometric functions like arccos do not give the correct angles on a calculator, because there are multiple possible correct answers mathematically. For that reason, it is usually best to start with a diagram like that in Figure 5.5 and use only positive values for each part. Finding the θ in the upper right of that figure,

$$\theta = \arccos\left(\frac{4\,\mathrm{m/s}}{5\,\mathrm{m/s}}\right) = 37°$$

...so, $37°$ South of West. Using Equation 5.4 we can find that the North/South component of the velocity is -3 m/s. Since velocity is constant in this problem, we can use Equation 4.4 to find the time needed to cross the river.

$$\Delta t = \frac{\overrightarrow{\Delta x}}{\overrightarrow{v_{avg}}} = \frac{-50\,\mathrm{m}\,\hat{N}}{-3\,\mathrm{m/s}\,\hat{N}} = 17\,\mathrm{s}$$

5.4 Kicking in a New Direction

Words

Since we have started considering two dimensions, we have only looked at motion, not forces, momentum, or energy. Now it's time to branch out.

A 0.45-kg soccer ball is initially moving at 20 m/s in a direction 30° West of North. A player kicks the ball so that it begins moving at 20 m/s in a direction 30° West of South. The time that the player's foot is in contact with the ball is 0.02 seconds. How can we describe the ball's acceleration, energy, momentum, and the force applied to it during the kick?

In the description, the velocity appears to be constant before and after the kick, so the momentum also must be constant before and after the kick, and it also means that there is no acceleration before and after the kick.

What happens during the kick?

The initial and final speeds are the same, but the direction changes, so the velocity changes during the kick. That means that there was an acceleration, since acceleration is a change in velocity over time. And the time involved, the time of the kick, is very short. That means the acceleration had to be very large to get a large change in velocity in a small amount of time.

Graphics

Figure 5.8: A soccer player kicking a ball.[11]

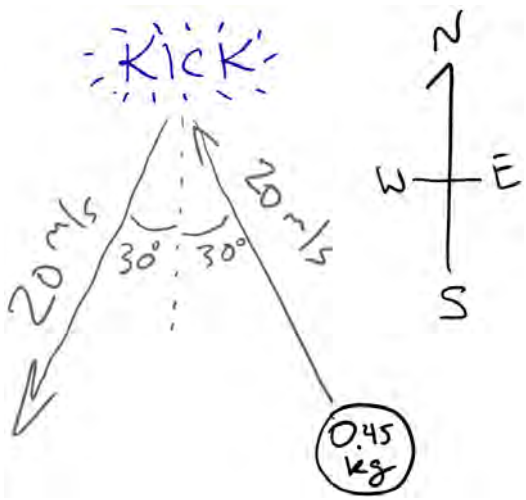


Figure 5.9: A sketch of the physical scenario.[1]

Numbers

Knowns	Unknowns
$m = 0.45\,\mathrm{kg}$	$\vec{a}$
$\vec{v_0} = 20\,\mathrm{m/s}$ @ $30°$ W of N	E
$\vec{v_f} = 20\,\mathrm{m/s}$ @ $30°$ W of S	$\vec{p}$
$\Delta t = 0.02\,\mathrm{s}$	$\overrightarrow{F_{foot\to ball}}$

We should start by breaking the velocities up into their component parts using Equations 5.2, 5.3, and 5.4. Since the directions are given in terms of North, South, East, and West, we can use that notation.

$$v_{0,N} = v_0 \cdot \cos\theta_0 = 17.3\,\mathrm{m/s}$$
$$v_{0,E} = -v_0 \cdot \sin\theta_0 = -10\,\mathrm{m/s}$$
$$v_{f,N} = -v_0 \cdot \cos\theta_f = -17.3\,\mathrm{m/s}$$
$$v_{f,E} = -v_0 \cdot \sin\theta_0 = -10\,\mathrm{m/s}$$

Where did the minus signs come from, and why weren't the angles converted to normal physics notation, with 0° pointing East? The sketch in Figure 5.9 was used to determine positive and negative signs, and to find appropriate triangles using the angles given in the question. This is a simpler approach than converting to normal physics notation, and avoids the ambiguity in angles that comes from trusting the output of a calculator.

If the velocity changes, that means that the momentum also changes during the kick, which means that a force must have been applied to the ball.

We know that a force applied in the direction of an object's motion speeds it up, and a force applied opposite the direction of motion slows it down. In this case, a force acted that didn't actually change the speed of the ball! So the force must have been in a direction other than the direction of motion.

The ball was initially moving North and West, and after being kicked it was moving South and West. So we know that there was definitely a change in the North/South direction. There must have been an acceleration to the South, which means that there must have been a force applied to the South, and a change in momentum to the South.

In order to determine whether a force was also applied to the East or West, details about the velocity or momentum in that direction need to be known.

If we assume that the ball stayed on the ground the whole time, we don't need to worry about gravitational energy. Kinetic energy depends only on the speed of the ball, and since the initial and final velocities are the same, the kinetic energy is the same before and after the collision.

To find a change in momentum, the momentum vectors can be subtracted graphically. So far, we have only learned about adding vectors tip-to-tail. Subtracting a vector is the same as adding its opposite, and the opposite of a vector is another vector with the same magnitude and opposite direction.

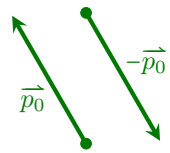


Figure 5.10: Finding the opposite of the initial momentum.[1]

Change in momentum is final momentum minus initial momentum, so...

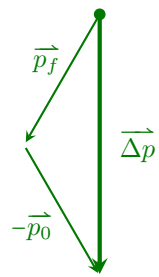


Figure 5.11: Finding the change in the momentum of the soccer ball.[1]

The change in momentum of the ball is due South. This means that the acceleration and the force applied to the ball are also due South.

If we assume a constant net force during the kick, we can use Equation 2.2 to find the acceleration:

$$a_E = \frac{v_{f,E} - v_{0_E}}{\Delta t} = 0\ \mathrm{m/s^2}$$

$$a_N = \frac{v_{f,N} - v_{0_N}}{\Delta t} = -1730\ \mathrm{m/s^2}$$

Combining these, we find $\vec{a} = -1730\ \mathrm{m/s^2}\ \hat{N}$

We can use Equation 1.3 to find initial and final momentum:

$$p_{0,N} = m \cdot v_{0,N} = 7.79\ \mathrm{kg \cdot m/s}$$

$$p_{0,E} = m \cdot v_{0,E} = -4.5\ \mathrm{kg \cdot m/s}$$

$$p_{f,N} = m \cdot v_{f,N} = -7.79\ \mathrm{kg \cdot m/s}$$

$$p_{f,E} = m \cdot v_{f,E} = -4.5\ \mathrm{kg \cdot m/s}$$

Subtracting initial from final momentum gives us the impulse caused during the kick: $\vec{\Delta p} = -15.6\ \mathrm{kg \cdot m/s}\ \hat{N}$.

Then Equation 1.7 can be used to find the force on the ball during the kick:

$$\overrightarrow{F_{net,foot \to ball}} = \frac{\vec{\Delta p}}{\Delta t} = -779\ \mathrm{N}\ \hat{N}$$

Since the speed is the same before and after the kick, we know that the kinetic energy is also the same before and after the kick. We can find kinetic energy using Equation 1.6:

$$E_{k,f} = E_{k,0} = \frac{1}{2} m \cdot v_0^2 = \frac{1}{2}(0.45\mathrm{kg}) \cdot (20\ \mathrm{m/s})^2 = 90\ \mathrm{J}$$

Using the velocity vector components instead of the speed gives the same result.

5.5 Checking In Ice Hockey

Words

Ice hockey players commonly run into each other on the ice, often intentionally. During any collision, external forces can generally be neglected because the force of the colliding bodies against each other is so much larger than any other forces that are acting on them.

If a 90-kg hockey player is moving South across the ice at 5 m/s and is struck by an 80-kg hockey player who is moving NorthEast at 6 m/s, what can we say about the collision and the two players after the collision? Assume that the collision is perfectly inelastic, so the two become entangled together, not bouncing off of each other.

Since this is a collision, we should begin by thinking about momentum. We know that momentum is conserved in any collision, whether it is elastic or inelastic.

Since the 90-kg hockey player was moving South, that player's initial momentum is to the South. The 80-kg hockey player was moving NorthEast, so that player's momentum was partially North and partially East.

After the two collide and become entangled, their total momentum has to be the same as their initial momentum, just as in the one-dimensional problems we have considered previously. But this time we have to think about both directions.

Graphics

Figure 5.12: Ice hockey players.[20]

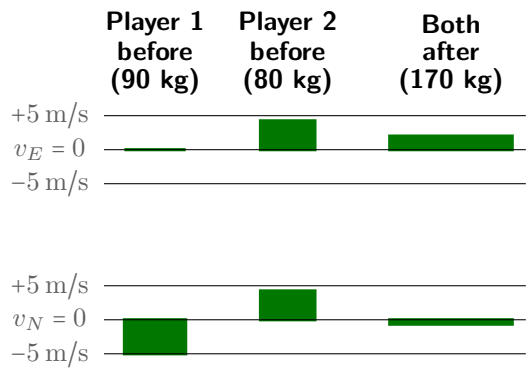


Figure 5.13: Two sets of momentum bars: One for the East/West direction and one for the North/South direction, in the earth's reference frame.[1]

Numbers

Knowns	Unknowns
$m_1 = 90\,\mathrm{kg}$	???
$\overrightarrow{v_{1,i}} = -5\,\mathrm{m/s}\,\hat{N}$	
$m_2 = 80\,\mathrm{kg}$	
$\overrightarrow{v_{2,i}} = 6\,\mathrm{m/s}$ @ $45°$ N of E	

The $45°$ angle is given, since the direction is listed as NorthEast, which is half-way between North and East. $\overrightarrow{v_{2,i}}$ can be broken into its component parts using Equations 5.2 & 5.3.

$$\overrightarrow{v_{2,i}} = 4.24\,\mathrm{m/s}\,\hat{N} + 4.24\,\mathrm{m/s}\,\hat{E}$$

Since it is often easier to consider collisions in the center of mass reference frame, where the total momentum of the system is zero, we can begin by finding the velocity of the center of mass using Equation 3.6.Taking each component separately,. . .

$$v_{com,N} = \frac{m_1 \cdot v_{1,i,N} + m_2 \cdot v_{2,i,N}}{m_1 + m_2} = -0.65\,\mathrm{m/s}$$

$$v_{com,E} = \frac{m_1 \cdot v_{1,i,E} + m_2 \cdot v_{2,i,E}}{m_1 + m_2} = 2.00\,\mathrm{m/s}$$

Since the two are stuck together after the collision, these vector components represent the velocity of the two hockey players after the collision. We can convert this into speed and direction using Equations 5.1 & 5.4.

Let's start with the East/West direction. Initially, the first hockey player has no momentum in that direction, and the second hockey player has momentum to the East. After they collide, they will still have momentum to the East. So they will end up moving to the East, at a slower speed than the second hockey player had initially, because the momentum that hockey player had before the collision is shared by the two hockey players after the collision.

The North/South direction is more complicated. One hockey player initially has momentum to the North and the other has momentum to the South. With their different masses, different speeds, and the angle, it is difficult to determine whether after the collision they will be moving slightly North or slightly South, so the final direction should be mostly to the East.

Since the first hockey player was initially moving South and ended up moving mostly East, that player must have experienced a force during the collision that was North and East. That makes sense, because that was the direction the second player was moving before the collision. This means that the second hockey player would have experienced an equal and opposite force, South and West. South is easy to see, because the first hockey player was initially moving South, and there is a force to the West because some of that player's Eastward momentum was transferred to the other player.

There was a loss of kinetic energy during the collision; the lost kinetic energy was converted into thermal energy.

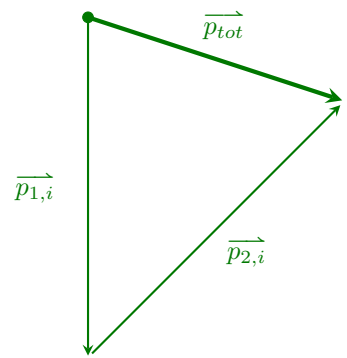


Figure 5.14: Finding the total momentum of the two hockey players.[1]

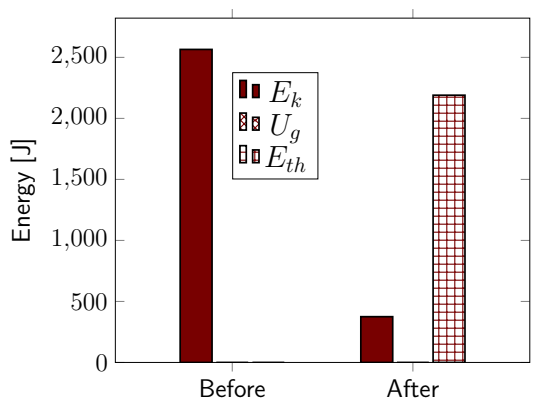


Figure 5.15: Energy bar graph for the collision. Initial thermal energy is taken to be zero. Gravitational energy is zero because both players are on the ground before and after the collision.[1]

$$v_f = \sqrt{(-0.65\,\mathrm{m/s})^2 + (2.00\,\mathrm{m/s})^2} = 2.10\,\mathrm{m/s}$$

$$\theta_f = \arctan\frac{v_{com,N}}{v_{com,E}} = -18^\circ$$

The final speed is correct, but we need to check the angle against a sketch of the situation to be sure. -18° is South and East, and knowing that velocity is always in the same direction as momentum we can use Figure 5.14 to see that this is the correct general direction, so our angle is correct.

We can use Equation 1.3 to find the final momentum of the hockey players:

$$\overrightarrow{p_f} = m_{tot} \cdot \overrightarrow{v_f} = 357\,\mathrm{kg \cdot m/s} \text{ @ } 18^\circ \text{ S of E}$$

Initial and final kinetic energy can be found using Equation 1.6.

$$E_{k,i} = \frac{1}{2}m_1 \cdot v_{1,i}^2 + \frac{1}{2}m_2 \cdot v_{2,i}^2 = 2565\,\mathrm{J}$$

$$E_{k,f} = \frac{1}{2}m_{tot} \cdot v_f^2 = 375\,\mathrm{J}$$

Total energy is conserved, and there is no spring energy or gravitational potential energy stored after the collision, so the lost kinetic energy is converted to thermal energy.

5.6 Standing on a Rope

Words

We have looked in detail at motion and momentum in two dimensions. Let's turn now to look at forces and energy. Consider the fiddler who is standing on a rope in Figure 5.16. We will assume that the fiddler is not moving. What are the forces that are acting on him? And how much work is he doing?

Take the mass of the fiddler to be 70 kg. The rope in front of him and behind him is angled up $25°$ above the horizontal.

First, it is important to establish the meaning of the question. One way to look at the situation is to say that the only two forces that are affecting the fiddler are the force of gravity, directed downward, and the normal force of the rope pushing up on his feet. That makes this problem essentially the same as the rock sitting motionless on the ground from Section 1.3, which is a one-dimensional question that by now is something that we should be able to answer easily.

The intent of this question is to investigate the impact of having a rope going off at an angle, so we will use as our system the fiddler and also the section of the rope on which he is standing. We aren't given the mass of the rope, and we will assume that it is negligible compared to the mass of the fiddler.

Graphics

Figure 5.16: A fiddler balancing on a rope.[21]

Figure 5.17: Sketch of the fiddler balancing on a rope. The blue dotted line indicates the "system" to be considered when making a free body diagram.[1]

Numbers

Knowns	Unknowns
$m = 70\,\mathrm{kg}$	Forces on fiddler
$\theta_1 = 25°$ above horizontal	W
$\theta_2 = 25°$ above horizontal	
$g = 9.8\,\mathrm{m/s^2}$	

Note that the angles are not necessarily the standard angles in math–they are as described in the text and in Figures 5.17 & 5.18.

To analyze this system, we need to separate the forces shown in into $\hat{x}$ & $\hat{y}$ components, using Equations 5.2 & 5.3. There are three forces to consider, as can be seen in Figure 5.18.

Using the normal directions of $+\hat{x}$ to the right and $+\hat{y}$ up...

$$F_{net,x} = F_{t,2} \cdot \cos 25° - F_{t,1} \cdot \cos 25°$$

$$F_{net,y} = F_{t,2} \cdot \sin 25° + F_{t,1} \cdot \sin 25° - F_g$$

Since the fiddler is not moving in this scenario, his acceleration is zero. That means, according to Equation 1.8, that $\overrightarrow{F_{net}} = 0$. So...

$$F_{t,2} = F_{t,1}$$

...and so...

Once we have decided on our system, the forces that we need to consider are only the forces that act on our system, not the forces acting inside the system or the forces of the system acting on something else. One force that acts on the system is the force of gravity, which depends on the mass of the system (in this case, the mass of the fiddler, since we are neglecting the mass of the rope).

There are also two ropes that connect our system to things that are outside of the system, and each of those ropes can have a tension force pulling on the system. Tension is always a pulling force, just as the normal force is always a pushing force. Tension is in the direction of the rope, string, chain, *etc*, and if the rope or other object that is being used to pull is light (often referred to as massless), the tension is the same along the entire length of the rope.

Looking at the sketch in Figure 5.17, we can see that there is symmetry. The angles are the same in front of the fiddler and behind the fiddler, so if the fiddler turned around the picture would be exactly the same. That symmetry tells us that whatever the tension is in the rope in front of the fiddler is the same as the tension behind the fiddler.

When considering the amount of work the fiddler is doing, it was already stated that the fiddler is not moving, and since work is caused by a force acting over a distance, if there is no distance moved then there is no work that is being done.

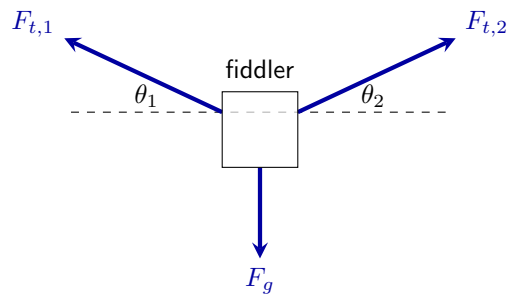


Figure 5.18: FBD for the fiddler[1]

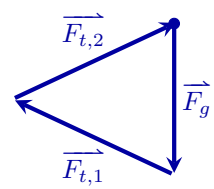


Figure 5.19: Adding the force vectors graphically. The net force is zero, so we know that the arrows have to end in the same place where they started.[1]

Graphically adding the vectors shows that the horizontal components of the two tension forces have to cancel out, because the gravitational force does not have a horizontal component. Graphical addition also shows that the tension force has approximately the same magnitude as the gravitational force, since all three arrows are approximately the same length.

$$2 \cdot F_{t,1} \cdot \sin 25^\circ = F_g$$

Equation 1.4 tells us that...

$$F_g = m \cdot g = 686\ \mathrm{N}$$

Solving for $F_{t,1}$, we get...

$$F_{t,1} = \frac{F_g}{2 \cdot \sin 25^\circ} = 812\ \mathrm{N}$$

Because of the angles involved, the tension in the rope is actually larger than the force of gravity!

We can use Equation 2.3 to find the work that the person is doing.

$$W = \vec{F} \cdot \overrightarrow{\Delta x}$$

And since there is no motion $\overrightarrow{\Delta x} = 0$, so $W = 0$. No work is being done.

5.7 Pulling a Sled

Words

This image shows a moose that is connected to a sled. Imagine that the moose is pulling the sled across level ground at a constant velocity of 3 m/s to the left. The strap that it pulls with is at an angle of 30° above the horizontal, the mass of the loaded sled is 200 kg, and the magnitude of the frictional force between the sled and the ground is 800 N.

Analyze this physical scenario as thoroughly as you can.

We are given very little information about the moose, so we should instead focus on the sled for our analysis. Since the velocity is constant, there are several things that we know immediately:

- The momentum has to be constant.
- The kinetic energy has to be constant.
- The acceleration, which is the change in velocity over time, has to be zero.
- Since the momentum is constant (or since the acceleration is zero), the net force on the sled has to be zero.

Graphics

Figure 5.20: A moose strapped to a sled.[22]

Motion Map

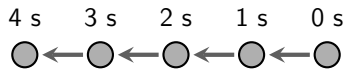


Figure 5.21: Motion map of the sled moving to the left at constant velocity.[1]

We could draw a single momentum bar, or an energy bar graph with only one bar, but usually those are only useful when we want to show a change in something. Since momentum and energy are both constant, drawing them is not very useful.

Numbers

Knowns	Unknowns
$m = 200\,\mathrm{kg}$	???
$\theta = 30°$ above horizontal	
$\vec{v} = -3\,\mathrm{m/s}\,\hat{x}$	
$F_f = 800\,\mathrm{N}$	

We can find the momentum of the sled from Equation 1.3:

$$\overrightarrow{p_{sled}} = m \cdot \vec{v} = -600\,\mathrm{kg} \cdot \mathrm{m/s}\,\hat{x}$$

Kinetic energy is easy to find, from Equation 1.6:

$$E_{k,sled} = \frac{1}{2} m \cdot v^2 = 900\,\mathrm{J}$$

Acceleration is also easy to find, from Equation 2.2:

$$\vec{a} = \frac{\overrightarrow{\Delta v}}{\Delta t} = 0$$

Since $\vec{a} = 0$, Equation 1.8 tells us that the net force is also zero. So the sum of the forces in both the $\hat{x}$ and the $\hat{y}$ directions must be zero.

$$F_{net,x} = F_f - F_t \cdot \cos\theta = 0$$

We are given F_f, so we can solve for F_t:

$$F_t = \frac{F_f}{\cos\theta} = 923\,\mathrm{N}$$

The force of gravity is balanced partly by the normal force but also partly by the upward part of the tension in the strap, so the normal force should be less than the gravitational force in this case. The forward part of the tension force needs to be just large enough to balance the force of friction.

The net work that is being done is zero, since the speed is not changing, but in fact the moose is doing work by pulling the sled, and the frictional force is doing negative work to try to stop the sled.

Work is a force being applied through a displacement, and the displacement is constantly increasing in time since the sled is moving at a constant velocity. That means that the moose is doing a constant amount of work per time.

In physics we have a special name for work per time: Power. Power is measured in Watts, where 1 Watt is 1 Joule per second.

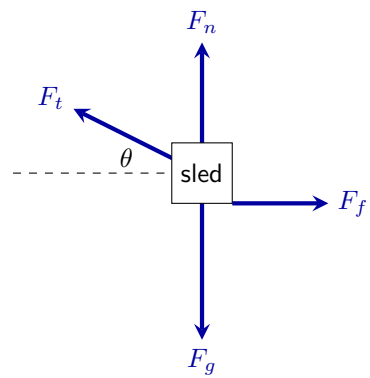


Figure 5.22: FBD for the sled, with guesses about the magnitudes of each arrow.[1]

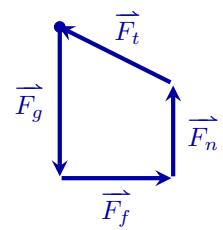


Figure 5.23: Adding the force vectors graphically. The net force is zero, so we know that the arrows have to end in the same place where they started.[1]

Just from adding the force vectors graphically with correct directions but without doing calculations for magnitudes, we can see that the only way the net force can be zero is for the normal force to be smaller than the gravitational force and for the horizontal component of the tension force to exactly cancel the frictional force.

Also,

$$F_{net,y} = F_n + F_t \cdot \sin\theta - F_g = 0$$

F_g is the mass times the acceleration of gravity, or 1960 N, so the only thing we don't know in that equation is F_n, so we can solve for it.

$$F_n = F_g - F_t \cdot \sin\theta = 1500\,\mathrm{N}$$

The force of friction is opposing the direction of motion, so as the sled moves, friction is doing negative work on the sled. The moose is doing positive work on the sled, so that the net work is zero. The amount of work done by the moose is given by Equation 4.3:

$$W = \vec{F} \cdot \overrightarrow{\Delta x} = F \cdot \Delta x \cdot \cos\theta$$

But we don't have a displacement in this scenario. The displacement increases over time, so the work also increases over time. The moose is doing a constant amount of work per unit time, or supplying a constant power:

$$\boxed{P = \frac{W}{\Delta t} = \frac{\vec{F} \cdot \overrightarrow{\Delta x}}{\Delta t} = \frac{F \cdot \Delta x \cdot \cos\theta}{\Delta t}} \tag{5.5}$$

...where P is power, and the equation is valid while the force is constant. Power can also be expressed as...

$$\boxed{P = \vec{F} \cdot \vec{v} = F \cdot v \cdot \cos\theta} \tag{5.6}$$

So the power supplied by the moose is:

$$P_{moose} = F_t \cdot v \cdot \cos\theta = 2400\,\mathrm{W}$$

5.8 Pulling a Frictionless Sled

Words

Let's consider the same physical scenario as in Section 5.7, but this time consider what would happen if there were no friction between the sled and the ground.

If the moose started at rest and pulled the 200 kg sled with the same force as before, 923 N, at an angle of $30°$ above the horizontal, across a horizontal distance of 50 m, what would happen?

Again, we are given very little information about the moose, so we should instead focus on the sled for our analysis.

The moose is generating a force that is acting over a certain distance, so a good starting point would be to consider the work that the moose is doing. The moose does work on the sled, which increases the kinetic energy of the sled. At first the sled is not moving, so it begins with zero kinetic energy. The sled stays on level ground the entire time, and there are no springs to worry about, so all of the work done by the moose is converted into kinetic energy.

Graphics

Figure 5.24: A moose strapped to a sled.[22]

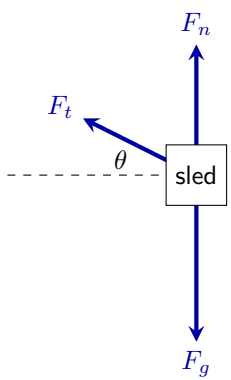


Figure 5.25: FBD for the sled, with arrow lengths based on the forces found in Section 5.7.[1]

Numbers

Knowns	Unknowns
$m = 200\,\mathrm{kg}$	???
$F_t = 923\,\mathrm{N}$	
$\theta = 30°$ above horizontal	
$\Delta x = -50\,\mathrm{m}\,\hat{x}$	
$F_f = 0\,\mathrm{N}$	

Since there is no acceleration in the vertical direction, our analysis of the y components of force in Section 5.7 where $F_{net,y} = 0$ remains valid. But we need to reconsider $F_{net,x}$

$$F_{net,x} = -F_t \cdot \cos\theta = -800\,\mathrm{N}$$

Since we know the mass of the sled, we can also find its acceleration, using Equation 1.8:

$$\vec{a} = \frac{\overrightarrow{F_{net}}}{m} = -4\,\mathrm{m/s}$$

Given net force and displacement, we can use Equations 2.1 & 2.3 to find final kinetic energy:

$$W_{net} = F_{net} \cdot \Delta x \cdot \cos\theta = \Delta E_k = E_{k,f} - E_k, i$$

The net force on the sled gives it an acceleration in the same direction as the net force, to the left. Since the sled started at rest, its speed will increase to the left over the entire time that the moose is pulling. That also means that the momentum will be increasing to the left the entire time that the moose is pulling.

In the physical situation examined in Section 5.7, the velocity was constant, so the power supplied by the moose was constant. In this situation, the speed is increasing, so every second the moose is covering more distance than it did in the previous second. That means every second the moose is doing more work than it did in the previous second.

So the amount of power that the moose is producing increases as its speed increases. For any real creature or machine, the power it can produce limits the amount of force that it is able to apply to an object. Typically a larger force can be applied when the object is moving at a low speed, and as speed increases power becomes the limiting factor.

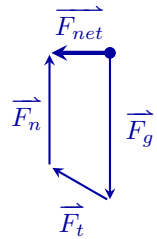


Figure 5.26: Adding the force vectors graphically shows that there is a net force to the left.[1]

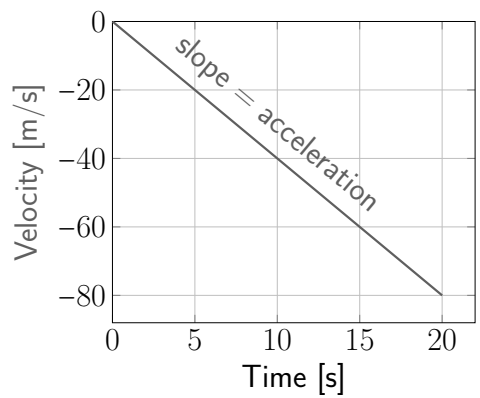


Figure 5.27: Velocity of the sled.[1]

...where θ is the angle between $\overrightarrow{F_{net}}$ and $\overrightarrow{\Delta x}$, not the angle of the strap that the moose is using to pull the sled.

$$E_{k,f} = (800\ \mathrm{N}) \cdot (50\ \mathrm{m}) \cdot \cos 0^\circ = 40\ 000\ \mathrm{J}$$

From here, we can easily find the final speed of the sled, using Equation 1.6:

$$E_{k,f} = \frac{1}{2} m \cdot v_f^2$$

Interestingly, since we found acceleration earlier we could have found the velocity in a more direct route. Multiplying the above equation for W_{net} by 2, dividing by m , and rearranging it slightly gives:

$$\boxed{2a \cdot \Delta x \cdot \cos\theta = v_f^2 - v_i^2} \tag{5.7}$$

This is commonly considered to be a standard equation of motion.

Once we have the final velocity and the acceleration, we can also find the time needed for the moose to pull the sled 50 m.

The power that the moose uses to pull the sled increases over time, even though the force stays the same. This can be seen from Equation 5.6, knowing that the velocity is constantly increasing in time since the force is constant.The maximum power will be needed at the maximum speed:

$$P_{max} = F_{net} \cdot v_f \cdot \cos 0^\circ$$

5.9 Waterslide

Words

The waterslide shown in Figure 5.28 has an extremely steep slope at the top. Water is used to lubricate the slide during operation, making it essentially frictionless.

The slide has a height of 9 m and the slope goes down at an angle $70°$ from the horizontal. If a 50 kg rider starts at the top at rest and slides down the steep slope, what is her acceleration down the steep slope, what is her speed when she reaches the bottom of the steep slope, and how much time does it take her to reach the bottom of the steep slope?

We need to consider the acceleration down the slide. Acceleration is a vector, so it includes direction. We could specify the direction of the acceleration using x and y components, or a magnitude in some direction, but it would be easier if we just create a way to refer to the direction down the slope. We can call that the "parallel" direction, and then the "perpedicular" direction would be at a right angle to the direction of the slope.

If the slope were completely flat, then the rider would just sit there, not accelerating. If the slope were completely vertical then the rider would be in free-fall, accelerating down the slope with the acceleration of gravity g. For slopes in between these extremes, the acceleration should be between zero and g. The steeper the slope, the closer the acceleration would be to g.

Graphics

Figure 5.28: A waterslide.[23]

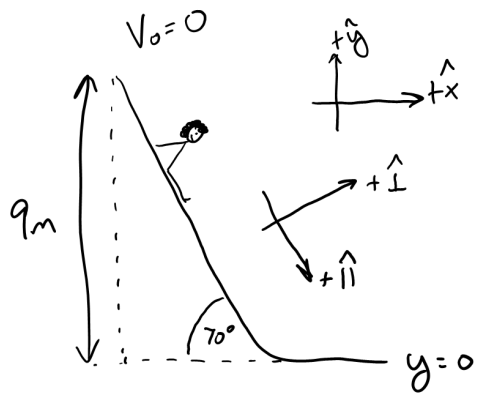


Figure 5.29: A sketch of the rider on the waterslide. $\hat{x}$ & $\hat{y}$ axes are shown, as are $\hat{\parallel}$ & $\hat{\perp}$ axes.[1]

Numbers

Knowns	Unknowns
$m = 50\,\mathrm{kg}$	$\vec{a}$
$\theta = 70°$ above horizontal	v_f
$y_0 = 9\,\mathrm{m}$	Δt
$y_f = 0\,\mathrm{m}$	
$v_0 = 0\,\mathrm{m/s}$	
$F_f = 0\,\mathrm{N}$	

We can use work and energy, Equation 2.1, to easily find the speed at the bottom of the slope. There are no external forces or springs, and no friction to generate thermal energy, so we only need to consider kinetic and gravitational potential energy.

$$E_{k,i} + U_{g,i} = E_{k,f} + U_{g,f}$$

Taking ground level to be $y = 0$...

$$0 + m \cdot g \cdot y_0 = \frac{1}{2} m \cdot v_f^2 + 0$$

$$v_f = \sqrt{2g \cdot y_0} = 13.3\,\mathrm{m/s}$$

If we knew the rider's displacement from the top to the bottom of the steep slope, we could use Equation 5.7 to find her acceleration down the slope. The magnitude of the displacement is the length of the steep slope, which we can find from Figure 5.29 using Equation 5.2:

$$\Delta x = \frac{y_0}{\sin 70°}$$

If there is really no friction on the waterslide, then the only forces affecting the rider are gravity and the normal force. And since the motion is perpendicular to the normal force, the normal force doesn't do any work on the rider. So it is only gravity that affect's the rider's energy. Since energy is conserved, all of the rider's gravitational potential when she is at the top of the waterslide will be converted to kinetic energy when she reaches the bottom. So that means it doesn't actually matter how steep the slope is–the speed at the bottom will be the same if the rider has dropped the same vertical distance.

The amount of time that the rider takes to reach the bottom will depend on how steep the slope is. If the slope is steep then the distance traveled will be short and the acceleration will be large. If the slope is shallow, then the distance traveled will be longer and the acceleration will be smaller, and both of these changes will mean a longer time to the bottom with a more shallow slope.

In our everyday experience, we would say that the rider would be going faster at the bottom if the slope is steep. That is correct, because in everyday life the friction is never really zero, and in fact the frictional force would be smaller when the slope is steeper.

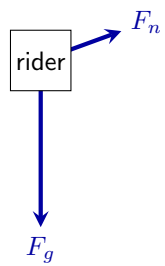


Figure 5.30: FBD of the rider on the waterslide.[1]

Since the acceleration of the rider is in the parallel direction, the net force also has to be in the parallel direction, giving us the relative magnitudes of F_g, F_n, and F_{net}.

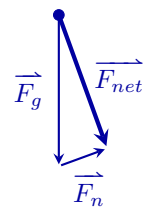


Figure 5.31: Adding the force vectors graphically.[1]

Since the displacement and the acceleration are both in the positive parallel direction...

$$2a_{\parallel} \cdot \Delta x_{\parallel} \cdot \cancel{\cos 0}^{\,1} = v_f^2 - 0^2$$

$$a_{\parallel} = \frac{v_f^2}{2\Delta x_{\parallel}} = \frac{2gy_0}{2\frac{y_0}{\sin 70^\circ}} = g \cdot \sin 70^\circ$$

Which means that the force due to gravity down a slope is given by

$$\boxed{F_{g,\parallel} = m \cdot g \cdot \sin\theta} \tag{5.8}$$

...where the positive parallel direction is down the slope and θ is the angle of the slope above the horizontal.

In the direction perpendicular to the slope, the gravitational force pointing into the slope exactly cancels the normal force out of the slope. Using Figure 5.31 and Equation 5.3...

$$\boxed{F_{g,\perp} = -m \cdot g \cdot \cos\theta} \tag{5.9}$$

...where the positive perpendicular direction is up out of the slope and θ is the angle of the slope above the horizontal.

5.10 Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

General

- For our analyses, the two dimensions being considered must be at right angles to each other.
- The two dimensions can be considered completely independent from each other in terms of forces, momentum, and motion. For example, a force in the vertical direction does not affect motion in the horizontal direction.
- Vectors can be added graphically by combining them "tip to tail," with the resultant vector starting at the tail of the first vector and ending at the tip of the last vector. This provides a good estimate of the resultant vector if the drawing is made and measured carefully.
- Vectors can be added numerically by breaking them up into components and adding the components separately.
- In a problem involving compass directions, it is often convenient to call South "negative North" and West "negative East" while doing calculations, but the final answer should be given in positive values North, South, East, and/or West.
- SouthEast, NorthEast, *etc.* are specific directions exactly halfway between, for example, South and East. "South and East" can refer to any direction between South and East.
- The bearings on a compass are not the same as the standard angles used in mathematics and physics. In physics, angles are normally measured counterclockwise from the positive x axis.
- It is important to sketch a physical situation when dealing with angles, so you have a general idea of the directions. Trusting in a calculator alone to give a correct answer for an angle is risky, because for every trigonometric function there are multiple angles that give the same answer.
- Subtracting a vector is the same as adding the opposite of the vector. The opposite of a vector is another vector with the same magnitude and opposite direction.

Forces

- Force vectors in two dimensions can be added graphically or numerically.
- If a force is applied in a direction that is not in the direction of an object's motion and not opposite the direction the object's motion, the speed of the object may not change, but the direction of its motion will change.
- Tension is a pulling force that is in the direction of the string, rope, chain, *etc.*
- If a rope or other object that is being used to pull is light (often referred to as massless), the tension is the same along the entire length of the rope.

Motion

- The mathematical models that are used to compare relative motion between different objects also work when one object is riding on another.
- Displacement vectors in two dimensions can be added graphically or numerically.
- Velocity vectors in two dimensions can be added graphically or numerically.
- Acceleration vectors in two dimensions can be added graphically or numerically.

Momentum

- Momentum vectors in two dimensions can be added graphically or numerically.
- Momentum bars can be used in two dimensions if separate sets of bars are used for the different directions.

Energy

- Since kinetic energy does not depend on the direction of an object's motion but on its speed, it doesn't matter whether you calculate the kinetic energy using vector components separately or simply using the speed.
- Power is work per time. It is measured in Watts.
- 1 Watt is 1 Joule per second.
- For any real creature or machine, power limits the force that it can produce. Typically a larger force can be applied at low speed and power becomes the limiting factor at high speeds.

Mathematical Models

equation		restrictions on the validity of the equation
$A^2 = A_x^2 + A_y^2$	(5.1)	-none-
$\sin\theta = \frac{A_y}{A}$	(5.2)	calculator may give wrong theta–confirm with a sketch
$\cos\theta = \frac{A_x}{A}$	(5.3)	calculator may give wrong theta–confirm with a sketch
$\tan\theta = \frac{A_y}{A_x}$	(5.4)	calculator may give wrong theta–confirm with a sketch
$P = \frac{W}{\Delta t} = \frac{\vec{F}\cdot\overrightarrow{\Delta x}}{\Delta t} = \frac{F\cdot\Delta x\cdot\cos\theta}{\Delta t}$	(5.5)	only valid when the force is constant
$P = \vec{F}\cdot\vec{v} = F\cdot v\cdot\cos\theta$	(5.6)	only valid when the force is constant
$2a\cdot\Delta x\cdot\cos\theta = v_f^2 - v_i^2$	(5.7)	only valid when the net force is constant
$F_{g,\parallel} = m\cdot g\cdot\sin\theta$	(5.8)	-none-
$F_{g,\perp} = -m\cdot g\cdot\cos\theta$	(5.9)	-none-

5.11 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

Level 1 - Remember

5.1 **[G & N]** Which gives the most accurate result, adding vectors graphically or numerically?

5.2 **[W & G]** Describe the direction that the boat in Figure 5.2 is moving, in terms of compass directions.

5.3 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

Level 2 - Understand

5.4 **[W & N]** Why is it especially important to sketch physical scenarios instead of just trusting a calculator when it comes to angles?

5.5 **[N]** How much thermal energy was generated in the collision described in Section 5.5?

5.6 **[W & G]** Motion maps can be drawn in two dimensions. Create a motion map showing the soccer ball from Section 5.4 from three seconds before the kick to three seconds after the kick.

5.7 **[N]** What are the initial momenta for each of the ice hockey players in Section 5.5, in the earth's reference frame?

5.8 **[G & N]** Equation 5.6 includes an angle θ, but the text doesn't say what the angle is. Based on what you know about Equation 2.3, what would this θ be measured between? Make a sketch that shows the angle θ in relation to the other variables in this mathematical model.

5.9 **[N]** In the calculation of $E_{k,f}$ in Section 5.8, $0°$ is used for θ, but in the "knowns" it says $\theta = 30°$. Is this an error in the text? Explain your answer.

Level 3 - Apply

5.10 **[G & N]** For the physical scenario considered in Section 5.2, the velocity of the boat as seen by the dock is calculated in component form, and the magnitude of the velocity is also found, but the direction is not found. Find the direction.

5.11 **[N]** At the end of Section 5.4, a statement is made that using velocity vector components instead of speed gives the same result when calculating kinetic energy. Verify this statement by calculating the initial and final kinetic energy of the soccer ball in Section 5.4 using the $\hat{N}$ and $\hat{E}$ components of the velocity and comparing to the given result that was calculated using the speed.

5.12 **[G & N]** If everything else stayed the same, what initial speed of the first hockey player in Section 5.5 would have resulted in a final momentum that was due East?

5.13 **[G & N]** If everything else stayed the same, what initial speed of the second hockey player in Section 5.5 would have resulted in a final momentum that was due East?

5.14 **[G]** Create a motion map representing the position of the sled in Section 5.8 as the moose pulls it 50 m.

5.15 **[G & N]** The final velocity and maximum power are not calculated in Section 5.8. Calculate values for them.

Level 4 - Analyze

5.16 **[W, G, & N]** If the river in Section 5.2 were 50 m wide like the river in Section 5.3, how much time would the boat need to cross the river if it were pointed directly South? What is the best direction in which to angle a boat if you want to cross the river as quickly as possible with no regard for the distance the boat travels downstream?

5.17 **[W & G]** First, draw the motion map for the soccer ball from the question in the "Level 2 - Understand" section. Then, explain how the motion map shows the acceleration during the kick.

5.18 **[G & N]** Find the direction of the force that would have to be applied to the soccer ball in Section 5.4 in order for its final velocity to be...

(a) ...due East at 20 m/s.

(b) ...due East at 10 m/s.

(c) ...20 m/s @ $30°$ East of South.

5.19 **[N]** Find the magnitudes and directions of the forces on each hockey player in the collision described in Section 5.5, if the length of time that the two players are crashing into each other is 0.05 seconds.

5.20 **[W, G]** Redraw Figures 5.17, 5.18, and 5.19 for the rope at the following angles, and describe how the magnitude of the tension force in the ropes would be different from that with the rope at the original angle.

(a) $\theta = 75°$ above the horizontal

(b) $\theta = 5°$ above the horizontal

Level 5 - Evaluate

5.21 **[W, G, & N]** According to Section 5.2, for a boat moving at a speed of 5 m/s relative to a river whose speed is 4 m/s, the highest possible speed for the boat as seen from the dock is 9 m/s, and the slowest possible speed is 1 m/s. Is there any direction that the boat could travel such that its speed relative to the water is still 5 m/s, and the speed of the boat relative to the dock is also 5 m/s? If not, explain why not. If so, find the direction graphically or numerically.

5.22 **[W, G, & N]** Describe using words and either graphics or numbers how the acceleration, final energy, and final momentum of the soccer ball in Section 5.4 would have been different if the force applied to the ball during the kick were doubled and the time remained the same.

5.23 **[W, G, & N]** Consider the physical scenario described in Section 5.5. Are the speeds and masses reasonable? Is the assumption about the collision being perfectly inelastic reasonable? Give reasons for your answers.

5.24 **[W]** In Section 5.6, an assumption is made that the mass of the section of rope on which the fiddler is standing is negligible.

(a) Is this a reasonable assumption? Explain your answer.

(b) If a question were asked about the fiddler in which the mass of the entire length of rope were neglected, would that be a reasonable assumption? Explain your answer.

5.25 **[W, G, & N]** Analyze the physical scenario of the moose pulling the sled in Section 5.8 if the mass of the sled were to be cut in half. What affect would this have on the motion, momentum, forces, energy, and power?

5.26 **[W, G, & N]** Analyze the physical scenario of the moose pulling the sled in Section 5.8 if the mass of the sled were to be cut in half and the pulling force of the moose were also cut in half. What affect would this have on the motion, momentum, forces, energy, and power?

5.27 **[W, G, & N]** Is the final speed attained by the moose and sled in Section 5.8 reasonable? Explain your answer. If it is not reasonable, what faulty assumptions or unrealistic starting parameters result in an unreasonable answer?

Level 6 - Create

5.28 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Add the main ideas from this chapter to the concept map that you began for the question at the end of Chapter 1.

5.29 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

5.30 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

Chapter 6

Variable Forces

So far, we have considered only situations where force is constant. That makes life easier, and it allows us to use reasonably simple mathematical models. But the world isn't simple, and now it is time to begin addressing situations where forces change.

In the example of the spring scale shown in Figure 6.1, the distance between the ends of the scale is related to the amount of force applied by the scale. That's what makes it function as a scale–you can read the position to know the amount of force. So with a spring, force is dependent upon displacement.

Another example is frictional force. Friction opposes the relative motion between two objects, so if the direction of motion changes, the direction of the frictional force also changes. If the two objects are not moving relative to each other, the frictional force changes depending on the net force created by all other forces, adjusting to keep the total net force at zero so the objects remain stationary relative to each other.

A third example is gravitational force. We have been using g as the acceleration due to gravity, but that is only true at the surface of the earth. In fact, the force of gravity changes with distance from the center of the earth...and with distance from the center of everything else in the universe.

Figure 6.1: A spring scale, which changes in length as the tension force changes.[24]

6.1 Spring Scale

Words

Springs are physical objects that are usually made from metal because of many metals' ability to return to their original shape after being flexed. Springs can produce force, and they can also store potential energy. For an ideal spring, the force required to stretch or compress it is proportional to the amount of extension or compression. This is referred to as "Hooke's Law." We will consider only ideal springs, which will be a good approximation to springs in the real world in most circumstances. Real springs are more complicated. They do not obey Hooke's Law when the compression or extension is large, and can become permanently deformed if stretched too far.

Let's consider the spring inside the scale shown in Figure 6.2. If the markings represent the mass of an object that is hanging from the hook, measured in kg, and the spacing between each number is 0.01 m, what is the stiffness of the spring, and how much potential energy is stored in the spring if a 5 kg mass is hanging from the hook?

It is important to notice that the numbers on the scale do not refer to the length of the spring. They refer to how far the spring is compressed from its natural length.

Graphics

Figure 6.2: A spring scale and a schematic of the way the spring is connected inside.[25]

The ring at the top of the scale is connected to the orange body of the scale, which is connected to the bottom of the (purple) spring inside. The black hook at the bottom of the scale is connected to a rod that passes up through the spring and is connected to the top of the spring. Pulling down on the hook presses down on the top of the spring.

Numbers

Knowns	Unknowns
0.01 m/kg	k_s
	U_s for 5 kg

k_s is called the spring constant, and it is a measure of the stiffness of a spring. It is defined using Hooke's Law:

$$\boxed{\vec{F_s} = -k_s \cdot \vec{\Delta x}} \tag{6.1}$$

...where $\vec{F_s}$ is the force of the spring acting on whatever is extending or compressing it, k_s is the spring constant, and $\vec{\Delta x}$ is the displacement of the end of the spring from its unstretched and uncompressed "equilibrium" position.

k_s always has a positive value. The minus sign in Equation 6.1 tells us that the force applied by the spring is opposite to the direction of the displacement. For example, if a spring is stretched to the right, it will be pulling to the left.

Hanging one kg from this scale compresses the spring by 0.01 m. We can use this information to find the spring constant with Equation 6.1. The free body diagram in Figure 6.3 shows us that the force of the spring is upward and has the same magnitude as the force of gravity that is acting on the 1 kg mass. We only need to consider the vertical direction.

The stiffness of the spring is measured in terms of its spring constant, in Newtons per meter. A stiffer spring (higher spring constant)will compress less when a force is applied.

If a spring is not stretched or compressed, it cannot do any work, so it has no spring potential energy. But if work is done on the spring to stretch or compress it and none of that work is converted to final kinetic energy, all of the work gets stored as spring potential energy. The farther a spring is stretched or compressed, the more energy it stores. A stiffer spring also stores more energy than a less stiff spring if both are stretched or compressed by the same amount.

An ideal spring also stores the same amount of energy whether it is stretched or compressed, if the amount of extension or compression is the same. Spring potential energy is another type of mechanical energy.

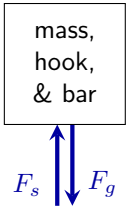


Figure 6.3: FBD of the 1 kg mass, hook, and metal bar when the system is in equilibrium.[1]

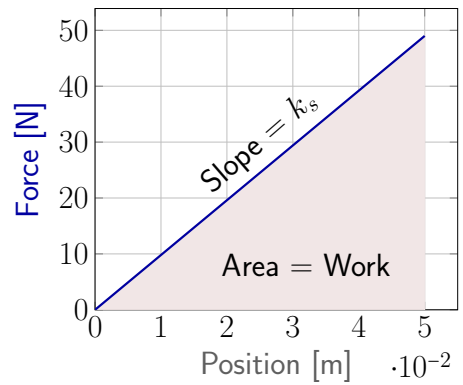


Figure 6.4: As the distance the spring is stretched (or compressed) increases, the force required to stretch (or compress) it also increases in linear fashion.[1]

$$F_{s,y} = m \cdot g = -k_s \cdot (-0.01\ \mathrm{m})$$

$$k_s = \frac{m \cdot g}{0.01\ \mathrm{m}} = \frac{(1\ \mathrm{kg}) \cdot (9.8\ \mathrm{m/s^2})}{0.01\ \mathrm{m}} = 980\ \mathrm{N/m}$$

We have not only found the value for the spring constant; we have also found that the unit for the spring constant is [N/m].

To find energy stored in the spring, we need to consider how much work is used to stretch or compress the spring. Work is done by a force acting over a distance, so we need the distance that the spring is compressed when a 5 kg mass is hung from it. We could use Hooke's law, but it is simpler to note that we found k_s by knowing that the spring compresses by 0.01 m/kg. So a simple ratio tells us the distance:

$$5\ \mathrm{kg} \cdot 0.01\ \mathrm{m/kg} = 0.05\ \mathrm{m}$$

We can use Figure 6.4 to find the work that was done in compressing the spring by finding the area under the curve. It is triangular, so the area is $\frac{1}{2} b \cdot h$.The base is $0.05\ \mathrm{m}$, and the height is the force associated with the maximum distance:

$$F = -F_s = k_s \cdot \Delta x$$

Note that we are using $-F_s$ because we are looking for the force on the spring, not the force caused by the spring. The area under the curve is therefore:

$$W = \frac{1}{2} \cdot \Delta x \cdot k_s \cdot \Delta x = \frac{1}{2} \cdot k_s \cdot \Delta x^2$$

So the potential energy stored in a spring is given by...

$$\boxed{U_s = \frac{1}{2} \cdot k_s \cdot \Delta x^2} \tag{6.2}$$

6.2 Bouncing Ball

Words

The collision between a tennis ball and a concrete driveway was recorded using high-speed video at a frame rate of 240 frames per second. Five consecutive frames are shown from left to right in Figure 6.5. The tennis ball has a mass of 57 g and a diameter of 6.6 cm. Analyze the motion, forces, energy, and momentum involved in this situation.

The ball appears to have taken damage during the collision, but in fact it was thoroughly chewed by a dog before the video! It was not damaged during this collision.

By using the tennis ball as a length scale, we should be able to determine the initial velocity from the first two images and the final velocity from the last two images.

We can also use the distance above the ground to determine the gravitational potential energy in each frame.

We do not know the exact length of the time that collision lasted, the time that the ball was touching the ground, but we can see that the collision had not yet started in the second image and was finished by the fourth image, so the time of the collision was less than 1/120 s.

Graphics

Figure 6.5: Still frames of a tennis ball bouncing off of concrete. The images, left to right, were taken at a rate of 240 frames per second. Note the flattening of the ball in the center frame.[1]

The still frames in Figure 6.5 are shifted compared to each other. The ball is not actually moving to the right. Careful comparison with the background shows that in fact the ball has a small horizontal velocity to the left. For this analysis, the slight horizontal motion is neglected and the ball is assumed to be moving only vertically.

Numbers

Knowns	Unknowns
m =0.057 kg	v_i
d =0.066 m	v_f
t_{frame} = $\frac{1}{240}$ s	$\vec{a}$
$t \leq \frac{1}{120}$ s	$\vec{F}$'s
y_0 = 0.11 m	E's & U's
y_1 = 0.06 m	$\vec{\Delta p}$
y_2 = 0.02 m	
y_3 = 0.05 m	
y_4 = 0.08 m	

t is used for the duration of the collision. The different values for y of the center of the ball have been estimated from the images and the diameter of the ball. There is not significant motion in the $\hat{x}$ direction, so we will restrict our analysis to the $\hat{y}$ direction.

$$v_i = \frac{y_1 - y_0}{t_{frame}} = \frac{(0.06 - 0.11)\,\mathrm{m}\,\hat{y}}{\frac{1}{240}\mathrm{s}} = -12\,\mathrm{m/s}\,\hat{y}$$

$$v_f = \frac{y_4 - y_3}{t_{frame}} = \frac{(0.08 - 0.05)\,\mathrm{m}\,\hat{y}}{\frac{1}{240}\mathrm{s}} = +7.2\,\mathrm{m/s}\,\hat{y}$$

$$\overrightarrow{a_{avg}} = \frac{\overrightarrow{\Delta v}}{\Delta t} = \frac{(+7.2 - (-12))\,\mathrm{m/s}\,\hat{y}}{3 \cdot \frac{1}{240}\,\mathrm{s}} = 1\,540\,\mathrm{m/s^2}\,\hat{y}$$

Looking at the series of images, it is clear that the most significant changes revolve around the time of the collision. The ball is badly misshapen when it is touching the ground, and the direction of motion (and the direction of the momentum) reverses during that time as well. So we will focus our attention on the center frame.

The ball must be pressing down very hard on the concrete at that time, so there must also be a correspondingly large normal force pushing up on the ball from the concrete. This is what is able to change the direction of motion (and therefore also the momentum)of the ball.

What about the energy during that center frame? Before and after the center frame, the ball clearly has kinetic energy. But in the center frame the ball does not. Also before and after the center frame the ball has more gravitational potential energy than in the center frame.

In all of our previous examples, when kinetic and gravitational potential energy were reduced it meant that the energy was converted to thermal energy. And that is partly true here as well. In the last frame the ball apparently has less kinetic energy and less gravitational potential energy than in the first frame. So some was lost to thermal energy. But not all!

The rest of the energy in the center frame was stored as spring (sometimes called elastic) potential energy in the ball. Like a compressed spring, a compressed ball also stores spring potential energy.

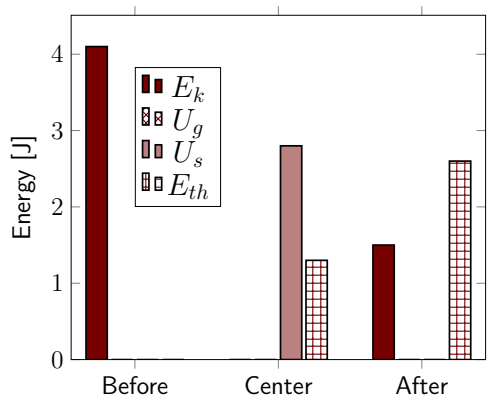


Figure 6.6: Energy bar graph for the collision. Initial thermal energy is taken to be zero. [1]

The series of images provides a good estimate of all types of energy before and after the collision. During the collision, in the center frame, it is clear that kinetic and potential energy should both be at or near zero. But it is not clear for the center frame how the energy should be split between spring and thermal. For this bar graph it has been assumed that the thermal energy for the center frame is roughly half of the thermal energy that is present after the collision.

The time used above for the calculation of acceleration was three frames because v_i was calculated between frames 0 & 1 and v_f was calculated between frames 3 & 4, so three frames later. The acceleration found is so much larger than the acceleration due to gravity that we can safely ignore the effects of gravity, so in fact nearly all of the acceleration occurs between frames 2 & 4. The maximum acceleration is given by

$$\overrightarrow{a_{max}} \geq \frac{\overrightarrow{\Delta v}}{\Delta t} = \frac{(+7.2 - (-12))\ \mathrm{m/s}\ \hat{y}}{2 \cdot \frac{1}{240}\ \mathrm{s}} = 2\,300\ \mathrm{m/s^2}\ \hat{y}$$

Using Equation 1.8 gives us the maximum force during the collision:

$$\overrightarrow{F_{net,max}} = m \cdot \overrightarrow{a_{max}} = 131\ \mathrm{N}\ \hat{y}$$

The initial and final kinetic energy is given by Equation 1.6:

$$E_{k,i} = \frac{1}{2} m \cdot v_i^2 = 4.1\ \mathrm{J}$$

$$E_{k,f} = \frac{1}{2} m \cdot v_f^2 = 1.5\ \mathrm{J}$$

The gravitational potential energy when the ball is at the highest point in these frames is given by Equation 1.5:

$$U_{g,max} = m \cdot g \cdot y_0 = 0.06\ \mathrm{J}$$

Again with gravity, its contribution to energy is so small compared to the other types of energy that it can safely be ignored in this scenario.

6.3 Pushing a Barrel

Words

A group of sailors in Figure 6.7 are attempting to slide a barrel across the deck of an aircraft carrier using water from a hose.

If the force that the water applies to the initially motionless 180 kg barrel starts at zero and slowly increases, the barrel remains in place until the applied force reaches $1\,200\,\mathrm{N}$, at which point it begins to slide with an acceleration of $3\,\mathrm{m/s^2}$.

Find the force of friction between the barrel and the deck of the aircraft carrier over the time that the sailors are spraying water at it.

Before the water is sprayed at the barrel, the barrel is motionless. Since the barrel is motionless, we know that the net force on it has to be zero, so the normal force exactly balances the force of gravity in the vertical direction. In the horizontal direction, there are no forces, so the force of friction at that time is zero.

That sounds wrong, because if you were to push on the barrel the friction would probably be strong enough to prevent you from moving it. But the key here is that phrase, *if you were to push*. If you push to the left, friction will push to the right. If you push to the right, friction will push to the left. Friction will do whatever it has to do to keep the barrel in place. If nothing else tries to push the barrel, friction does nothing!

Graphics

Figure 6.7: Sailors trying to move a barrel using water from a fire hose.[26]

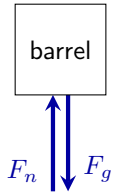


Figure 6.8: FBD of the barrel with no applied force from the water.[1]

Numbers

Knowns	Unknowns
m =180 kg	$\overrightarrow{F_f}$
$g = 9.8\,\mathrm{m/s^2}$	
$0 \le F_{x,applied} \le 1\,200\,\mathrm{N}$	
$\overrightarrow{a_f} = 3\,\mathrm{m/s^2}\,\hat{x}$	
$\overrightarrow{a} = 0$ when $F_{x,applied} < 1\,200\,\mathrm{N}$	

There are two types of frictional force, "static" $\left(\overrightarrow{F_{f,s}}\right)$ when the objects are not moving with respect to each other and "kinetic" $\left(\overrightarrow{F_{f,k}}\right)$ when they are.

The magnitude of the maximum static frictional force between two objects is...

$$\boxed{F_{f,s,max} = \mu_s \cdot F_n} \tag{6.3}$$

...where μ_s is the coefficient of static friction and F_n is the magnitude of the normal force between the two objects.

The magnitude of the kinetic frictional force between two objects is...

$$\boxed{F_{f,k} = \mu_k \cdot F_n} \tag{6.4}$$

...where μ_k is the coefficient of kinetic friction and F_n is the magnitude of the normal force between the two objects.

When the water is sprayed at the barrel, it applies a force to the right. As this applied force increases, the frictional force increases along with it, preventing the barrel from moving...until the force from the water reaches $1\,200\,\mathrm{N}$. Then something changes. The friction is no longer strong enough to hold the barrel in place–we have found the maximum "static" frictional force between the barrel and the deck.

Static frictional force is present when the two objects are not moving with respect to each other. Once they start moving, it is "kinetic" frictional force that takes over. Unlike static fricional force, the magnitude of the kinetic frictional force does not depend upon any other applied forces. Kinetic frictional force is always less than or equal to the maximum static frictional force–typically it is much lower.

Since this situation is on an aircraft carrier, we can also consider the situation where the deck is rising and falling. If the deck is accelerating downward, the normal force will be smaller than the force of gravity; in other words, the "apparent weight" (the normal force required to keep the barrel on the deck) will be smaller. This reduces the force of friction between the surfaces, making the barrel easier to move. On the other hand, if the deck is accelerating upward the apparent weight of the barrel will be larger, increasing the frictional force and making it more difficult to move.

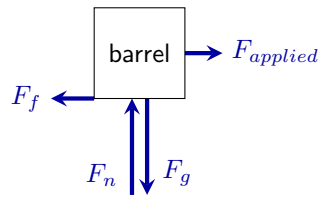


Figure 6.9: FBD of the barrel with 600 N of applied force.[1]

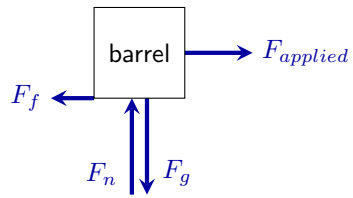


Figure 6.10: FBD of the barrel with 1200 N of applied force.[1]

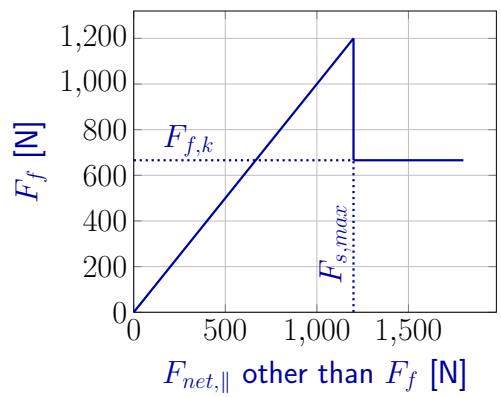


Figure 6.11: Force of friction as a function of net force from other sources parallel to a surface. [1]

In this physical scenario, the magnitude of the normal force is the same as the magnitude of the force of gravity and the maximum static frictional force is $1\,200\,\mathrm{N}$, because when the applied force reaches this value the barrel begins to slide. Using Equation 6.3 we can find μ_s:

$$\mu_s = \frac{F_{f,s,max}}{F_n} = \frac{1\,200\,\mathrm{N}}{180\,\mathrm{kg}\cdot 9.8\,\mathrm{m/s^2}} = 0.68$$

To find μ_k, we need to know $F_{f,k}$, which we can find by applying Equation 1.8 to Figure 6.10.

$$F_{net,x} = F_{applied} - F_{f,k} = m \cdot a_x$$

Solving for $F_{f,k}$ gives 660 N. Then, using Equation 6.4...

$$\mu_k = \frac{F_{f,k}}{F_n} = \frac{660\,\mathrm{N}}{180\,\mathrm{kg}\cdot 9.8\,\mathrm{m/s^2}} = 0.37$$

Notice that there are no units on these coefficients, and that μ_k is considerably smaller than μ_s, which is typical. Tables of μ_k & μ_s for various combinations of surfaces can be found by searching the internet for a "table of coefficients of friction."

6.4 Sledding at White Sands

Words

At White Sands National Monument in New Mexico, USA, the sand is slippery enough that children can slide down the sand dunes on plastic sleds. If the angle of the slope is less than $35°$ from the horizontal, a 25 kg child cannot slide down. But at $35°$ from the horizontal the child can slide down with an acceleration of $2\,\mathrm{m/s^2}$.

What are the static coefficient of friction and the kinetic coefficient of friction for the surface between the sled and the sand? What is the final speed of the child upon reaching the bottom of a 2.5-m-long sand dune if they started at the top with zero speed? What else can we find for this physical scenario?

Coefficients of friction are ratios of the frictional force to the normal force between two surfaces. Frictional force is not dependent upon the surface area. That is important in this example, because the curved bottom of the sleds make it difficult to determine the surface area of the sled that is in contact with the sand.

If surface area doesn't affect friction, why are car tires and most shoes are patterned with treads? The main function of the treads is to allow the shoe or the tire to reach the ground firmly if there is water on the surface. A completely smooth car tire would have the same amount of friction as a treaded tire but would easily go out of control, "hydroplaning" if there were any water on the ground.

Graphics

Figure 6.12: Children sledding at White Sands National Monument.[27]

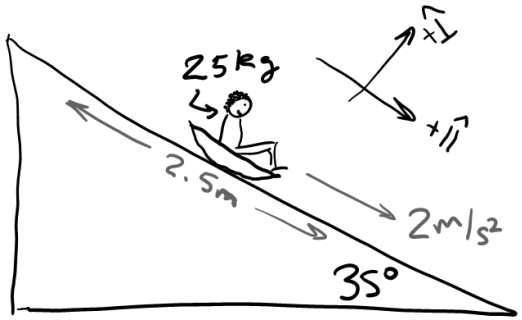


Figure 6.13: Sketch of a child sledding down the sand dune.[1]

Numbers

Knowns	Unknowns
$\theta = 35°$	μ_s
$g = 9.8\,\mathrm{m/s^2}$	μ_k
$\vec{a} = 2\,\mathrm{m/s^2}\,\hat{\parallel}$	v_f
$\Delta x_\parallel = 2.5\,\mathrm{m}$	???
$v_0 = 0$	
$m = 25\,\mathrm{kg}$	

We can find v_f by using Equation 5.7:

$$2a_\parallel \cdot \Delta x_\parallel = v_f^2 - v_i^2$$

$$v_f = \sqrt{2a_\parallel \cdot \Delta x_\parallel} = 3.16\,\mathrm{m/s}$$

By comparing the figures in this section and applying Equation 5.8, it can be seen that the net force down the slope is given by:

$$F_{net,\parallel} = F_{g,\parallel} - F_f = m \cdot g \cdot \sin\theta - F_f$$

Which F_f we use depends on whether the sled is moving or not. Since the angle given is just at the point where the child starts to slide, we can use it to find $F_{s,max}$ and if we include the acceleration given at this angle we can also use it to find $F_{f,k}$. Since $a_\perp = 0$, we can see from the figures in this section that $F_n = F_{g,\perp}$. This is needed to help us find $F_{f,s,max}$ and $F_{f,k}$:

$$F_{f,s,max} = \mu_s \cdot F_n = \mu_s \cdot F_{g,\perp} = \mu_s \cdot m \cdot g \cdot \cos\theta$$

Some shoes, for example golf shoes or crampons that are used for ice climbing, are built with very sharp spikes on the bottom. These do not actually increase the friction with the surface, but increase traction by breaking into the surface, making vertical surfaces where the golfer or climber can apply horizontal normal forces instead of relying on friction.

If we consider the work and energy involved in this scenario, the child starts out not moving at the top of a sand dune, so they have gravitational potential energy but no kinetic energy. As they slide down they are accelerating in the direction of their motion, so they are speeding up, increasing their kinetic energy. They are also losing gravitational potential energy as they go down.

The frictional force is fighting against the child's motion as they go down. A force opposite the direction of motion does negative work on the system, so in this case the friction is doing negative work on the child, removing kinetic energy. That energy is transformed into thermal energy.

So at the bottom of the sand dune the child has kinetic energy but no gravitational potential energy. And thermal energy has also been released in the process, warming the sand and the sled.

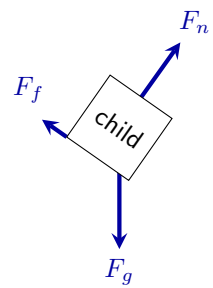


Figure 6.14: FBD of child on sand dune.[1]

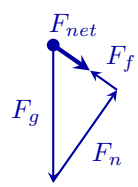


Figure 6.15: Sum of forces for the child.[1]

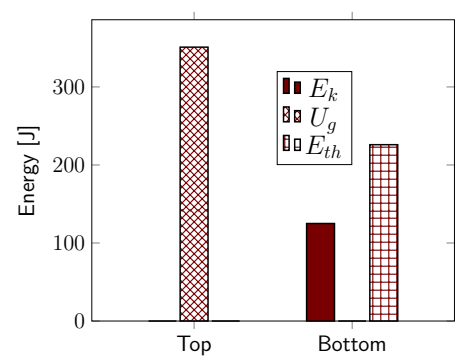


Figure 6.16: Energy bar graph for the child.[1]

Similarly, $F_{f,k} = \mu_k \cdot m \cdot g \cdot \cos\theta$. $F_{net,\parallel} = 0$ when considering μ_s, so combining the equations above...

$$0 = m \cdot g \cdot \sin\theta - \mu_s \cdot m \cdot g \cos\theta$$

Solving for μ_s and using the trigonometric identity that $\tan\theta = \frac{\sin\theta}{\cos\theta}$...

$$\mu_s = \tan\theta = 0.70$$

Following the same steps for μ_k and using Equation 1.8 since the sled is accelerating gives...

$$m \cdot a_\parallel = m \cdot g \cdot \sin\theta - \mu_k \cdot m \cdot g \cos\theta$$

$$\mu_k = \frac{g \cdot \sin\theta - a_\parallel}{g \cdot \cos\theta} = 0.45$$

We can also consider the energy transformations as the child sleds down the dune. $E_{k,i} = 0$; there are no external forces; no springs, so we don't need to consider U_s; if we set $y = 0$ at the bottom of the slope then $U_{g,f} = 0$; and we can also use $E_{th,i} = 0$. Then Equation 2.1 gives...

$$\cancelto{0}{E_{k,i}} + \cancelto{0}{U_{s,i}} + U_{g,i} + \cancelto{0}{E_{th,i}}$$
$$= E_{k,f} + \cancelto{0}{U_{s,f}} + \cancelto{0}{U_{g,f}} + E_{th,f}$$

We can find $E_{k,f}$ from v_f and $U_{g,i}$ from Figure 6.13 using Equation 5.2. Putting these into the equation above gives...

$$E_{th,f} = U_{g,i} - E_{k,f}$$

$$E_{th,f} = m \cdot g \cdot \left(\Delta x_\parallel \cdot \sin\theta\right) - \frac{1}{2} m \cdot v_f^2 = 226\ \mathrm{J}$$

6.5 The Truth About Gravity

Words

So far, we have considered gravity to be an acceleration with a magnitude of $g = 9.8\,\mathrm{m/s^2}$, caused by a force that is always pointing downward.

That's not actually true. It is a good approximation of the force of gravity on the surface of the earth, which is where most of us will probably spend most of our lives. So it is an approximation that works well in many situations that we will face. But when you leave the surface of the earth, going up or down, that model doesn't work any more.

We will now consider the force of gravity acting on a 1 kg mass that is placed either at the exact center of the earth or one earth radius above the surface of the earth.

At the earth's surface, the force of gravity from the earth acting on a 1-kg mass is 9.8 N, pointing downward. But what if you were at the center of the earth? There is no longer a "downward" direction! Imagine the earth being two pieces, the Southern and Northern hemispheres. At the center of the earth, the gravitational force caused by the Southern hemisphere would pull the mass toward the South pole, but the Northern hemisphere would pull with an equal but opposite force toward the North pole. These forces cancel, so we can use the symmetry of the situation to show that the earth's gravitational force acting on a mass at the center of the earth is zero. Symmetry arguments like this often provide helpful insights into physical situations.

Graphics

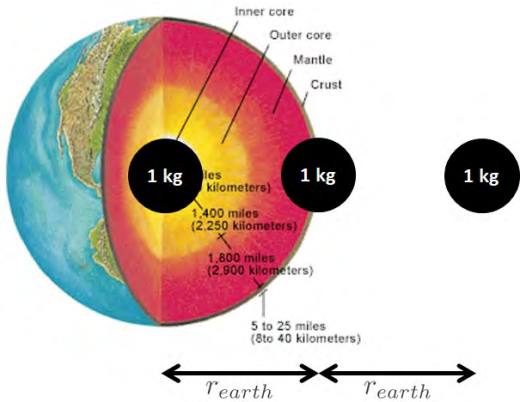


Figure 6.17: Cutaway of the earth, with a 1 kg mass shown at three different positions.[28]

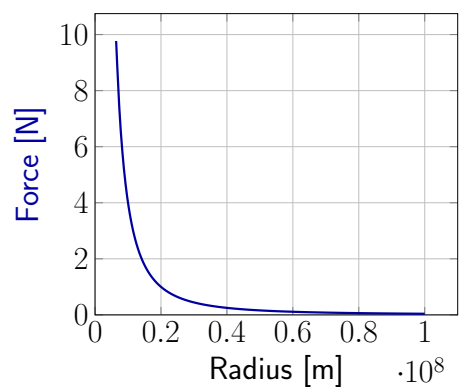


Figure 6.18: Magnitude of gravitational force between the earth and a 1 kg mass, as a function of distance.[1]

Numbers

Knowns	Unknowns
$m = 1\,\mathrm{kg}$	$\vec{F_g}$ at position r_0
$r_0 = 0$	$\vec{F_g}$ at position $\vec{r_1}$
$r_1 = 2r_{earth}$	

In the table above, radii "r" are measured from the center of the earth to the center of the 1 kg mass. The $+\hat{r}$ direction is directly away from the center of the earth.

For any location other than the surface of the earth, the mathematical model $\vec{F_g} = -\,m \cdot g\,\hat{y}$ does not work well. Instead, we need to use Newton's Law of Universal Gravitation:

$$\boxed{F_g = \frac{G \cdot m_1 \cdot m_2}{r^2}} \tag{6.5}$$

...where G is the Universal gravitation constant $6.67 \times 10^{-11}\,\frac{\mathrm{N{\cdot}m^2}}{\mathrm{kg^2}}$, m_1 & m_2 are the masses of two spherical objects between which the force is acting, and r is the distance between the centers of the two objects. Gravitational force is always attractive, so the force on each object is directed toward the other object. We can use Equation 6.5 to find the magnitude of the force when the 1 kg mass is one radius above the surface of the earth, so at a distance of two earth radii from the center of the earth.

$$F_g = \frac{\left(6.67 \times 10^{-11}\,\frac{\mathrm{N{\cdot}m^2}}{\mathrm{kg^2}}\right) \cdot \left(5.97 \times 10^{24}\,\mathrm{kg}\right) \cdot \left(1\,\mathrm{kg}\right)}{\left(2 \cdot 6.37 \times 10^{6}\,\mathrm{m}\right)^2}$$

As we move from the center of the earth to the surface of the earth, the force of gravity on our 1 kg mass increases smoothly, reaching a force of 9.8 N at the surface of the earth.

Above the surface, gravity follows an "inverse square law," meaning that the strength of the force is inversely proportional to the square of the distance from the earth's center. At *one* earth radius above the *surface* of the earth, we are *two* earth radii from the *center* of the earth. Our distance has doubled, so the force of gravity is reduced by one fourth $\left(\frac{1}{2^2}\right)$.

What about gravitational potential energy? Work is required to lift a 1 kg mass up away from the center of the earth, regardless of the distance from the center of the earth. So gravitational potential energy is at a minimum at the center of the earth and increases as we move away from the center.

These descriptions are valid for anything that has mass; the gravitational force caused by anything is proportional to its mass.

If you feel betrayed because of the lies this book made you believe about gravity, it gets worse. The description of gravity in this section is consistent with 19th century understanding. In the 20th century, gravity began to be understood as a curvature of space-time. And in the 21st century perhaps quantum gravity will give us an entirely new and better model for gravity.

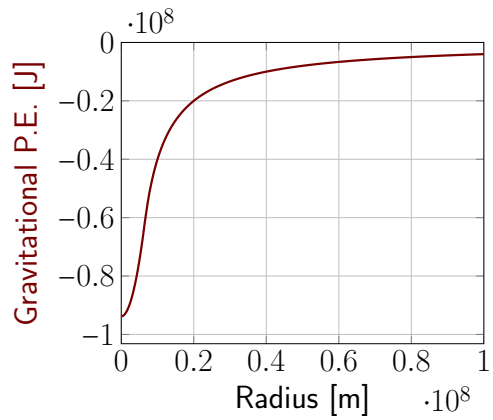


Figure 6.19: Gravitational potential energy of the earth and a 1 kg mass, as a function of distance.[1]

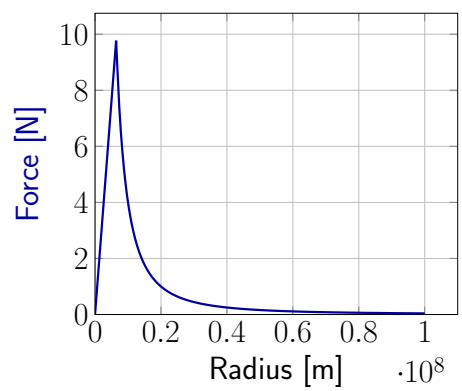


Figure 6.20: Magnitude of gravitational force between the earth and a 1 kg mass, as a function of distance, including when the mass is inside the earth.[1]

$$\vec{F_g} = -2.45\,\mathrm{N}\,\hat{r}$$

Let's try Equation 6.5 again to find the magnitude of the force at the center of the earth.

$$F_g = \frac{\left(6.67\times10^{-11}\,\frac{\mathrm{N\cdot m^2}}{\mathrm{kg^2}}\right)\cdot\left(5.97\times10^{24}\,\mathrm{kg}\right)\cdot\left(1\,\mathrm{kg}\right)}{\left(0\,\mathrm{m}\right)^2}$$

$$F_g = \infty\,\mathrm{N}$$

An infinitely large force! That cannot be correct. This model only works outside of the mass distribution of the objects themselves, so not, for example, inside the earth.

Along with this new model of gravitational force, we have a new expression for gravitational potential energy, which is also only valid outside of the mass distributions of the objects themselves:

$$\boxed{U_g = \frac{-G\cdot m_1\cdot m_2}{r}} \tag{6.6}$$

Notice that with this expression we no longer have the option of choosing our height where $U_g = 0$. Instead, $U_g = 0$ at $r = \infty$, and U_g is negative everywhere else.

Inside a uniform mass distribution, the magnitude of the force is smaller than the value calculated from Equation 6.5, increasing linearly with radial distance. The gravitational potential energy inside a mass distribution is higher than the value calculated from Equation 6.6.

6.6 Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

General

- The stiffness of a spring is defined by its spring constant k, which always has a positive value and is measured in units of Newtons per meter.

Forces

- Ideal springs follow Hooke's Law, where force is proportional to the amount of extension or compression.
- The force of friction between two surfaces depends upon the surfaces themselves and the normal force that the surfaces exert on each other.
- If two surfaces are motionless with respect to each other, the static force of friction between them will be exactly enough to cancel out all other forces in the direction parallel to the surfaces, unless the sum of all of the other forces is larger than the maximum static force of friction, in which case the surfaces begin to move with respect to each other.
- If two surfaces are moving with respect to each other, the kinetic force of friction between them will be in the direction opposing the motion.
- If the floor on which an object sits is accelerating upward, the apparent weight of the object as measured by the normal force is larger than if the floor were stationary or moving at constant velocity.
- If the floor on which an object sits is accelerating downward, the apparent weight of the object as measured by the normal force is smaller than if the floor were stationary or moving at constant velocity.
- Frictional force does not depend upon the contact area of the surfaces.
- The gravitational force between two objects is proportional to the masses of the objects and inversely proportional to the square of the distance between their centers.
- Gravitational force is always attractive.
- Gravitational force drops to zero as the radius gets smaller if one object is inside the mass distribution of the other object.

Motion

- A position vector can be defined in a radial direction, along a radius from a center point. The positive direction is away from the center point.

Energy

- Stretched or compressed springs store spring (also called elastic) potential energy.
- Spring potential energy is a form of mechanical energy.
- Deformed objects can store spring potential energy.
- Gravitational potential energy of two objects increases as the distance between the objects increases.

Mathematical Models

equation		restrictions on the validity of the equation
$\overrightarrow{F_s} = -k_s \cdot \overrightarrow{\Delta x}$	(6.1)	For objects that obey Hooke's Law
$U_s = \frac{1}{2} \cdot k_s \cdot \Delta x^2$	(6.2)	For objects that obey Hooke's Law
$F_{f,s,max} = \mu_s \cdot F_n$	(6.3)	-none-
$F_{f,k} = \mu_k \cdot F_n$	(6.4)	-none-
$F_g = \frac{G \cdot m_1 \cdot m_2}{r^2}$	(6.5)	Outside of the mass distribution of spherical objects.
$U_g = \frac{-G \cdot m_1 \cdot m_2}{r}$	(6.6)	Outside of the mass distribution of spherical objects.

6.7 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

Level 1 - Remember

6.1 **[W]** The kinetic force of friction is only relevant if the two surfaces are _____ relative to one another.

6.2 **[W]** In what direction does the kinetic force of friction act?

6.3 **[W]** The static force of friction is only relevant if the two surfaces are _____ relative to one another.

6.4 **[W]** In what direction does the static force of friction act?

6.5 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

Level 2 - Understand

6.6 **[G]** Draw free body diagrams for the tennis ball in the left, center, and right frames of Figure 6.5.

6.7 **[W, G, & N]** Find the momentum of the child at the top and bottom of the sand dune in Section 6.4. Momentum is conserved for an isolated system, so if the child's momentum changed, what caused that change?

Level 3 - Apply

6.8 **[G & N]** The potential energy stored in the spring in Section 6.1 is never actually calculated. What is it?

6.9 **[W & N]** Find the change in the momentum of the tennis ball during the collision in Section 6.2. Use the change in momentum to find the force applied to the tennis ball during the collision. Does it agree with the $\overrightarrow{F_{net,max}}$ that is found in the text? Explain why or why not.

6.10 **[W & N]** If the mass of the barrel in Section 6.3 were doubled, what affect would that have on the maximum static frictional force and the kinetic frictional force?

6.11 **[W, G, & N]** Now that we have Newton's Universal Law of Gravitation, does that mean that we can no longer use $9.8\,\mathrm{m/s^2}$ as the acceleration due to gravity at the earth's surface? Explain why or why not.

Level 4 - Analyze

6.12 **[W, G, & N]** If the deck of the ship in Section 6.3 started accelerating downward at $1\ \mathrm{m/s^2}$ because of stormy seas...

(a) ...how would that affect the horizontal acceleration of the barrel if it had already started moving?

(b) ... how would that affect the static force of friction if the barrel was not yet moving and there was no applied force in the horizontal direction?

(c) ... how would that affect the static force of friction if the barrel was not yet moving and the applied force in the horizontal direction was 200 N?

(d) ... what would happen if the barrel was not yet moving and the applied force in the horizontal direction was $1\,100\,\mathrm{N}$?

6.13 **[W & N]** Given the physical scenario described in Section 6.4, is there an angle at which a child would go down the slope at constant speed if they were given an initial push to start them moving? Explain why or why not. If it is possible, find the angle.

Level 5 - Evaluate

6.14 **[G & N]** If the spring constant were doubled and the same mass was hung from the scale in Section 6.1, how would that affect the amount of energy stored in the spring? Explain your reasoning.

6.15 **[G & N]** If the mass were doubled and the spring constant was kept the same in Section 6.1, how would that affect the amount of energy stored in the spring? Explain your reasoning.

6.16 **[W, G, & N]** The height of the bars in the "Center" position of Figure 6.6 are not well defined just from examining Figure 6.5. What are the maximum and minimum possible heights for the bars representing spring potential energy and thermal energy?

6.17 **[W & N]** In Section 6.4 the coefficients of friction are found. Given those coefficients of friction, if the child's mass were doubled, how would that affect the angle at which the maximum static frictional force can no longer keep the child from sliding down the sand dune? Explain your reasoning.

6.18 **[W & N]** In Section 6.4 the coefficients of friction are found. Given those coefficients of friction, if the child's mass were doubled, how would that affect the acceleration of the child as they slid down the 35° slope? Explain your reasoning.

Level 6 - Create

6.19 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Add the main ideas from this chapter to the concept map that you began for the question at the end of Chapter 1.

6.20 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

6.21 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

Chapter 7

Curving Paths

We have already learned that if an object experiences a net force in the direction in which it is already moving, the force does positive work on the object, increasing its kinetic energy, its momentum, and also its speed. Conversely, if an object experiences a net force opposite the direction in which it is already moving, the force does negative work on the object, decreasing its kinetic energy, its momentum, and also its speed.

Now it is time to start looking at forces that are pointing in directions that are not parallel to the motion of an object. When that happens, the object follows a curved path.

Looking at the image of the fire dancer, try to imagine the forces that are involved. The dancer is holding two flaming balls that are hanging from chains. She spins them around, making intricate patterns in the air. The flaming balls experience forces due to both gravity and the tension in the chains. How do those combined forces make the balls move in such complicated patterns?

What is happening to the momentum of the balls as they follow these curved paths?

Figure 7.1: A fire poi dance. The flaming ball on the near side of the dancer traces out a circular path, while the ball on the far side of the dancer follows a more complicated path.[29]

7.1 Cliff Diving

Words

Figure 7.2 shows a diver jumping horizontally off of the edge of a cliff. After leaving the cliff, the diver's body is in free fall, affected only by the force of gravity. Because of the initial horizontal velocity and the vertical acceleration, the diver's body follows a curved path.

If the camera took four images per second and the diver jumped horizontally at 3 m/s, for how much time was the diver in the air before hitting the water, what is the height of the cliff, at what speed did the diver enter the water, and what was the horizontal displacement of the diver over that time?

The key to understanding this situation is realizing that the horizontal direction is independent of the vertical direction. This idea was explored in Chapter 5, where the focus was on velocity, momentum, and forces. The same principle applies to position as well.

The force of gravity pulls the diver down into the water in exactly the same amount of time as if she had fallen straight down into the water from the same height. This is perhaps most easily understood by thinking of it in terms of frames of reference. If you are standing on a motionless train and you hold a ball straight out and drop it to the floor, the ball will be in the air for a specific amount of time before hitting the floor.

If you are on the same train, holding the same ball and dropping it in the same way, but the train is

Graphics

Figure 7.2: A cliff diver jumps horizontally off of a cliff. The individual images of the diver are at equal time intervals.[30]

Numbers

Knowns	Unknowns
$\vec{v_0} = -3\,\mathrm{m/s}\,\hat{x}$	t_{tot}
$t_{image} = 0.25\,\mathrm{s}$	h_{cliff}
$g = 9.8\,\mathrm{m/s^2}$	Δx

If we use the image where the diver has one foot on the cliff as $t = 0$, and subsequent images are at 0.25-s intervals, then the splash into the water occurs at

$$t_{tot} = 6 \cdot t_{image} = 1.5\,\mathrm{s}$$

To find the height of the cliff, we can separate Equation 1.2 into its $\hat{x}$ & $\hat{y}$ components:

$$x = x_0 + v_{0x} \cdot t + \frac{1}{2} a_x \cdot t^2$$

$$y = y_0 + v_{0y} \cdot t + \frac{1}{2} a_y \cdot t^2$$

In free-fall, the only force acting on an object is gravity, so as long as the object is near the surface of the earth $\vec{a} = -g\,\hat{y}$. So the horizontal and vertical position of an object in free-fall can be described by

$$\boxed{x = x_0 + v_{0x} \cdot t} \tag{7.1}$$

and

$$\boxed{y = y_0 + v_{0y} \cdot t - \frac{1}{2} g \cdot t^2} \tag{7.2}$$

Note that the only connection between the $\hat{x}$ & $\hat{y}$ directions is the time. Often questions about

moving at a constant velocity, from your reference frame the ball will behave in exactly the same way. And it will hit the floor in exactly the same amount of time.

In the reference frame of somebody watching the train go by, the ball starts in your hand not at rest but moving with the same horizontal velocity as the train. When you drop the ball, it will continue to move horizontally along with the train, but it will drop vertically. Each person sees the ball taking a different path in their reference frame, but both see the ball in the air for the same amount of time.

The horizontal displacement that the ball travels while falling in the reference frame of the person on the train is zero, because the horizontal velocity is zero in that reference frame. But in the reference frame of the person watching the train go by, the horizontal displacement of the ball would be the velocity of the train multiplied by the time that the ball is in the air, since velocity is displacement over time.

In Figure 7.3, the horizontal spacing of the images are almost equally spaced–the slight decrease in spacing on the left is most likely due to the angle of the camera. The vertical spacing, on the other hand, increases with each successive image. This shows acceleration in the vertical direction, due to the force of gravity. It is also clear from the lengths of the arrows that the diver's speed is increasing during the fall.

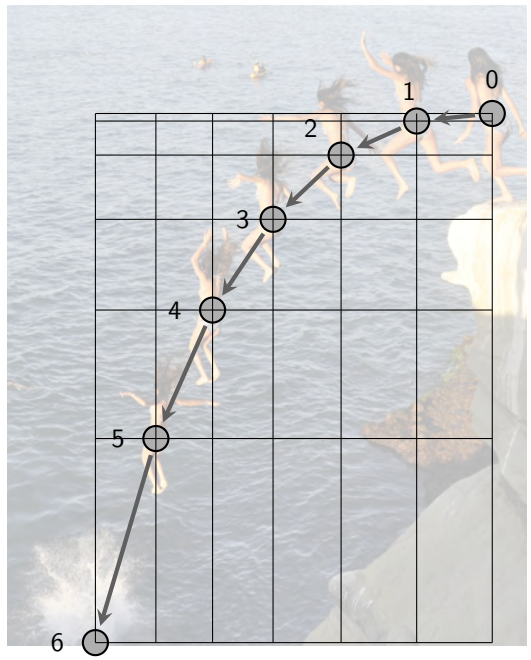


Figure 7.3: 2-D motion map of the diver[31]

two-dimensional physical scenarios can be answered by using information known about one direction to solve for time, and then using that time to solve for information about the other direction. In this case we are given the time and can use it to solve for information about both directions.

The height of the cliff is the opposite of the displacement in the y direction, $y_0 - y$, since the diver starts at the top and ends at the bottom. So...

$$h_{cliff} = \frac{1}{2} g \cdot t_{tot}^2 - \cancelto{0}{v_{0y} \cdot t_{tot}} = 11\,\mathrm{m}$$

The diver's horizontal displacement comes from Equation 7.1:

$$\Delta x = v_{0x} \cdot t = -4.5\,\mathrm{m}$$

Now we can find the speed of the diver by calculating $v_{x,f}$ and $v_{y,f}$ and using Equation 5.1. There is no horizontal acceleration, so $v_{x,f} = v_{0x}$. In the vertical direction, using Equation 2.2...

$$v_{f,y} = v_{0y} + a_y \cdot t_{tot} = 0 - g \cdot t_{tot} = -14.7\,\mathrm{m/s}$$

$$v_f = \sqrt{v_{f,x}^2 + v_{f,y}^2} = 15\,\mathrm{m/s}$$

7.2 Basketball Bounce

Words

Figure 7.4 shows a ball bouncing, with images captured using a stroboscopic light source that flashed 25 times per second. We will also assume that the photograph was taken somewhere near the surface of the earth.

We will analyze the photograph using all of our tools to see what we can learn just from the images.

This photograph looks like a 2-D map of the motion of the ball, but it doesn't contain arrows like a normal motion map. Can we determine which direction the ball is moving? The biggest indicator is the height of the peaks. The first peak is much higher than the second, and the speed of the ball when at the peaks looks like it is probably about the same for each peak.

So the ball has more mechanical energy at the top of the left peak than it has at the top of the right peak. Most likely some of the initial energy was transformed to thermal energy during an inelastic collision with the floor, so the peak on the left must be the first one, and the ball is moving to the right.

The image of the ball on the far right is considerably smaller than the image of the ball on the left, so it must also be moving away from the camera. That means any measurements of angles or distances will not be exact.

Graphics

Figure 7.4: This single photograph of a bouncing ball was taken using a stroboscopic light source that flashed 25 times per second .[32]

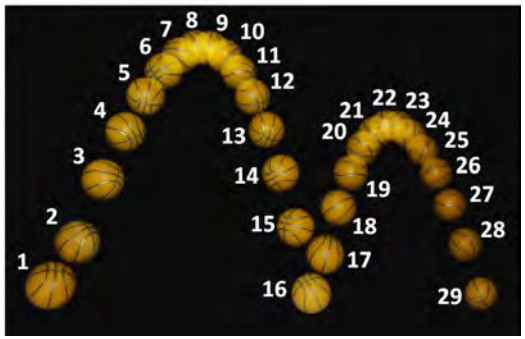


Figure 7.5: The same photograph as above, with the images numbered for reference.[32]

Numbers

Knowns	Unknowns
$t_{image} = 0.04\,\mathrm{s}$	???
$g = 9.8\,\mathrm{m/s^2}$	

We can begin by counting the images in each bounce. For the bounce on the left, there are 15 spaces between the ball, so...

$$t_{left} = 15 \cdot t_{image} = 0.6\,\mathrm{s}$$

Similarly, $t_{right} = 0.52\,\mathrm{s}$.

We can use this "time of flight" for each bounce to find initial and final vertical velocities and heights. If we measure the angles we can also find information about the horizontal direction. In this particular situation, when the initial and final heights are the same and the object is in free-fall, there are three "equations of projectile motion" that can be used. These are all derived from the equations of motion that we have already been using.

$$\boxed{t_{flight} = \frac{2v_0 \cdot \sin\theta}{g}} \tag{7.3}$$

$$\boxed{h_{max} = \frac{v_0^2 \cdot \sin^2\theta}{g}} \tag{7.4}$$

$$\boxed{R = \frac{v_0^2 \cdot \sin(2\theta)}{g}} \tag{7.5}$$

The path of motion that the ball follows is in the shape of a parabola. This is the expected shape whenever an object is in free-fall near the surface of the earth. Other names for this type of motion are "projectile motion" and "ballistic motion."

If we negect air resistance, which is typically very small at low speeds, then the only force that acts on the ball while it is in the air is the force of gravity, acting downward. When the ball is moving upward, gravity is doing negative work on the ball, slowing it down. When the ball is moving downward, gravity is doing positive work on the ball, speeding it up.

When the ball hits the floor, the ball applies a large normal force downward onto the floor and the floor pushes up on the ball with an equally large normal force. During the collision with the floor, the vertical part of the velocity of the ball changes drastically, from a large downward speed to a large upward speed. Looking at the horizontal direction, however, shows that the horizontal velocity did not change much. The ball continues to move to the right at a fairly uniform rate for the whole time that it was being imaged.

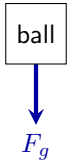


Figure 7.6: FBD of the ball for any time when it is in the air, images 2-15 and 17-28 in Figure 7.5.[1]

Figure 7.7: FBD of the ball when it is in contact with the ground, images 16 & 29 in Figure 7.5. Lengths of the arrows are not exact, but are intended to signify that F_g is the same as in Figure 7.6 and that $F_n > F_g$.[1]

...where θ is the angle between v_0 and the horizontal direction, approximately $70°$ in this case for both bounces; t_{flight} is the total time in free-fall; h_{max} is the vertical distance from the initial position to the peak; and R is the horizontal range. It is important to remember that these are not "magic" equations that always give correct answers when you don't know what to do. They are the equations of motion in the specific situation of free-fall near the surface of the earth when initial height is equal to final height. They will not work in any other situation.

Using these equations, we find:

	Left Bounce	Right Bounce
v_0	$3.1\,\mathrm{m/s}$	$2.7\,\mathrm{m/s}$
h_{max}	$0.88\,\mathrm{m}$	$0.66\,\mathrm{m}$
R	$0.64\,\mathrm{m}$	$0.48\,\mathrm{m}$

Some of these numbers are surprising, if you know something about the size of a basketball. The range for the left bounce is 0.64 m, and appears in the photograph to be roughly 6 diameters of the ball, so the ball diameter is between 10 and 11 cm, half the size of a basketball! In fact, this is not a real basketball in the photograph, but a child's toy ball that is made to look like a basketball.

Since we don't know anything about the mass of the toy ball, we cannot calculate force, momentum, or energy.

7.3 At the Peak

Words

Let's take a closer look at what is happening just at the very peak of the flight of the ball from Section 7.2. We know that the path followed by the ball is parabolic in shape. At the top of the parabola, the motion of the ball is horizontal.

At every point on the parabola, there is a constant net force, which is simply the gravitational force, pointing downward. As a result, at every point on the parabola the ball is constantly accelerating downward. And since acceleration is a change in velocity over time, the velocity of the ball is changing at a constant rate everywhere on the parabola.

The ball is moving upward on the left side of the peak, and since the acceleration is downward that means it is slowing down. We can say that the gravitational force is doing negative work, decreasing the kinetic energy as the ball is moving upward. The ball is moving downward on the right side of the peak, and since the acceleration is downward that means it is speeding up. We can say that the gravitational force is doing positive work, increasing the kinetic energy as the ball is moving downward.

When the ball is right at the peak, its velocity is perpendicular to the net force, so no work is being done on it, positive or negative. That means its *speed is not changing*. But, it is still accelerating the same as everywhere else, so the *velocity is changing*!

Graphics

Figure 7.8: A close-up of the top of the first peak for the bouncing ball from Section 7.2.[32]

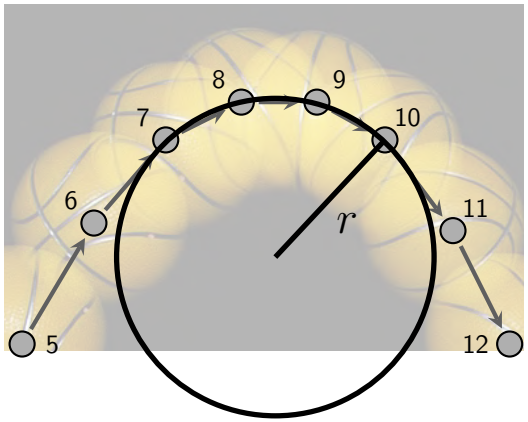


Figure 7.9: At the very peak of the ball's motion, the path briefly follows a circular path with radius of curvature r.[32]

Numbers

Knowns	Unknowns
$t_{image} = 0.04\,\mathrm{s}$	???
$g = 9.8\,\mathrm{m/s^2}$	

The work that is done on the ball at the top of the parabola is given by Equation 2.3:

$$W_{net} = F_{net} \cdot \Delta x \cdot \cos\theta = F_g \cdot \Delta x \cdot \cos 90^\circ$$

$\cos 90^\circ = 0$, so...

$$W_{net} = 0$$

...at the top of the parabola where the force is perpendicular to the direction of motion. Then Equation 2.3 tells us that...

$$W_{net} = \Delta E_k = 0$$

...so we also know that the speed v is constant at the top of the parabola. And yet the acceleration due to gravity is constant everywhere on the parabolic arc, so $\vec{v}$ is changing. The only way to change $\vec{v}$ while keeping v constant is to change the direction of $\vec{v}$.

How can the velocity change while the speed stays the same? Velocity is a vector, which includes speed and direction. When the net force (and therefore the acceleration) is perpendicular to the velocity, it causes a change in direction but not a change in speed. The path curves along a circular arc with a radius that is called the "radius of curvature."

This perpendicular net force is called the "centripetal" force, from the Latin words for "toward the center." The centripetal force is not a new type of force; it is simply a name for whatever force is causing an object to follow a curved path. In the case of the ball at the top of the arc, the centripetal force is the force of gravity. In other situations it could be a tension force, a friction force, a normal force, a combination of these, or any other type of applied force.

Imagine now a force that changes direction as the velocity changes direction. If a constant net force were kept always perpendicular to the direction of motion, the object would move in a complete circle. This is what happens, for example, in an Olympic hammer throw. The "hammer," a heavy ball on the end of a flexible cord, is spun in a circle. The tension in the cord supplies the centripetal force radially inward, and the velocity of the ball is in the "tangential" direction, perpendicular to the radius.

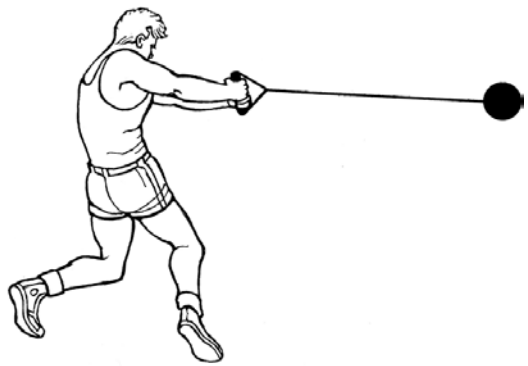


Figure 7.10: An Olympic hammer throw.[34]

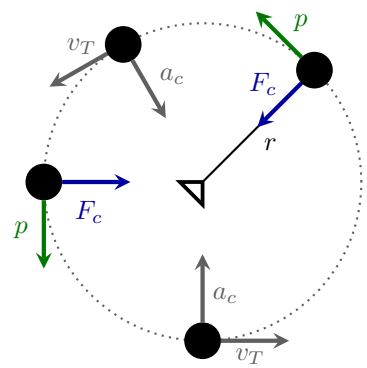


Figure 7.11: Looking down on the hammer throw. The ball is following the dotted line counterclockwise. $\vec{F_c}$, $\vec{a_c}$, $\vec{p}$, & $\vec{v}$ are shown at various positions.[1]

The radius of curvature r in Figure 7.9 depends on two things: the acceleration due to gravity and the horizontal speed at the top. The radius would increase if gravity were weaker, allowing the ball to stay up longer, and the radius would also increase if the speed were higher, allowing the ball to move farther horizontally in the time needed for gravity to pull the ball down.

The magnitude of the centripetal acceleration a_c, the tangential (perpendicular to the radius) velocity v_T, and the radius of curvature r are related by:

$$\boxed{a_c = \frac{v_T^2}{r}} \tag{7.6}$$

In Section 7.2, the speed and initial direction of the ball were found, which give us the horizontal velocity using Equation 5.3. The horizontal velocity is the tangential velocity at the top of the parabola, so we can use it to find the radius of curvature.

$$r = \frac{v_T^2}{a_c} = \frac{(v_0 \cdot \cos\theta)^2}{g} = 0.12\,\mathrm{m}$$

That is roughly the diameter of the ball, and in Figure 7.9 we can see that in fact the radius of curvature is very close to the diameter of the ball. Making this type of comparison can give us confidence that our work is correct. Using the relationship between force and acceleration given by Equation 1.8, we could also find the centripetal force if we knew the mass:

$$\boxed{F_c = m \cdot a_c = \frac{m \cdot v_T^2}{r}} \tag{7.7}$$

7.4 Earth's Orbit

Words

The earth orbits the sun at a distance of approximately 150 million kilometers, and it takes approximately 365 days to make one complete orbit.

Use this information to find the speed of the earth as it orbits around the sun and also the mass of the sun and any other information that can be found.

The speed of the earth can be found by considering that the earth makes one complete circuit around the sun every year. The speed of the earth is simply the path length traveled around the circular path divided by one year.

It seems surprising to think that we would not need to know the mass of the earth in order to find the mass of the sun for this question, but the force of the earth's gravity gives objects at the earth's surface a constant acceleration regardless of their mass. Just as the mass of the earth is much larger than the mass of anything on the surface of the earth, the mass of the sun is much larger than the mass of the earth. So it should make sense that the sun's gravitational force causes a constant acceleration at a given distance, at least as long as the other object's mass is much less than the mass of the sun.

Graphics

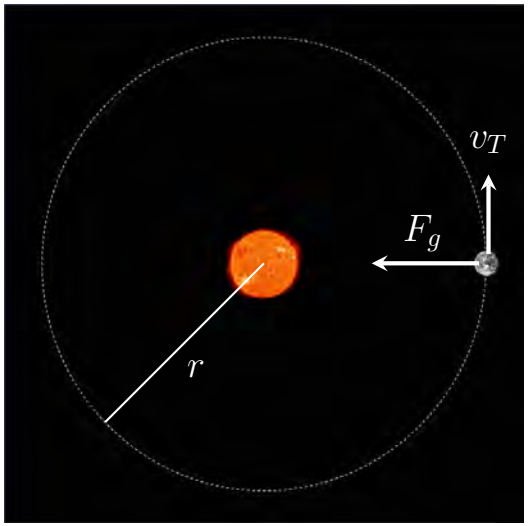


Figure 7.12: The earth's orbit around the sun is nearly circular.[35]

Note that the earth's speed is nearly constant as it orbits the sun; but its velocity, which includes direction, is always changing. It is more convenient to use *angular* velocity when describing a situation like this, because the earth's angular velocity, like its speed, is nearly constant.

Numbers

Knowns	Unknowns
$r = 1.5 \times 10^{11}$ m	v_T
$t_{orbit} = 3.15 \times 10^7$ s	m_{sun}
	???

In 365 days, the earth orbits the sun once, a path length that is the circumference of a circle whose radius is the distance from the earth to the sun:

$$s_{orbit} = 2\pi \cdot r = 9.4 \times 10^{11} \text{ m}$$

To find the tangential speed of the earth in its orbit we need to consider the path length traveled:

$$v_T = \frac{s_{orbit}}{t_{orbit}} = 3 \times 10^4 \text{ m/s}$$

Now that we know the tangential speed of the earth, we can use that to determine the centripetal acceleration, or with mass the centripetal force. At these distances, we need to use Equation 6.5 for the force of gravity. It is this force which is the centripetal force that keeps the earth in a circular path, so we can use Equation 7.7 as well.

$$F_g = F_c$$

$$\frac{G \cdot m_{sun} \cdot m_{earth}}{r^2} = \frac{m_{earth} \cdot v_T^2}{r}$$

$$\frac{G \cdot m_{sun}}{r} = v_T^2$$

While it is interesting to know the speed of the earth as it goes around the sun, that knowledge doesn't have a great impact on our daily lives. The thing that most affects our lives about the orbit of the earth around the sun is the passing of the seasons, which is related to angular position. This is an example when expressing position in terms of an angle, and how the angle changes over time, is more important than expressing position in terms of distances and how the distance changes over time.

There are also many other situations where angular changes are more important and easier to think about than changes in distance. For example, many vehicles have tachometers that display the angular speed of the engine measured in RPM (revolutions, sometimes called rotations, per minute).

When discussing angular motion, the units used for angles are radians. One radian is the angle created by an arc whose arc length is equal to the radius of the circle. So one full circle contains 2π radians. The angle then becomes a ratio of an arc length over a radius. Length per length. In other words, the unit "radian" is in some sense dimensionless. Expressing angles in radians greatly simplifies problem solving precisely because of this dimensionless quality of the angle measurement.

Just as we have learned about position, velocity, acceleration, momentum, *etc.* in linear form, we will now start to learn about angular position, angular velocity, angular acceleration, angular momentum, *etc.*

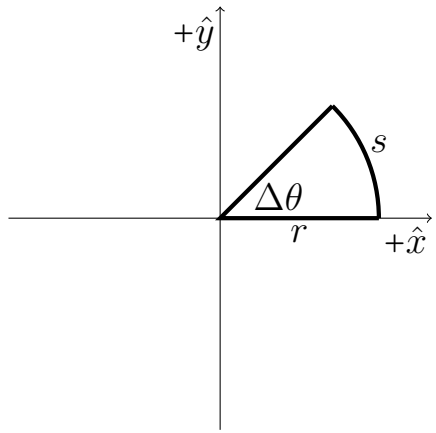


Figure 7.13: The angle in radians is the ratio of the arc length s to the radius r.[1]

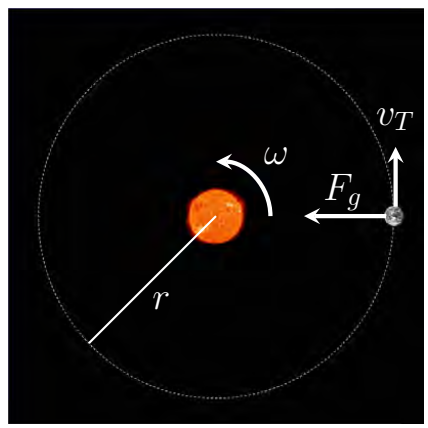


Figure 7.14: The earth's speed as it orbits the sun can be described in terms of angular velocity ω.[35]

$$m_{sun} = \frac{v_T^2 \cdot r}{G} = 2.0 \times 10^{30}\ \mathrm{kg}$$

It is often useful to describe a system in terms of angles, so for the earth we could give an angular velocity around the sun instead of a tangential velocity. The SI unit for angles is the radian [rad], illustrated in Figure 7.13.

$$\boxed{\Delta\theta = \frac{s}{r}} \tag{7.8}$$

...where s is the arc length and r is the radius of a circle. $\theta = 0$ is normally at the positive x axis and the positive θ direction is counter-clockwise.

Similarly, we can define angular velocity ω in terms of tangential velocity and radius:

$$\boxed{\omega = \frac{v_T}{r}} \tag{7.9}$$

The angular velocity of the earth around the sun can then be described as:

$$\omega_{earth} = \frac{3 \times 10^4\ \mathrm{m/s}}{1.5 \times 10^{11}\ \mathrm{m}} = 2 \times 10^{-7}\ \mathrm{rad/s}$$

The SI unit for angular velocity is therefore [rad/s]. We can also find the period T of the earth's orbit around the sun, the amount of time for one complete revolution. When ω is constant,

$$\boxed{T = \frac{2\pi}{\omega}} \tag{7.10}$$

So for the earth, $T = 3.14 \times 10^7\ \mathrm{s}$.

7.5 Frictionless Puck on a String

Words

For situations where angular velocity is easier to think about than linear velocity, it would be convenient to also have angular versions of kinetic energy and momentum, which are strongly connected to velocity. Take, for example, a 0.2 kg frictionless puck that is connected by a 1.4-m-long string to a bolt that is solidly mounted in the floor.

If the puck is moving at constant angular speed of 4 rad/s in a circle around the bolt, its kinetic energy, which depends only on mass and speed, but not on direction, is constant.

We now have two different ways to look at the motion and the kinetic energy of the puck:

- We can see the puck as a moving object with a linear (tangential) velocity that is always changing direction because of the centripetal acceleration caused by the tension in the string. In this view, the puck has the same kind of kinetic energy that we are used to dealing with, which is usually called "translational" kinetic energy.
- Or we can see the puck as part of a puck-bolt system that does not have any linear velocity (since the bolt is solidly fixed in place), but that is spinning about an axis at a constant angular speed. In this view, the puck-bolt system has *rotational* kinetic energy, but *no translational* kinetic energy.

Graphics

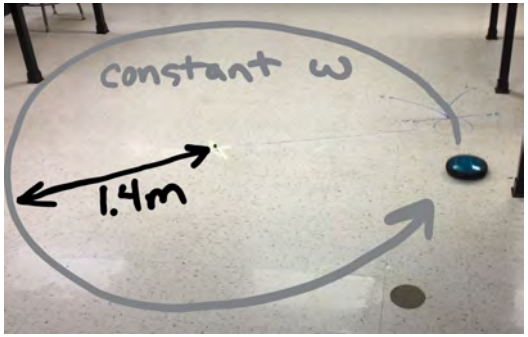


Figure 7.15: A frictionless puck moving in a circle at constant angular speed.[33]

Figures 7.15 & 7.16 are based on a screenshot from a YouTube video: https://youtu.be/mNdLRySeh9o.

Something really interesting happens in the video at a place where the string falls off of the bolt. The puck stops moving in a circle with constant angular velocity, since the source of the centripetal force has disappeared. With no forces in the horizontal direction, the puck instead begins to move in a straight line with constant velocity.

Numbers

Knowns	Unknowns
$r = 1.4\,\mathrm{m}$	E_k
$\omega = 4\,\mathrm{rad/s}$	rotational kinetic energy?
$m = 0.2\,\mathrm{kg}$	angular momentum?

We need to use Equation 1.6 to find the kinetic energy of the puck, recognizing that its speed is simply v_T. But we don't have v_T, so we will also need to use Equation 7.9:

$$E_k = \frac{1}{2} m \cdot v_T^2 = \frac{1}{2} m \cdot (r \cdot \omega)^2$$

This is referred to as the rotational kinetic energy $E_{k,r}$ of a point mass, and the expression is normally grouped in a slightly different way:

$$E_{k,r} = \frac{1}{2}\left(m \cdot r^2\right) \cdot \omega^2$$

For the puck in this example...

$$E_{k,r} = \frac{1}{2}\left(0.2\,\mathrm{kg} \cdot (1.4\,\mathrm{m})^2\right) \cdot (4\,\mathrm{rad/s})^2 = 3.14\,\mathrm{J}$$

The kinetic energy could also be found for the puck by finding v_T and calculating kinetic energy as we have done before. Whether we choose to consider the puck as having only translational kinetic energy in the tangential direction or only rotational kinetic energy around the pivot, we will get the same result for its kinetic energy.

This type of thinking also works for momentum.

- We can see the puck as a moving object with a linear momentum that is always changing direction because of the tension in the string.
- Or we can see the puck as part of a system that does not have any linear momentum, but that has constant angular momentum around a pivot point at the location of the bolt.

Angular momentum, like linear momentum, is conserved for any isolated system. That is what makes it such a useful concept. An isolated system, remember, is one that is not affected by any outside forces.

By watching the video associated with the figures in this section, we can learn something surprising about angular momentum: even an object that is moving in a straight line can have angular momentum! When the string falls off of the bolt, there is no force applied to the puck, so its angular momentum can't change at that time–it has to keep the same amount of angular momentum that it had just before the string fell off.

For a pointlike object like this puck, the angular momentum depends on its mass, angular velocity, and distance from the pivot; or, if it is moving in a straight line then it depends on the mass, linear velocity, and the "lever arm," which is the perpendicular distance from the pivot to the line along which it is traveling.

Figure 7.16: When the string falls off of the bolt, the puck begins to move in a straight line.[33]

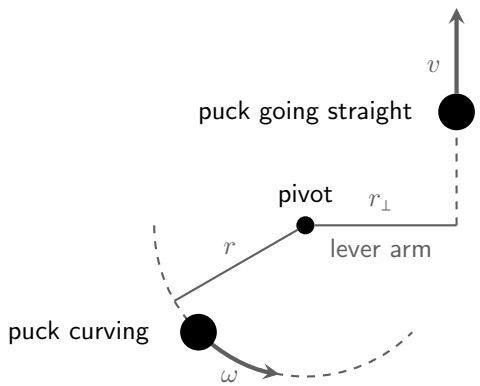


Figure 7.17: Motion of the puck, showing a lever arm, $r_\perp$, for an object moving in a direction that is not aligned with a pivot point.[1]

Angular momentum L is conserved for an isolated system, so...

$$\boxed{L_f = L_i} \tag{7.11}$$

For an object moving in a straight line that is not aligned with a pivot point, as shown when the puck is going straight in Figure 7.17, L is given by...

$$\boxed{L = m \cdot v \cdot r_\perp} \tag{7.12}$$

...where $r_\perp$ is the lever arm.

From Figure 7.17, we can see that in the case of a small object traveling on a circular path, the lever arm is simply r and the velocity is simply v_T. So for that situation, angular momentum can be described in angular terms using...

$$\begin{aligned} L &= m \cdot v_T \cdot r \\ &= m \cdot (\omega \cdot r) \cdot r \\ &= \left(m \cdot r^2\right) \cdot \omega \end{aligned}$$

Notice that $\left(m \cdot r^2\right)$ has made another appearance. And as we explore rotation we will see more of this same type of grouping of mass and radius. That is why they are usually grouped together in expressions for rotational motion.

Now we can find the angular momentum of the puck:

$$\begin{aligned} L &= \left(m \cdot r^2\right) \cdot \omega \\ &= \left(0.2\,\mathrm{kg} \cdot (1.4\,\mathrm{m})^2\right) \cdot 4\,\mathrm{rad/s} \\ &= 15.7\,\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s} \end{aligned}$$

7.6 Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

General

- When an object is following a curved path, the radial direction is in the direction of the radius of curvature and the tangential direction is perpendicular to the radius of curvature.
- The time needed for an object to move in a complete circle is called the period.
- A "lever arm" is the perpendicular distance from a straight line to a pivot point.

Forces

- A net force applied perpendicular to an object's velocity causes the object to change direction but maintain constant speed.
- Centripetal force is always directed in the radial direction toward the center of the circular path.
- Centripetal force is not a new kind of force, but is used to describe the force that is causing circular motion.

Motion

- Motion in the horizontal direction and motion in the vertical direction are independent of each other. The thing that connects them is time.
- An object that is in free-fall near the surface of the earth follows a path that is shaped like a parabola.
- Free-fall motion is sometimes called projectile motion or ballistic motion.
- When an object's acceleration is perpendicular to its velocity, the direction of the object's motion changes. It follows a path described by an arc of a circle with a "radius of curvature" r.
- Centripetal acceleration is always directed in, toward the center of circular path.
- Speed is path length traveled over time.
- The positive direction of angular quantities is defined as the counter-clockwise direction.
- The SI unit for angle is the radian [rad]. Zero radians is usually at the positive x axis.
- The SI unit for angular velocity is [rad/s].

Momentum

- Angular momentum is conserved for any isolated system.
- The angular momentum of an object moving in a straight line depends on its mass, linear velocity, and lever arm.
- The SI unit for angular momentum is [$mathrmkg \cdot m/s$].

Energy

- Objects can have translational kinetic energy or rotational kinetic energy.

Mathematical Models

equation		restrictions on the validity of the equation
$x = x_0 + v_{0x} \cdot t$	(7.1)	For objects in free-fall
$y = y_0 + v_{0y} \cdot t - \frac{1}{2} g \cdot t^2$	(7.2)	For objects in free-fall near the surface of the earth
$t_{flight} = \frac{2 v_0 \cdot \sin\theta}{g}$	(7.3)	For objects in free-fall near the surface of the earth with final height the same as initial height
$h_{max} = \frac{v_0^2 \cdot \sin^2\theta}{g}$	(7.4)	For objects in free-fall near the surface of the earth with final height the same as initial height
$R = \frac{v_0^2 \cdot \sin(2\theta)}{g}$	(7.5)	For objects in free-fall near the surface of the earth with final height the same as initial height
$a_c = \frac{v_T^2}{r}$	(7.6)	-none-
$F_c = m \cdot a_c = \frac{m \cdot v_T^2}{r}$	(7.7)	-none-
$\Delta\theta = \frac{s}{r}$	(7.8)	-none-
$\omega = \frac{v_T}{r}$	(7.9)	-none-
$T = \frac{2\pi}{\omega}$	(7.10)	When ω is constant
$L_f = L_i$	(7.11)	For an isolated system
$L = m \cdot v \cdot r_\perp$	(7.12)	For pointlike objects with linear velocity

7.7 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

Level 1 - Remember

7.1 **[W]** What is the difference between free-fall, projectile motion, and ballistic motion?

7.2 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

Level 2 - Understand

7.3 **[W]** Explain how the photograph in Figure 7.5 shows that the normal force is much larger than the gravitational force when the ball is touching the ground in image 16.

7.4 **[W]** Describe how the momentum of the ball in Figure 7.5 changes over the course of time during which it is imaged.

7.5 **[W & G]** What type of force creates the centripetal force that keeps the ball moving in a circle in Figure 7.11?

Level 3 - Apply

7.6 **[N]** If the mass of the ball in Section 7.2 were doubled, how would that affect the radius of curvature at the top of the left peak that was found in Section 7.3? Explain your reasoning.

7.7 **[W & G]** If the hammer in Figure 7.11 were released when the cord was to the South compared to the thrower, in what direction would the hammer travel?

7.8 **[N]** Use the same analysis that was used in Section 7.4 to determine which planets in our solar system have a higher tangential velocity than the earth.

Level 4 - Analyze

7.9 **[W, G, & N]** Describe the energy of the diver in Section 7.1 at the top of the cliff and just before they hit the water.

7.10 **[G]** Make energy bar graphs for the ball in Figure 7.5 for images 1 (where the ball is close to the ground but not touching the ground), 4, 8, 13, 16 (where the ball is on the ground and not moving), 22, and 29 (where the ball is on the ground and not moving). Without a mass for the ball, it is not possible to calculate the actual energies, so just make the relative heights of the bars as accurate as possible.

7.11 **[W & N]** An angular velocity of the earth around the sun is calculated at the end of Section 7.4. Convert that number into revolutions per year. Does your answer make sense? Explain.

7.12 **[W & G]** Look at the photograph of a fire poi dance at the beginning of Chapter 7. Assuming that the balls of fire are moving at constant speed, identify places in the balls' paths where the centripetal acceleration has a large magnitude and places where it has a small magnitude.

7.13 **[N]** Find the amount of angular momentum that the earth has due to its orbit around the sun. Use the position of the sun as the pivot point. Do the calculation in two different ways: One using the angular speed of the earth and the other using its tangential speed. Verify that the result is the same either way.

7.14 **[N]** Find the amount of kinetic energy the earth has due to its orbit around the sun. Do the calculation in two different ways: One using the angular speed of the earth and the other using its tangential speed. Verify that the result is the same either way.

Level 5 - Evaluate

7.15 **[G & N]** What effect would each of the follow changes have on the total time, final speed, and horizontal displacement of the diver in Section 7.1?

(a) Doubling the initial velocity of the diver and keeping everything else the same

(b) Doubling the height of the cliff and keeping everything else the same

(c) Keeping the same initial speed but jumping up and out instead of just horizontally out from the cliff, and keeping everything else the same

Level 6 - Create

7.16 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Add the main ideas from this chapter to the concept map that you began for the question at the end of Chapter 1.

7.17 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

7.18 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

Chapter 8

Rotation

Interesting things happen when objects are allowed to rotate. Ice skaters start a spin and then go faster and faster seemingly without any effort. Children's tops fall over unless they are spinning.

At a playground or amusement park virtually every piece of equipment involves rotation in one way or another. Spinning wheels are used in almost every form of modern transportation. Even transportation that doesn't involve wheels, like flying in a helicopter or walking, involve rotation around a joint or axle.

This chapter focuses on a new set of tools–still forces, motion, momentum, and energy,–but specifically applied to rotating objects.

Many things have not changed. We will see that angular force (called torque) changes angular momentum, causes angular acceleration, and does work, just as before. Energy and angular momentum are still conserved in an isolated system. And most of the mathematical models that we have used will apply equally well to rotation after just a few changes.

Figure 8.1: Sasha Cohen performs an I-spin at the 2009 Stars on Ice in Halifax, Nova Scotia.[36]

8.1 Kind of the Same

Words

To look at the similarities between linear and rotational motion, forces, and momentum, we will consider a thin, light (e.g. massless) rod with a small, massive ball on one end, rotating around a fixed pivot location at the opposite end of the rod.

Linear motion is described by position, velocity, and acceleration. Velocity is a change in position (also called displacement) over time, and acceleration is a change in velocity over time. Angular motion will be described in exactly the same way. The position is replaced by angular position, or simply angle. Angular velocity is a change in angular position over time, and angular acceleration is a change in angular velocity over time.

Each of these angular quantities are also directly related to their corresponding linear quantities in the tangential direction. For example, if the length of the rod were doubled but the angular velocity stayed the same, the ball would trace out a circle with twice the radius in the same amount of time, so the tangential velocity would double; if the length of the rod stayed the same but the angular velocity (rate of spin) was doubled, the ball would trace out a circle of the same size in half the time, so again the tangential velocity would double. So the angular quantities are a combination of the linear quantities and the radius.

Graphics

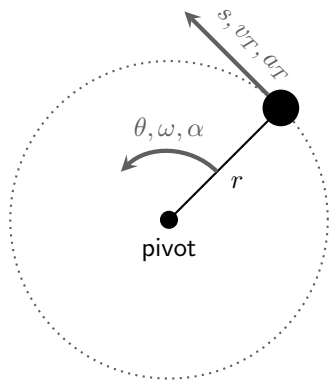


Figure 8.2: Looking down on the rod and ball. The ball is following the dotted line counterclockwise. s, v_T, & a_T are the linear quantities corresponding to the angular quantities θ, ω, & α, respectively.[1]

Numbers

$\Delta\theta$ and ω were already given in terms of s and v_T in Section 7.4. Now, to do the same for the angular acceleration α, whose SI unit is $[\mathrm{rad/s^2}]$:

$$\boxed{\alpha = \frac{a_T}{r}} \tag{8.1}$$

Every equation of motion that we have learned works equally well for angular motion, simply by replacing the linear quantities with their angular counterparts. So, for example, Equation 1.2 becomes

$$\boxed{\theta = \theta_0 + \omega_0 \cdot t + \frac{1}{2}\alpha \cdot t^2} \tag{8.2}$$

Equation 4.2 becomes

$$\boxed{\alpha_{avg} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{\Delta t}} \tag{8.3}$$

. . . and Equations 4.4 & 4.5 become

$$\boxed{\omega_{avg} = \frac{\Delta\theta}{\Delta t}} \tag{8.4}$$

and. . .

$$\boxed{\omega_{avg} = \frac{1}{2}\left(\omega_i + \omega_f\right)} \tag{8.5}$$

When we looked at linear motion, we saw that an acceleration only exists if there is a net force. So we would again expect that to be true for rotational motion. An angular acceleration only exists if there is a net angular force, which is called a "torque." The units of torque are $[\mathrm{N} \cdot \mathrm{m}]$.

Several forces are shown in Figure 8.3, all with the same magnitude. Which of them would create the largest angular acceleration?

Two of the forces, F_1 and F_2, are acting right at the pivot point. Since physical scenario states that the pivot point is at a fixed location, it can't move. That means neither of those two forces can create any angular acceleration around the pivot.

F_3 looks like a very good choice, and in fact this is where you would probably apply the force if you were in this situation trying to accelerate the ball. Of all of the forces in Figure 8.3, F_3 is most effective at creating angular acceleration, because it creates the greatest amount of torque. F_4 and F_5 also create torque and will cause angular acceleration of the rod an ball, but not as much as F_3. That is because F_3 has the largest lever arm.

Just as forces cause a change in linear momentum over time, torques cause a change in angular momentum over time.

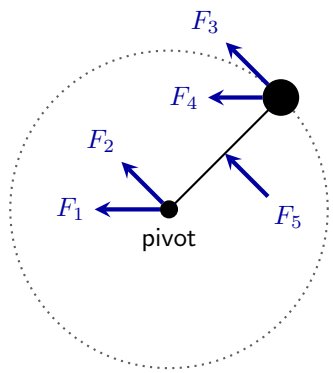


Figure 8.3: Several forces, all with the same magnitude, applied to a ball attached to a pivot by a light rod.[1]

The lever arms are zero for F_1 & F_2 since they are both at the pivot. The lever arm for F_3 is the length of the rod, and the lever arm for F_5 is half the length of the rod. The lever arm for F_4 is a bit more difficult to find.

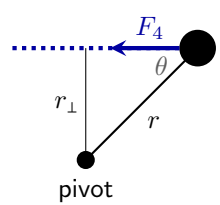


Figure 8.4: Finding the lever arm for F_4 from Figure 8.3.[1]

And, finally, Equation 5.7 becomes

$$\boxed{2\alpha \cdot \Delta\theta = \omega_f^2 - \omega_i^2} \tag{8.6}$$

The angular version of force is torque τ. Just as with angular momentum, torque increases with the length of the lever arm:

$$\boxed{\tau = F \cdot r_\perp} \tag{8.7}$$

As seen in Figure 8.4, the lever arm can be found from the distance r to the pivot and the angle θ between the vector being considered (in this case a force vector) and the vector $\vec{r}$. Using Equation 5.2, we have...

$$\boxed{r_\perp = r \cdot \sin\theta} \tag{8.8}$$

Mathematically, torque affects rotating systems in the same way that force affects linear systems. That includes the ability to change angular momentum over time:

$$\boxed{\tau = \frac{\Delta L}{\Delta t}} \tag{8.9}$$

8.2 Points and Hoops

Words

In Section 8.1, we considered a ball whose mass is concentrated in a small point some distance away from the pivot. Now we will explore what happens to rotational kinetic energy when the mass is distributed across a larger area.

In each of these examples of different mass distributions, we will assume that the total mass of the system m is the same, and that the mass is first distributed and then the system is rotated at the same angular speed ω.

To begin, instead of a single mass m at the end of a light rod with length r, we will consider two masses, each with a mass of $m/2$, connected by a light rod, still keeping each mass at a distance r from the center pivot point, rotating at a constant angular speed ω.

For each of these masses, the tangential speed will be the same as it was for the original single mass. We will need to combine their kinetic energies. Instead of a single mass at a certain speed, we have two half masses at the same speed. The kinetic energy should be the same, whether the mass is in one location or split into two locations.

What if we split the mass into three, or four, or six, or ten, or even more pieces, and kept all of them the same distance from the pivot?

Graphics

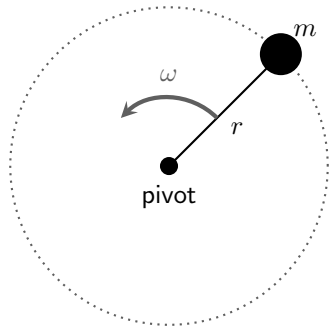


Figure 8.5: The original situation, with a single point mass rotating around a pivot.[1]

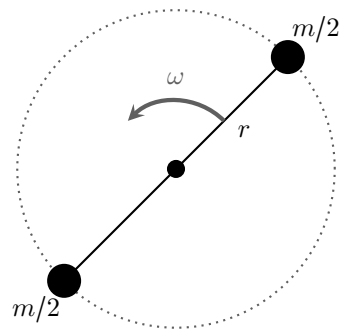


Figure 8.6: The second situation, with the same mass split into two parts, with both parts at the original distance from the pivot.[1]

Numbers

In Section 7.5 we found that the rotational kinetic energy for a mass distribution like that shown in Figure 8.5 is...

$$E_{k,r} = \frac{1}{2}\left(m \cdot r^2\right) \cdot \omega^2$$

For the mass distribution shown in Figure 8.6, we need to consider each mass separately and add their energies to find the total rotational kinetic energy.

$$E_{k,r} = \frac{1}{2} \cdot \frac{m}{2} \cdot r^2 \cdot \omega^2 + \frac{1}{2} \cdot \frac{m}{2} \cdot r^2 \cdot \omega^2$$

$$E_{k,r} = \frac{1}{2}\left(m \cdot r^2\right) \cdot \omega^2$$

The result is the same as if all of the mass were at a single point.

Using the same analysis for a mass distribution where the same mass is divided into a huge number N of small pieces, all kept at the same distance from the pivot, we have to add up N energies:

$$E_{k,r} = N \cdot \frac{1}{2} \cdot \frac{m}{N} \cdot r^2 \cdot \omega^2 = \frac{1}{2}\left(m \cdot r^2\right) \cdot \omega^2$$

If N is large enough, all of the mass could be spread out to be completely touching all of the way around the circle, creating a hoop, without affecting the rotational kinetic energy.

If the angular speed remains the same, each piece has the same tangential speed as the original single mass. When we combine their kinetic energies, the kinetic energy of the whole system is the same whether the mass is in one location or split into many locations. In fact, the kinetic energy would be the same even if you spread the mass out into a thin hoop at the same distance r from the pivot.

Let's try one more thing. We will split the mass into five equal parts and space them out equally in a line so that the ones on the ends are at the original distance r and the middle one is at the center.

The two pieces on the ends still move at the same tangential speed as before, but the pieces that are farther in move at a slower speed and the piece in the center doesn't move at all. So this time, when we add up the kinetic energies for the five pieces we will end up with less energy than before.

So this time the way the mass is distributed has reduced the "moment of inertia" of the system of masses. If an object has a large moment of inertia, that means it is difficult to change its angular velocity. This is analogous to saying that if an object has a large mass, it is difficult to change its linear velocity.

The closer the mass is to the pivot, the smaller the moment of inertia will be.

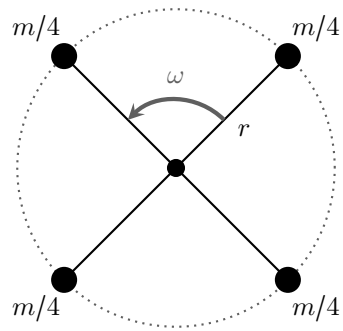


Figure 8.7: The third situation, with the same mass split into four parts, with all parts at the original distance from the pivot.[1]

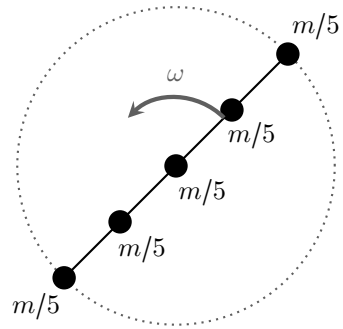


Figure 8.8: The fourth situation, with the same mass split into five parts at varying distances from the pivot.[1]

For the situation shown in Figure 8.8, we have two masses at radius r, two at $r/2$, and one at the pivot point, so radius zero. When we add their kinetic energies we get:

$$E_{k,r} = 2 \cdot \frac{1}{2} \cdot \frac{m}{5} \cdot r^2 \cdot \omega^2 + 2 \cdot \frac{1}{2} \cdot \frac{m}{5} \cdot (r/2)^2 \cdot \omega^2 + 0$$

$$E_{k,r} = \frac{1}{2} \cdot \left(\frac{1}{2} m \cdot r^2\right) \cdot \omega^2$$

This is a different result! If we want to keep the form of our equation for rotational kinetic energy, the expression in the parentheses is not always going to be $m \cdot r^2$.

The expression in the parentheses is called the "moment of inertia," I, and it is the angular counterpart to mass. There will always be a mass and a length squared in I, but it also contains a multiplier that depends upon the distribution of the mass. The multiplier gets smaller when the mass is brought closer to the pivot.

Replacing linear variables with their angular counterparts, we now have a few more mathematical models for rotating systems:

$$\boxed{E_{k,r} = \frac{1}{2} I \cdot \omega^2} \tag{8.10}$$

$$\boxed{L = I \cdot \omega} \tag{8.11}$$

$$\boxed{\tau = I \cdot \alpha} \tag{8.12}$$

8.3 Helicopter Blades

Words

A typical helicopter blade has a mass of approximately 60 kg and a length of approximately 10 m, and the blades typically rotate at approximately 500 RPM. Assume that the blades of the two-blade helicopter shown in Figure 8.9 can go from motionless to full angular speed in 5 seconds, with constant angular acceleration over that time. What can we find from this information?

Since we are dealing with rotational motion, we should be thinking in terms of angular quantities: angular position, angular velocity, angular acceleration, angular momentum, torque, and rotational kinetic energy.

Initially the blade is not moving, so no angular velocity, no angular momentum, and no rotational kinetic energy. But once the blade is spinning it has all three of those, so that means that a torque must have been acting on the blade, coming from the engine inside the helicopter.

The torque does several things. For one, it causes angular acceleration, increasing the angular velocity. The amount of angular acceleration depends not only on the torque but also on the moment of inertia of the blades. Increasing torque will increase the angular acceleration, but increasing the moment of inertia will decrease the angular acceleration.

Graphics

Figure 8.9: A helicopter with two blades.[37]

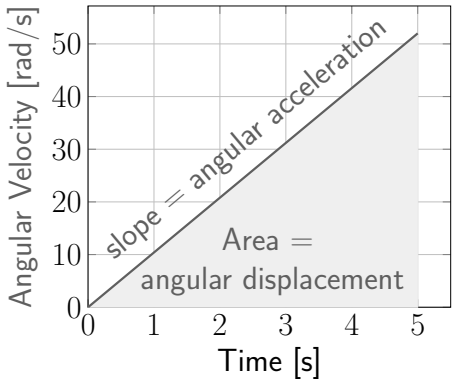


Figure 8.10: Similarly to linear velocity and linear acceleration, the slope of an angular velocity vs time graph is angular acceleration.[1]

Numbers

Knowns	Unknowns
$m_{blade} = 60\,\mathrm{kg}$	???
$l_{blade} = 10\,\mathrm{m}$	
$\omega_i = 0\,\mathrm{rad/s}$	
$\omega_f = 52\,\mathrm{rad/s}$	
$t = 5\,\mathrm{s}$	
2 blades	

The value for ω_f in the table above was found by converting from rotations per minute (RPM).

We can begin by finding the average angular acceleration using Equation 8.1:

$$\alpha_{avg} = \frac{\omega_f - \omega_i}{\Delta t} = \frac{(52-0)\ \mathrm{rad/s}}{5\,\mathrm{s}} = 10.4\,\mathrm{rad/s^2}$$

We can also find the total angle that the blades rotate through as they come up to full speed by using Equations 8.4 & 8.5:

$$\Delta\theta = \omega_{avg} \cdot \Delta t = \frac{1}{2}(\omega_i + \omega_f) \cdot \Delta t$$
$$= \frac{1}{2}(0+52)\ \mathrm{rad/s} \cdot 5\,\mathrm{s} = 130\,\mathrm{rad}$$

The torque also changes the angular momentum of the blades. The change in angular momentum depends only on the net torque and the time over which that torque is applied.

The torque also changes the rotational kinetic energy of the blades. That is because the torque is doing work on the blades, just as a force can do work on an object that has linear velocity. Since the blades are moving in the same direction as the torque, the work is positive, increasing the kinetic energy.

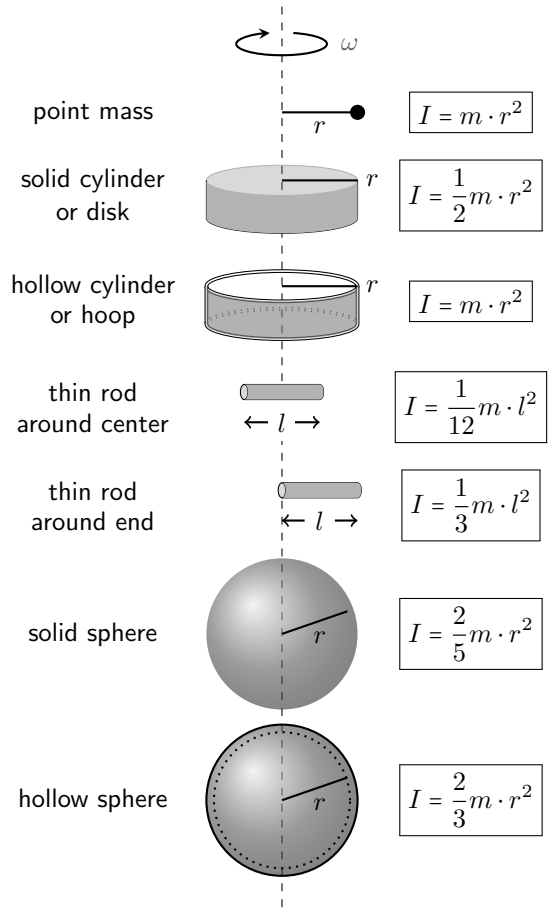


Figure 8.11: Moments of inertia for various shapes, each with mass m and the same radius r or length l.[1]

Since we are dealing with rotation, it will be helpful to find the moment of inertia. The moments of inertia for several shapes are shown in Figure 8.11. We need to for the shape that is most similar to the blades.

In this case, if we consider the two blades as a single object with twice the length of one blade we can use the "thin rod around center" from Figure 8.11.

$$\begin{aligned} I &= \frac{1}{12} m_{tot} \cdot l_{tot}^2 \\ &= \frac{1}{12} (120\,\mathrm{kg}) \cdot (20\,\mathrm{m})^2 \\ &= 4\,000\,\mathrm{kg} \cdot \mathrm{m}^2 \end{aligned}$$

Now we can use Equation 8.7 to find the net torque applied to the blades:

$$\tau = I \cdot \alpha = 4\,000\,\mathrm{kg} \cdot \mathrm{m}^2 \cdot 10.4\,\mathrm{rad/s}^2 = 41\,600\,\mathrm{N} \cdot \mathrm{m}$$

We can also use Equation 8.11 to find the final angular momentum of the blades:

$$\begin{aligned} L_f &= I \cdot \omega_f = 4\,000\,\mathrm{kg} \cdot \mathrm{m}^2 \cdot 52\,\mathrm{rad/s} \\ &= 208\,000\,\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s} \end{aligned}$$

And finally, we can use Equation 8.10 to find the final rotational kinetic energy:

$$\begin{aligned} E_{k,r,f} &= \frac{1}{2} I \cdot \omega_f^2 = \frac{1}{2} \cdot 4\,000\,\mathrm{kg} \cdot \mathrm{m}^2 \cdot (52\,\mathrm{rad/s})^2 \\ &= 5.4 \times 10^6\,\mathrm{J} \end{aligned}$$

8.4 Figure Skating

Words

Describe what happens if a 60 kg skater who is 1.7 m tall with a torso diameter of 0.3 m is spinning at a rate of 1.5 rad/s in the position shown in Figure 8.12 and then brings themselves into a vertical position, as in Figure 8.13.

When we are dealing with linear motion, we know that the more mass something has, the more difficult it is to move. But once you get it moving, it will have a large amount of momentum and kinetic energy. For rotational motion, it is not just the mass but how it is distributed that is important. This combination of mass and distance is called the moment of inertia. Figure skaters are expert at changing their moments of inertia, which is what is happening when they go from a horizontal position to a vertical one.

For someone who has observed a spinning figure skater, it is hard to forget what happens when they do that. The figure skater starts a spin, and then spins faster and faster, apparently without any effort, without even pushing off again on the ice.

The only forces on the skater are gravity and the normal force from the ice, exactly the same as for the motionless rock we considered so long ago. The rock didn't suddenly start spinning faster and faster. So how can a figure skater's angular speed increase when there is no torque?

Graphics

Figure 8.12: Tangxu Li skating at Lillehammer in 2016.[38]

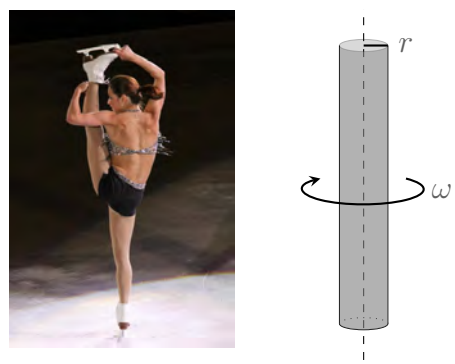


Figure 8.13: Sasha Cohen skating at Stars on Ice in 2009, and an approximation of her shape for finding her moment of inertia.[36]

Numbers

Knowns	Unknowns
$m = 60\,\mathrm{kg}$	???
height $l = 1.7\,\mathrm{m}$	
radius $r = 0.15\,\mathrm{m}$	
$\omega_i = 1.5\,\mathrm{rad/s}$	

We need to find the shape from Figure 8.11 that is most similar to the skater, and use the moment of inertia for that shape. In this case, we have two different shapes to consider.

The final position shown in Figure 8.13 is simpler, so we will start there. Sasha Cohen's body does not form a perfect solid cylinder, but there is no better option among the shapes in Figure 8.11. It should be noted that the location and direction of the axis of rotation is important. The "thin rod" is rotating in such a way that its ends go around, or one end stays in position while the other end goes around. The "solid cylinder" is rotating along the axis of the cylinder, which is the direction in which Sasha is rotating.

Approximating Sasha as a solid cylinder, her moment of inertia is

$$I_f = \frac{1}{2}m \cdot r^2 = \frac{1}{2}60\,\mathrm{kg} \cdot (0.15\,\mathrm{m})^2 = 0.675\,\mathrm{kg} \cdot \mathrm{m}^2$$

For an isolated system, angular momentum is conserved. That is the key to what is happening to the figure skater. The figure skater has no linear velocity, but they have angular velocity. Angular momentum of a rotating object is the moment of inertia times the angular velocity, just as linear momentum is the mass times the velocity.

So when the skater is spinning with their body in a horizontal direction, their mass is spread out far from the vertical axis around which they are spinning. That gives them a large moment of inertia. Then, when they pull themselves into a vertical position they are bringing their mass in close to the axis around which they are spinning. This reduces their moment of inertia. Since angular momentum is conserved, reducing the moment of inertia creates a corresponding increase in angular velocity.

This increase in angular velocity also results in an increase in the rotational kinetic energy of the skater, so work was done on the skater. Where did the work come from? The skater does work on their own body. A centripetal force, toward the axis of rotation, is needed to keep something moving in a circle. In order to move something closer to the axis, the displacement is in the direction of the force, so the skater has to do work to come to a vertical position.

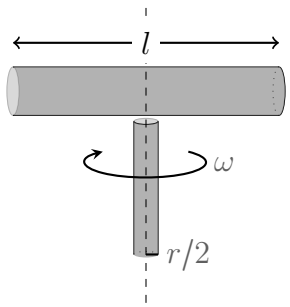


Figure 8.14: An approximation of the shape of Tangxu Li for finding his moment of inertia.[1]

The position that Tangxu is in is more difficult to analyze, because his body is going in two directions. His left leg is vertical, and so could be approximated as a cylinder just as we did with Sasha. But the rest of his body is horizontal and spinning end-to-end, which is much more like the "thin rod around center." To find Tangxu's moment of inertia, we can make both of these approximations, and add their moments of inertia. If we assume that his left leg contains 1/4 of the mass of his whole body and is 1/2 the radius of his torso, we have...

$$I_i = I_{leg} + I_{body} = \frac{1}{2}\left(\frac{1}{4}m\right)\cdot\left(\frac{r}{2}\right)^2 + \frac{1}{12}\left(\frac{3}{4}m\right)\cdot l^2$$

$$I_i = (0.04 + 10.8)\,\mathrm{kg}\cdot\mathrm{m}^2 = 10.8\,\mathrm{kg}\cdot\mathrm{m}^2$$

Apparently we didn't even need to take the vertical leg into account. It is so close to the axis compared to the rest of Tangxu's body that it doesn't significantly affect the moment of inertia. Now we can use conservation of angular momentum to find the final angular velocity of the skater:

$$L_f = L_i$$

$$I_f \cdot \omega_f = I_i \cdot \omega_i$$

$$\omega_f = \frac{10.8}{0.675}\cdot 1.5\,\mathrm{rad/s} = 24\,\mathrm{rad/s}$$

The angular velocity increases by more than a factor of 10! Now, using Equations 2.1 & 8.10...

$$W_{net} = \Delta E_k = \frac{1}{2}I_f\cdot\omega_f^2 - \frac{1}{2}I_f\cdot\omega_f^2 = 182\,\mathrm{J}$$

So 182 J of work was done by the skater to change position while spinning.

8.5 Charging a Radio

Words

In areas with limited access to electrical power, hand-cranked devices are often used. Figure 8.15 shows a boy cranking a radio to charge its battery. He has to push on the end of the 0.08-m-long handle of the crank with a force of 5 N to get it to move. If he cranks at a constant rate of 10 rad/s, how much time will it take him to store 150 J of energy in the battery, assuming that the charging system is 100% efficient?

In this situation, we are given a force that is applied a certain distance away from a pivot, so the boy is applying a torque to the crank. With linear motion, a force that is applied through a distance does work (and so can store energy). The rotational corrolary of this is that a torque that is applied through an angle does work.

The amount of work that the boy does will be proportional to the torque and also proportional to the angle. Since he is cranking at a constant rate, the angle will change linearly in time. The force he applies will have to be constant at 5 N so that there is no acceleration. So the applied torque is also constant. Since the angle changes linearly in time and the force is constant, the energy stored will also increase linearly in time. And since power is energy per time, the power the boy produces and stores is constant while he is cranking.

Graphics

Figure 8.15: A boy charging his hand-cranked radio[40]

Numbers

Knowns	Unknowns
$\omega = 10\,\mathrm{rad/s}$	t
$r = 0.08\,\mathrm{m}$	
$F_{applied} = -5\,\mathrm{N}\,\hat{\theta}$	
$\omega = -10\,\mathrm{rad/s}$	
$W = 150\,\mathrm{J}$	

The torque applied to the handle, as shown in Figure 8.16, is

$$\tau = F_{applied} \cdot r \cdot \sin\theta = -0.4\,\mathrm{N}\cdot\mathrm{m}$$

The angular version of Equation 4.3 for work is...

$$\boxed{W = \tau \cdot \Delta\theta} \tag{8.13}$$

...when the torque is constant. And since power is work per time, it can also be expressed as...

$$\boxed{P = \tau \cdot \omega} \tag{8.14}$$

From Equation 8.13 we can find the total angle that the boy has to crank through to charge the radio:

$$\Delta\theta = \frac{W}{\tau} = -375\,\mathrm{rad}$$

...or approximately 60 full revolutions. With the angular displacement we can find the time using Equation 8.4.

$$\Delta t = \frac{\Delta\theta}{\omega} = 37.5\,\mathrm{s}$$

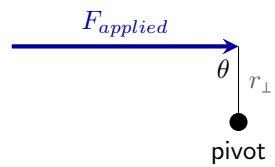


Figure 8.16: The boy applies a clockwise torque on the device, as seen from the left in Figure 8.15. Note that this theta refers to the angle between the force and the lever arm, not the rotation of the crank.[1]

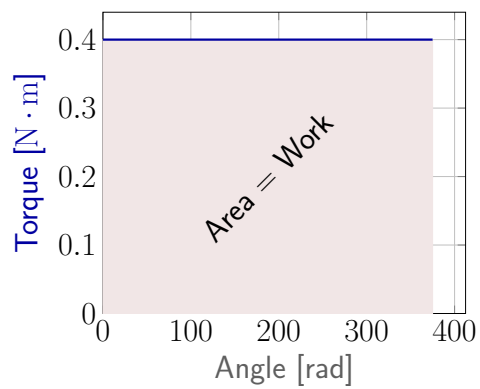


Figure 8.17: A torque in the direction of angular motion does an amount of work equal to the area under the curve in a Torque-vs-Angle graph. [1]

Torque and angle are taken to be positive for this graph. Using negative values as described in the "Numbers" section, the area would be negative (so under the x axis) but the angle would go right-to-left instead of left-to-right, which creates another negative. Looking at Figure 8.15 from the right instead of from the left would have made both torque and angle positive.

This question could also have been answered using a linear analysis in the tangential direction. Using Equation 2.3. . .

$$s = \frac{W}{F_T} = \frac{150\,\mathrm{J}}{-5\,\mathrm{N}} = 30\,\mathrm{m}$$

We aren't given a tangential velocity, but we can find it from the angular velocity using Equation 7.9.

$$v_T = \omega \cdot r = -0.8\,\mathrm{m/s}$$

Then we can find the time:

$$\Delta t = \frac{s}{v_T} = 37.5\,\mathrm{s}$$

Whichever way the analysis is done, the result is the same. The power generated by the boy is also the same either way:

$$P = \frac{W}{\Delta t} = 4\,\mathrm{W}$$

It should be noted that there are two different kinds of "W" in the expression above. The italicized one in the numerator (W) is Work; the one at the end that is not italicized is Watts.

8.6 Hoop Rolling

Words

In the children's game "hoop rolling" or "hoop trundling," a stick is used to start a hoop moving and then to keep it going. Let's say that the hoop starts at rest and accelerates uniformly across level ground to a speed of 5 m/s to the right in 2.5 seconds. The hoop has a mass of 0.4 kg and a radius of 0.2 m, and it rolls without slipping. Describe everything you can about this situation.

We know from considering other situations that velocity changes in the direction of acceleration, so acceleration is to the right. Momentum starts at zero and increases to the right along with the velocity, so there is a force to the right. Since the ground is level, we don't need to consider gravitational potential energy, and there also is no spring potential energy to consider. The hoop rolls without slipping, and usually thermal energy is generated by surfaces sliding together or by collisions. In this case, we don't have either. So the hoop starts with no kinetic energy and ends with kinetic energy. This means that positive work was done on the hoop, which makes sense because we have already established that there is a force to the right and the motion is to the right. Force in the direction of motion does positive work.

We have done all of that before. But that's not all that there is to the story. We haven't taken into consideration the fact that the hoop is rolling. This is a physical scenario that involves both translation (linear movement) and rotation.

Graphics

Figure 8.18: Statue of a boy playing "hoop rolling." [39]

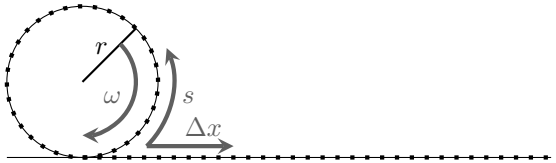


Figure 8.19: As the hoop rolls, it does not slide, so the distance it moves has the same magnitude as the path length on the circumference of the hoop, $\Delta x = s$. As the hoop rolls, each mark on the hoop will fall on its corresponding, equally-spaced mark on the ground.[1]

Numbers

Knowns	Unknowns
$\vec{v_0} = 0$	???
$\vec{v_f} = 5\,\mathrm{m/s}\,\hat{x}$	
$\Delta t = 2.5\,\mathrm{s}$	
$m = 0.4\,\mathrm{kg}$	
$r = 0.2\,\mathrm{m}$	
rolls without slipping	

It is possible to approach this question just as we would have after one or two chapters, using a linear analysis of motion, forces, *etc.* But this analysis will be done by analyzing it from an angular perspective. Our knowns are all linear, but they are connected to angular motion through tangential velocity. In the reference frame of a person sitting at the center of the hoop, the ground goes to the left at the speed at which the hoop is moving, and the bottom edge of the hoop moves along with it.

$$\omega_0 = -\frac{v_0}{r} = 0$$

$$\omega_f = -\frac{v_f}{r} = -25\,\mathrm{rad/s}$$

Why the minus signs? It is possible to keep all signs consistent mathematically, but it is often easier just to look at a sketch and see whether the motion (or angular momentum, or torque) is counterclockwise (+) or clockwise (-) and use the appropriate sign. In this example, we can see that the hoop is spinning clockwise.

At first, the hoop is not rolling, so zero angular velocity, but at the end it is rolling to the right, which means that it is turning in the clockwise direction. The standard "positive" angular direction in physics is counter-clockwise, so the final angular velocity is negative. That means that the angular acceleration is also negative.

Angular momentum starts at zero and increases along with angular velocity, in the negative direction, so there is a negative torque.

Torque is created by an off-center force. In this case, the child is pushing to the right at some point near the middle of the hoop, and friction with the ground prevents it from rolling by pushing left on the bottom of the hoop. It is the combination of these two forces that creates the torque.

We also have one additional type of energy to consider: rotational kinetic energy. Not all of the work that the child did in pushing the hoop went into linear kinetic energy; some went into rotational kinetic energy.

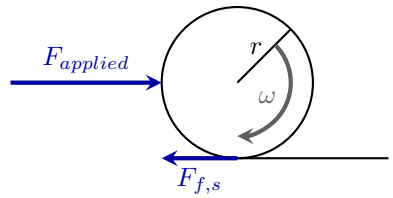


Figure 8.20: We know that the net force is to the right, because that is the direction of acceleration. But we also know that there is a force of friction to the left, because the hoop would slide to the right without this friction.[1]

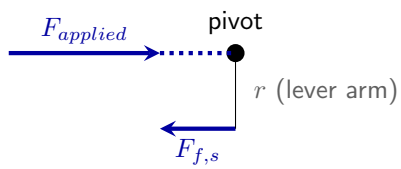


Figure 8.21: Force locations when considering torque.[1]

If we use the center of the hoop as the reference frame, the only motion of the hoop is rotational, around the center of the hoop. So we can use the center of the hoop as our pivot point. In this reference frame, it is the frictional force that is not aligned with the pivot but is separated by a lever arm distance. So it is the frictional force that creates the torque, making the hoop roll instead of sliding.

Knowing the time, we can use Equation 8.3 to find the angular acceleration:

$$\alpha = \frac{\Delta\omega}{\Delta t} = -10\,\mathrm{rad/s^2}$$

We can also use Equation 7.8 to find the total angle that the hoop rolls through as it comes up to speed in the 2.5 s.

$$\Delta\theta = \theta - \theta_0 = \cancel{\omega_0 t}^{\,0} + \frac{1}{2}\alpha \cdot t^2 = 31.3\,\mathrm{rad}$$

It should be clear that the appropriate moment of inertia is that of a hoop, so...

$$I = m \cdot r^2 = 0.016\,\mathrm{kg \cdot m^2}$$

With this information, we can easily find final angular momentum, torque, and final rotational kinetic energy:

$$L_f = I \cdot \omega_f = -0.4\,\mathrm{kg \cdot m^2/s}$$

$$\tau = I \cdot \alpha = -0.16\,\mathrm{N \cdot m}$$

$$E_{k,r,f} = \frac{1}{2} I \cdot \omega_f^2 = 5\,\mathrm{J}$$

So the total work done by the child on the hoop is...

$$W_{net} = \Delta E_k + \Delta E_{k,r} = \frac{1}{2} m \cdot v_f^2 + 5\,\mathrm{J} = 10\,\mathrm{J}$$

For a rolling hoop, half of its kinetic energy is rotational! We can use Equation 2.3 to find the force applied by the child.

$$F_{applied} = \frac{W_{tot}}{\Delta x} = \frac{10\,\mathrm{J}}{r \cdot \Delta\theta} = 1.6\,\mathrm{N}$$

8.7 Balancing

Words

Figure 8.22 shows a board and a brick balancing on the end of another board. The brick has a mass of 1.76 kg and its center of mass is 7.5 cm to the left of the upright board. The horizontal board has a length of 49 cm, and its left edge is 9.5 cm to the left of the upright board. We can use this information to find the mass of the horizontal board and the normal force that the upright board applies to the horizontal board.

The key to understanding this scenario is the rotational version of Newton's First Law. That law states that if the net force on a system is zero then the acceleration of that system is also zero. The rotational version of that statement would be that if the net torque on a system is zero then the angular acceleration of that system is also zero. Since the system is balancing motionless, the acceleration, both angular and linear, must be zero. That situation is called "static equilibrium."

There are actually two ways to approach this problem. One is by considering forces and torque as described above, and the other is by considering the center of mass. Let's first consider the center of mass.

Unless the brick, boards, and cement floor are somehow glued or bolted together (they aren't!), then to remain balanced they have to be supported at the center of mass. The center of mass of the brick is clearly above the horizontal board. If it weren't, for example if it were hanging more than

Graphics

Figure 8.22: Two boards and a brick balancing[1]

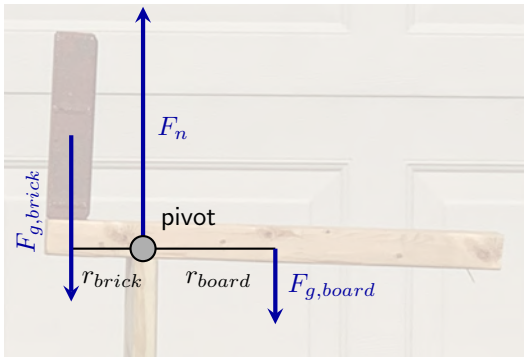


Figure 8.23: The forces and torques on the balancing brick and boards.[1]

Numbers

Knowns	Unknowns
$m_{brick} = 1.76\,\mathrm{kg}$	m_{board}
$r_{brick} = 0.075\,\mathrm{m}$	F_n
$r_{board} = 0.15\,\mathrm{m}$	
$g = 9.8\,\mathrm{m/s^2}$	

r_{board} was found by assuming that the center of mass of the board is at the center of the board, which is 0.245 m to the right of the left end of the board, and thus

$$r_{board} = (0.245\,\mathrm{m} - 0.095\,\mathrm{m}) = 0.15\,\mathrm{m}$$

For an object that is in static equilibrium, the net force and the net torque are both zero:

$$\boxed{\Sigma F = 0} \tag{8.15}$$

$$\boxed{\Sigma \tau = 0} \tag{8.16}$$

Calculation of the net torque around the pivot in Figure 8.23 gives

$$\Sigma \tau = \tau_{brick} + \cancelto{0}{\tau_n} - \tau_{board} = 0$$

$$F_{g,brick} \cdot r_{brick} \cdot \sin 90^\circ = F_{g,board} \cdot r_{board} \cdot \sin 90^\circ$$

$$m_{board} = m_{brick} \cdot \frac{r_{brick}}{r_{board}} = 0.88\,\mathrm{kg}$$

We can also use Figure 8.23 as a free body diagram to find F_n, since the sum of forces is zero.

$$\Sigma F = -F_{g,brick} + F_n - F_{g,board} = 0$$

halfway off of the end of the board, the brick would fall. The combined center of mass of the brick and the horizontal board has to be above the vertical board for the same reason. If it weren't, the horizontal board would fall. The combined center of mass of both boards and the brick also has to be above the bottom of the vertical board. If it were not, the vertical board would fall.

Now, we will consider the situation from the perspective of forces and torque. Since we are not given information about the vertical board, and not asked any questions about it, we will only use torque to examine the brick and horizontal board. If they were to fall, they would tip around a pivot at the top of the vertical board. It doesn't tip, so that means the net torque around that pivot has to be zero.

There is a gravitational force acting downward on the brick, and since the brick is to the left of the pivot, it would cause counterclockwise (or positive) rotational motion. So the brick creates a positive torque. The horizontal board must create an equal but opposite torque to keep the system balanced. When considering torque, all of an object's mass acts as if it is at a single point: the center of mass of the object. The center of mass of the horizontal board should be near the center of the board, which is clearly to the right of the pivot. So the mass of the horizontal board has to be just enough that its gravitational force creates enough torque to balance the torque created by the brick.

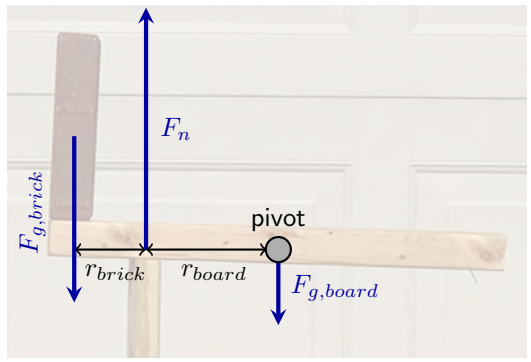


Figure 8.24: An alternate location for the pivot, allowing a solution for the normal force without knowing the mass of the board.[1]

For a system in static equilibrium, it isn't actually moving around any pivot point, so we are free to use any pivot point that is convenient. Notice that when we set the pivot at the point where the boards meet there was no torque generated by the normal force. If we instead chose a pivot at the center of mass of the horizontal board, its gravitational force will not generate any torque.

$$F_n = m_{brick} \cdot g + m_{board} \cdot g = 25.9\ \mathrm{N}$$

We could have found m_{board} using center of mass instead. Since we are free to choose our zero position, it is convenient to put it at the location of the pivot in Figure 8.23, so that the center of mass of the system is at $r = 0$. Using Equation 3.5...

$$r_{com} = 0 = \frac{m_{brick} \cdot r_{brick} + m_{board} \cdot r_{board}}{m_{brick} + m_{board}}$$

$$m_{brick} \cdot r_{brick} = -m_{board} \cdot r_{board}$$

$$m_{board} = \frac{(-1.76\ \mathrm{kg}) \cdot (-0.075\ \mathrm{m})}{0.15\ \mathrm{m}} = 0.88\ \mathrm{kg}$$

A negative value was assigned to r_{brick} because it is to the left of the zero point at the pivot.

We could also have found F_n without knowing m_{board} simply by choosing a different pivot point. This is illustrated in Figure 8.24.

$$\Sigma\tau = \tau_{brick} - \tau_n + \cancelto{0}{\tau_{board}} = 0$$

$$F_{g,brick} \cdot (r_{brick} + r_{board}) = F_n \cdot r_{board}$$

$$F_n = \frac{m_{brick} \cdot g \cdot (0.075\ \mathrm{m} + 0.15\ \mathrm{m})}{0.15\ \mathrm{m}} = 25.9\ \mathrm{N}$$

8.8 Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

General

- Objects have a "moment of inertia" that increases with mass and size. The farther an object's mass is distributed from the axis of rotation, the larger its moment of inertia.
- Moments of inertia can be added together for a complex shape.
- An object can only balance when its center of mass is supported.

Forces

- Torque $[\mathrm{N} \cdot \mathrm{m}]$ is the angular quantity that corresponds to force. It is a force applied in a direction that is not aligned with the pivot point, but separated from it by a lever arm distance.
- Torque causes angular acceleration, changes angular mometum over time, and does work through an angle.
- Static equilibrium is when an object is completely motionless. It only can occur when net force and net torque are both zero.

Motion

- Every equation of motion that we have learned works equally well for angular motion, simply by replacing the linear quantities with their angular counterparts.
- The SI unit for angular acceleration is $[\mathrm{rad}/\mathrm{s}^2]$.

Momentum

- A point mass has angular momentum if it has momentum in a direction that is not aligned with the pivot point, but separated from it by a lever arm distance.
- Angular momentum is related to moment of inertia and angular velocity in the same way that linear momentum is related to mass and linear velocity.
- Angular momentum is conserved for an isolated system.

Energy

- Rotating objects have rotational kinetic energy even if their center of mass is stationary.

Mathematical Models

equation		restrictions on the validity of the equation
$\alpha = \frac{a_T}{r}$	(8.1)	-none-
$\theta = \theta_0 + \omega_0 \cdot t + \frac{1}{2}\alpha \cdot t^2$	(8.2)	only valid when the net torque is constant
$\alpha_{avg} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{\Delta t}$	(8.3)	-none-
$\omega_{avg} = \frac{\Delta\theta}{\Delta t}$	(8.4)	-none-
$\omega_{avg} = \frac{1}{2}(\omega_i + \omega_f)$	(8.5)	only valid when the net torque is constant
$2\alpha \cdot \Delta\theta = \omega_f^2 - \omega_i^2$	(8.6)	only valid when the net torque is constant
$\tau = F \cdot r_\perp$	(8.7)	-none-
$r_\perp = r \cdot \sin\theta$	(8.8)	-none-
$\tau = \frac{\Delta L}{\Delta t}$	(8.9)	-none-
$E_{k,r} = \frac{1}{2} I \cdot \omega^2$	(8.10)	-none-
$L = I \cdot \omega$	(8.11)	-none-
$\tau = I \cdot \alpha$	(8.12)	-none-
$W = \tau \cdot \Delta\theta$	(8.13)	only valid when the torque is constant
$P = \tau \cdot \omega$	(8.14)	only valid when the torque is constant
$\Sigma F = 0$	(8.15)	constant velocity, static equilibrium
$\Sigma\tau = 0$	(8.16)	static equilibrium

8.9 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

Level 1 - Remember

8.1 **[W & N]** Make a table of linear quantities and their corresponding angular quantities.

8.2 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

Level 2 - Understand

8.3 **[N]** Calculate the moment of inertia I for the hammer as described in Section 8.1.

8.4 **[W, G, & N]** In Section 8.1, the lever arm for momentum is described in the "Words" column as being a distance that is related to the momentum vector but in the "Numbers" column it is instead described as being related to the velocity vector. How can both of these be correct, or was this an error?

8.5 **[W,G, & N]** Try to rank the objects in Figure 8.11 in order from those with the most mass concentrated near the axis of rotation to those with the most mass concentrated far from the axis of rotation. For the ones that have an r instead of an l, how do the numbers in the expressions for moment of inertia compare when you have put the shapes in order?

8.6 **[N]** At the end of Section 8.2 are three mathematical models with very little explanation. Replace the angular quantities in these three mathematical models with their linear counterparts, and verify that they are all valid mathematical models.

(a) Equation 8.10 corresponds to...
(b) Equation 8.11 corresponds to...
(c) Equation 8.7 corresponds to...

Level 3 - Apply

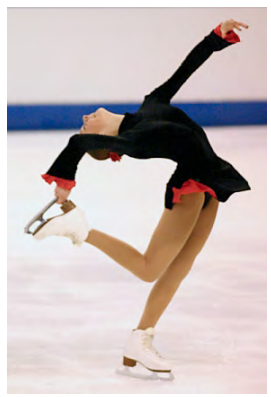
8.7 **[N]** Use dimensional analysis to show that Equation 8.11 and Equation 7.12 have the same units.

8.8 **[N]** If the ball described in Section 8.1 is spinning at a constant rate of 2.5 rad/s, find its kinetic energy and angular momentum, and the net torque that is being applied to keep it spinning at a constant rate.

8.9 **[N]** The analysis in Section 8.3 was done taking the two blades to be one object, a thin rod spinning about its center. That wouldn't work for a helicopter with three blades. Do the same analysis that was done in Section 8.3 but for the helicopter shown in the figure below. Take the mass of each blade, the length of each blade, the time, and the final angular speed to be the same as in Section 8.3.

8.10 **[W, G, & N]** If the Figure skater in Section 8.4 had a final position like the image of Elena Glebova shown here, how would the final angular momentum, final angular velocity, and final rotational kinetic energy compare to that for a position like that of Sasha Cohen in Figure 8.13? Explain. Elena is not in a shape that is very much like any of the shapes that are shown in Figure 8.11, but maybe the hollow sphere, with a radius of about 0.8 m would be close.

Three-bladed helicopter.[41]



Elena Glebova in a spin.[42]

8.11 **[G & N]** Use Figure 8.21 and the torque that was found in Section 8.6 to find the magnitude of the force of friction between the hoop and the ground.

Level 4 - Analyze

8.12 **[N]** Use dimensional analysis to find the SI unit for $E_{k,r}$. Is it the same unit that E_k has for linear motion? Should it have the same unit? Explain your answer.

8.13 **[W, G, & N]** What is the total moment of inertia for two thin rods, each of mass $m/2$ and length $l/2$, rotated around their ends? Compare your answer to that for the moment of inertia of a single rod with mass m and length l, rotated around its center. What do you notice? Explain this result.

Level 5 - Evaluate

8.14 **[N]** What effect does doubling angular velocity have on angular momentum and rotational kinetic energy?

8.15 **[W, G, & N]** In Section 8.6, a child is rolling a hoop. What would change if all of the knowns stayed the same, but the child was instead rolling a solid sphere?

8.16 **[W, G, & N]** In Section 8.6, a child is rolling a hoop. What would change if all of the knowns stayed the same, but the radius of the hoop doubled?

8.17 **[W, G, & N]** Try placing the pivot in Section 8.7 at the point where the brick is resting on the horizontal board. Does the torque equation also yield a valid solution at that location? Explain your answer.

8.18 **[W, G, & N]** In the analysis that was done in Section 7.5, it was noted that the angular momentum can't change when the string falls off of the bolt because there is no force to change the angular momentum. Now that we have learned more about angular momentum, we can see that the angular momentum was also not changing before the string fell off of the bolt. Why did the force of tension in the string, which was acting on the puck the whole time it was moving along a circular path, not change the angular momentum of the puck?

Level 6 - Create

8.19 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Begin a new concept map just for rotation.

8.20 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

8.21 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

Chapter 9

Stability and Oscillations

An object that is in static equilibrium is motionless, with zero net force and zero net torque acting on it. But what happens to such an object if a small net torque or net force is briefly applied?

In some situations, like that shown in the image of the acrobat balancing on a pile of chairs, that small torque will result in disaster. That's what makes acrobatic shows exciting to watch–we know that a tremendous amount of skill is needed to avert disaster.

In some situations, a small net torque or net force will cause an object's position to briefly shift, but then the object will quickly return to its original position.

And in some situations, a small net torque or net force will cause an object to rock or swing back and forth, until eventually friction brings everything back into static equilibrium. This back-and-forth motion is called an "oscillation."

In this chapter we will explore what conditions determine whether an object in static equilibrium will experience disaster, return to its original state, or begin to oscillate when it is disturbed, and we will see how motion, momentum, forces, and energy interact during oscillations.

Figure 9.1: A performer with Peking Acrobats performing in Nashville, Tennessee.[43]

9.1 The Great Pyramid of Giza

Words

The Great Pyramid of Giza is 137 m tall, 230 m long on each side, and has a mass of roughly 6 billion kg. Its center of mass is approximately 34 m above its base. For our purposes we will imagine that the sand it is sitting on is a hard, rough surface that can't be dented, and we will consider the pyramid to be one solid, unbreakable block, although in fact it is made up of roughly 2 million blocks of stone, and would break into roughly 2 million pieces if we attempted this with the actual pyramid!

First we will imagine briefly applying a force lifting one side of the pyramid to create a small net torque and see what happens to the pyramid. It would not be easy to apply such a force. When the pyramid is just sitting on the sand, there are two main forces acting on it: The force of gravity, which can be seen as acting downward on the center of mass of the pyramid, and the normal force which is pushing up from the sand all across the base of the pyramid but which is usually shown acting at a single point at the center.

You would have to use a force that is slightly more than half the weight of the pyramid, as shown in Figure 9.2. All of the normal force from the sand would be concentrated on the side opposite you, which is where the pyramid would pivot. So your force would have to create slightly more torque than the force of gravity, but your lever arm is twice as long as that for gravity.

Graphics

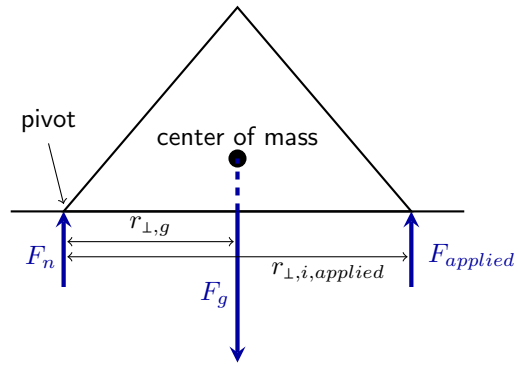


Figure 9.2: A huge applied force is needed to create enough torque around the pivot point to get the pyramid to move.[1]

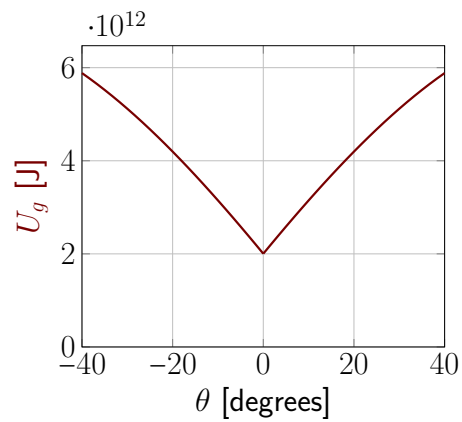


Figure 9.3: Gravitational potential energy is smallest with the pyramid flat on the ground.[1]

Numbers

Knowns	Unknowns
$m = 6 \times 10^9$ kg	$F_{applied,min}$
$h_{pyramid} = 137$ m	
$h_{com} = 34$ m	
$r_{\perp,g} = 115$ m	
$r_{\perp,applied} = 230$ m	

One thing we should be able to find in this situation is the minimum force needed to lift one side of the pyramid when it is sitting on its base, $F_{applied,min}$. To find that force, we need to analyze the situation shown in Figure 9.2 when it is in static equilibrium, just before the applied force is enough to lift the pyramid. The net torque around the pivot is zero in static equilibrium, so...

$$\tau_{net} = 0 = \tau_n + \tau_g + \tau_{applied}$$
$$= F_n \cdot \cancelto{0}{r_{\perp,n}} - F_g \cdot r_{\perp,g} + F_{applied} \cdot r_{\perp,applied}$$

Rearranging gives...

$$F_{applied} = \frac{F_g \cdot r_{\perp,g}}{r_{\perp,applied}} = \frac{F_g}{2}$$
$$= \frac{m \cdot g}{2} = 2.9 \times 10^{10} \text{ N}$$

This is the maximum force that keeps the pyramid in static equilibrium, which is equivalent to the minimum force to start moving it.

When you release this force, the pyramid would just fall back into its original position. So it takes a huge force to create enough torque to move the pyramid, and when this force is released, the pyramid quickly goes back to its original state. It is extremely stable, which is why it is still standing forty-five centuries after it was built!

It is also possible to consider stability from an energy perspective. Objects like to go to the place that will give them the smallest possible amount of potential energy. In the case of the pyramid sitting on its base, there is no other position that has less gravitational potential energy, so it is very stable.

Now let's imagine a different situation, in which the pyramid is upside-down. It can still be in static equilibrium if its center of mass is aligned perfectly above the tip, but if even the smallest amount of net torque is applied to the pyramid, the torque created by the force of gravity will continue to rotate the pyramid in the same direction as the initial torque, giving it more and more angular momentum,and it will come crashing down. This is an extremely unstable situation.

From an energy perspective, any shift in the angle of the upside-down pyramid will give it less gravitational potential energy, which means more kinetic energy. So it will move away from the unstable equilibrium position at higher and higher speed.

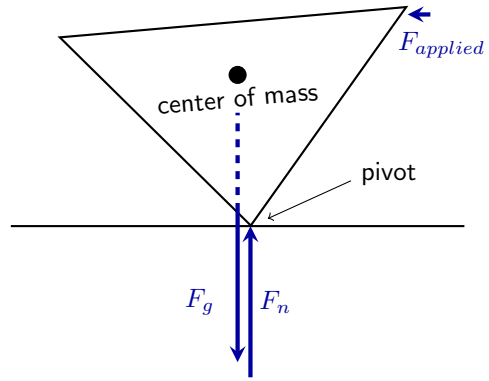


Figure 9.4: A small initial force creates torque around the pivot. This shifts the center of mass, so the gravitational force creates torque in the same direction.[1]

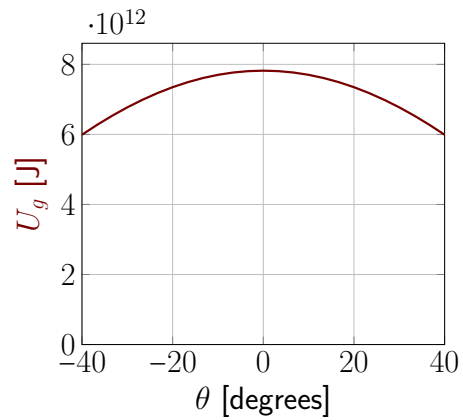


Figure 9.5: Gravitational potential energy is highest with the pyramid balanced on its tip. Here $0°$ is the upside-down position.[1]

There is no minimum force needed to start the pyramid in motion when it is standing on its tip; any applied force at all is enough to start the pyramid in motion.

Gravitational potential energy in Figures 9.2 & 9.4 are calculated using ground level as zero. Since the center of mass of the pyramid is always above ground level, the gravitational potential energy is always positive. The height above the ground is calculated as a function of the angle of rotation, keeping one corner of the pyramid on the ground.

9.2 A Horizontal Spring and Mass

Words

Now we will consider a horizontal system, ignoring any vertical forces because they cancel out and there is no vertical motion. A 2.4 kg block is motionless on a frictionless surface, connected to a solid wall by a spring with a spring constant of 90 N/m. Initially the block is resting at its equilibrium position, but then it is struck by an applied force that suddenly gives it a momentum of 1.2 kg·m/s to the left. What happens to the block?

If we look at this scenario in terms of energy, the block is given some kinetic energy and initially has no potential energy. But as the spring compresses, it stores potential energy, taking away the kinetic energy until the block completely stops moving. At that point, the spring begins pushing the block back toward its starting point, so the spring potential energy transforms back into kinetic energy. The block passes through its equilibrium position where it again has kinetic energy but there is no potential energy, and continues with the spring pulling against the motion until again all kinetic energy has been removed from the block and stored as spring potential energy. The block will continue to oscillate back and forth in this way.

As the block oscillates, it is not only the position that is continually changing in time, but also its velocity, acceleration, and momentum, and also the force that that spring is applying to it.

Graphics

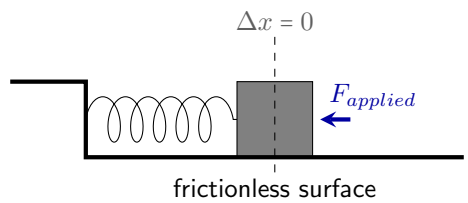


Figure 9.6: A mass on a spring is resting at its equilibrium position, and then is briefly hit with a horizontal applied force,giving it an initial momentum to the left.[1]

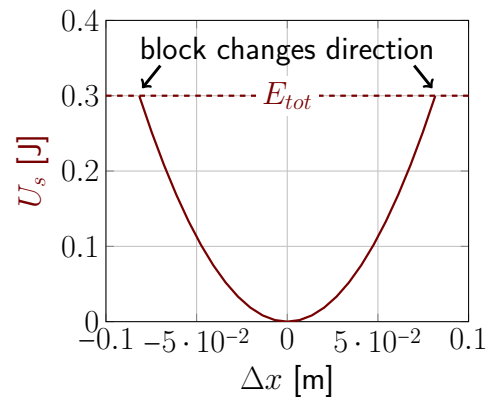


Figure 9.7: Spring potential energy is smallest at the equilibrium position, so the block oscillates around the equilibrium position.[1]

Numbers

Knowns	Unknowns
$m = 2.4\,\mathrm{kg}$	???
$k_s = 90\,\mathrm{N/m}$	
$\vec{p_i} = -1.2\,\mathrm{kg \cdot m/s}\,\hat{x}$	

We can use energy to analyze this situation, like we did for the pyramid. We are given the initial momentum and the mass, so we can use Equations 1.3 & 1.6 to find the initial kinetic energy.

$$E_k = \frac{1}{2}m \cdot v^2 = \frac{1}{2}m \cdot \frac{p^2}{m^2} = 0.3J$$

Rearranging gives kinetic energy in terms of momentum, which is a useful mathematical model even when we aren't dealing with oscillation:

$$\boxed{E_k = \frac{p^2}{2m}} \tag{9.1}$$

Now we can use conservation of energy to find how far the spring compresses while bringing the block to a stop.

$$E_f = E_i$$

$$\cancelto{0}{E_{k,f}} + \cancelto{0}{E_{th,f}} + U_{s,f} = E_{k,i} + \cancelto{0}{E_{th,i}} + \cancelto{0}{U_{s,f}}$$

$$\frac{1}{2}k_s \cdot \Delta x_f^2 = \frac{p_i^2}{2m}$$

Figure 9.8 shows the position, momentum, and net force on the block as a function of time, although we don't know what the time scale should be. All three of these follow a sine-wave type of pattern, although they are shifted with respect to each other. Notice that when The position is at a positive maximum the momentum is at zero and the force is at a negative maximum. When the position is zero the force is also zero and the momentum is at a maximum, either positive or negative.

We could also create graphs for velocity and acceleration of the block, but they would be very similar to the graphs for momentum and force. That is because velocity is the momentum divided by the mass, and acceleration is the net force divided by the mass.

The block and spring are in a stable equilibrium because the net force is always in a direction that pushes the block back toward the equilibrium position, never away from it. Or from an energy perspective we could say that the block and spring are in a stable equilibrium because the potential energy is always increasing and kinetic energy decreasing when the block is moving away from the equilibrium position.

This type of oscillation, with a force that is proportional to the distance from the equilibrium position and no friction-like forces, is called "simple harmonic motion."

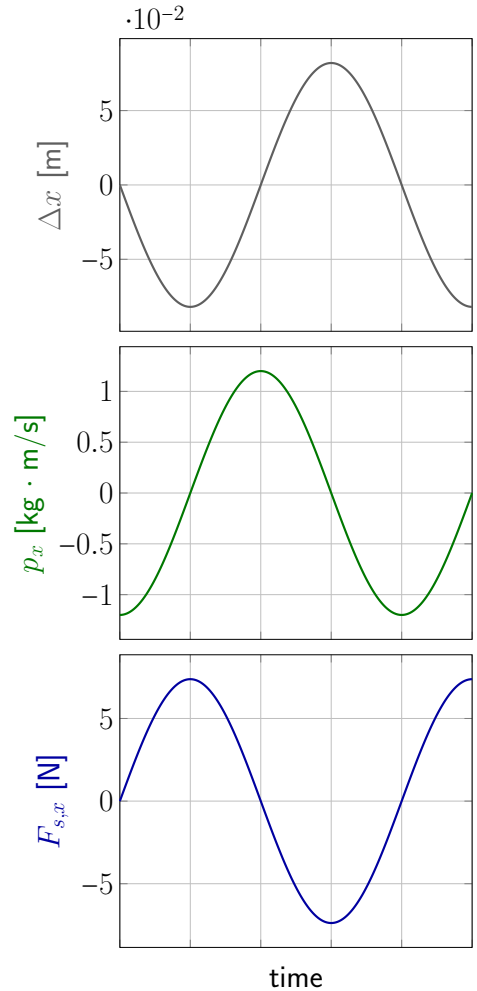


Figure 9.8: Position of the block, momentum of the block, and force on the block as a function of time. We don't yet know the time scale, but it is the same for all three graphs.[1]

Solving for Δx_f gives...

$$\begin{aligned}\Delta x &= \pm\sqrt{\frac{p_i^2}{k_s \cdot m}} \\ &= \pm\sqrt{\frac{(-1.2\,\mathrm{kg}\cdot\mathrm{m/s})^2}{90\mathrm{N/m}\cdot 2.4\mathrm{kg}}} = \pm 0.082\,\mathrm{m}\end{aligned}$$

Now that we know that the maximum displacement is 0.082 m, we can use Equation 6.1 to find the maximum force that is applied by the spring:

$$\begin{aligned}\overrightarrow{F_{s,max}} &= -\,k_s \cdot \overrightarrow{\Delta x_{max}} \\ &= -\,(90\,\mathrm{N/m}) \cdot \pm 0.082\,\mathrm{m}\,\hat{x} \\ &= \mp 7.38\,\mathrm{N}\,\hat{x}\end{aligned}$$

9.3 Jupiter's Moons

Words

Circular motion has a lot in common with oscillation. If viewed from the side, an object moving in a uniform circular path appears to be oscillating in exactly the same way as a mass on a spring. For example, the planet Jupiter has four moons that are large enough to see from the earth with an amateur telescope or even a good pair of binoculars. These four moons have roughly circular orbits around Jupiter, but they always appear from earth to be oscillating back and forth across Jupiter in a straight line. We will call that line the parallel direction.

The moon called "Callisto" is on the right of figure 9.9. Callisto's orbit has a radius of 1.9×10^9 m, and it completes one orbit every 17 earth days. We can use this information to determine the position, velocity, and acceleration in the parallel direction of Callisto as it orbits Jupiter.

If we take time $t = 0$ to be when Callisto is all of the way to the right as seen in Figure 9.10, then the initial position of Callisto is at its maximum positive value, $+r$. By the time it reaches the position at the top of Figure 9.10, its position in the parallel direction is zero. Then it goes to $-r$, back to zero, and finally back to $+r$ again when it completes one full period.

Graphics

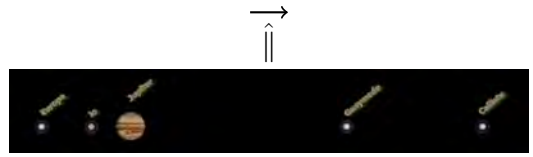


Figure 9.9: Jupiter and its four largest moons, as seen from Earth. The parallel direction $\hat{\parallel}$ is in the plane of the moons' orbits.[44]

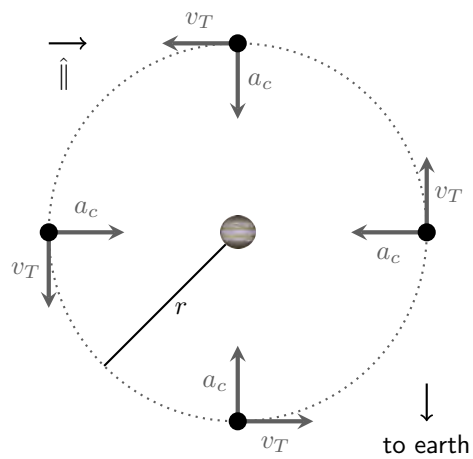


Figure 9.10: Callisto is shown at four different points in its near-circular orbit, but from earth we can only see it going back and forth in the horizontal $\left(\hat{\parallel}\right)$ direction, so we see only the parallel components of $\vec{x}$, $\vec{v}$, and $\vec{a}$.[45]

Numbers

Knowns	Unknowns
$r = 1.9 \times 10^9$ m	$x_\parallel$
$T = 17$ earthdays	$v_\parallel$
	$a_\parallel$

First we should convert the period of the orbit to SI units: $17\,\text{earthdays} = 1.5 \times 10^6$ s. We have used angular speed more than period, so we can use Equation 7.10 to find ω.

$$\omega = \frac{2\pi}{T} = 4.3 \times 10^{-6}\,\text{rad/s}$$

At all times in this full circle, its position in the parallel direction can be described by...

$$x_\parallel = r \cdot \cos(\omega \cdot t)$$

Velocity in the parallel direction also varies, but it starts at zero at $t = 0$, since it is only moving upward in Figure 9.10. Then it goes in the negative direction, goes back to zero, goes in the positive direction, and finally returns to zero as Callisto completes one period.

Centripetal acceleration is always pointed toward the center of the circle, so when the position is at its maximum positive value, acceleration is at its maximum negative value, and vice-versa.

When we analyzed the block on the spring, we considered momentum instead of velocity and net force instead of acceleration, but remember that momentum is proportional to velocity and net force is proportional acceleration, so if we graph them then the shapes of their graphs will look the same, just with a different scale.

Comparing Figures 9.8 and 9.11, we can see that their shapes are very similar. Really there are only two major differences. Their scales are different, and the curves in Figure 9.8 are all shifted horizontally by the same amount compared to the curves in Figure 9.11. That is because the block on the spring was initially at zero and moving to the left while Callisto was initially at the far right and not moving in the parallel direction.

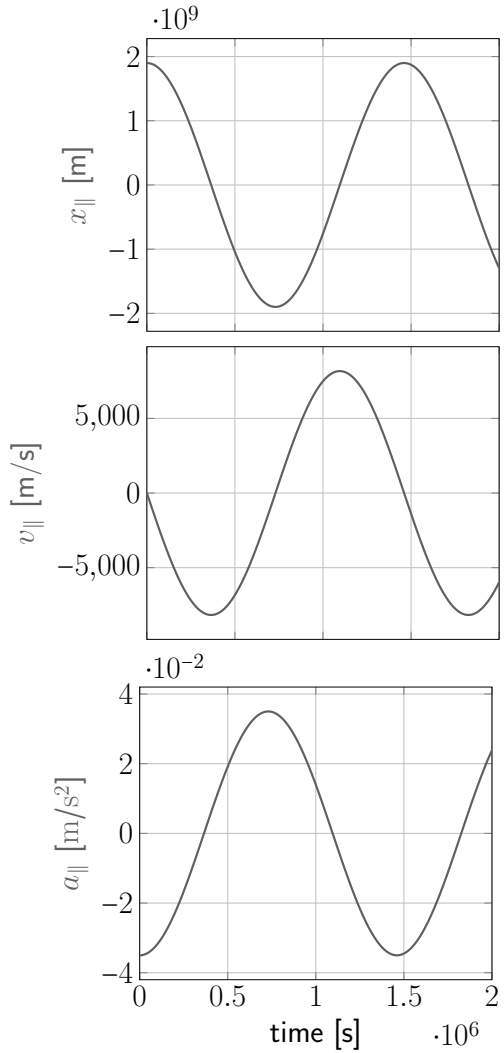


Figure 9.11: Parallel components of the position, velocity, and acceleration of Callisto.[1]

We know from Equation 7.9 that $v_T = r \cdot \omega$, and the maximum value of $v_{parallel}$ is v_T, so Callisto's velocity in the parallel direction can be described by...

$$v_\parallel = -r \cdot \omega \cdot \sin(\omega \cdot t)$$

Since our other mathematical models in this section depend on ω, it would be convenient to have a mathematical model for centripetal force that is also dependent upon ω. We can create this by combining Equations 7.6 & 7.9 to get...

$$\boxed{a_c = r \cdot \omega^2} \tag{9.2}$$

Using this for the magnitude of the acceleration, Callisto's acceleration in the parallel direction is...

$$a_\parallel = -r \cdot \omega^2 \cdot \cos(\omega \cdot t)$$

Combining Equation 9.2 with Equation 7.7 gives us an expression that can be used for centripetal force when we are given angular speed:

$$\boxed{F_c = m \cdot r \cdot \omega^2} \tag{9.3}$$

9.4 Simple Harmonic Motion

Words

Simple harmonic motion was described in Section 9.2 as being caused by a force that is proportional to the distance from the equilibrium position. Here we will consider more closely at what simple harmonic motion (SHM) looks like.

The position of an object that is undergoing SHM moves back and forth in a regular, periodic way. This means that if you wait for the object to go through one complete cycle of its motion, it will follow exactly the same motion through the next complete cycle. The amount of time that you have to wait for the object to go through one complete cycle is called the period. This is exactly the same way that a period is used when talking about circular motion: it is the amount of time needed for an object to complete one full rotation.

For SHM, during one period the object never experiences any sudden changes in motion, but is gradually speeding up and slowing down. Its position, velocity, acceleration, momentum, and the net force applied to it all follow the same type of curve, which could be described as a sine shape or a cosine shape.

If you look at a sine curve and a cosine curve, you can see that they both have the same shape, but they are shifted with respect to each other. For simple harmonic motion, you could change one to the other just by timing the motion starting from a different point in the cycle, effectively moving the point where time is defined as zero.

Graphics

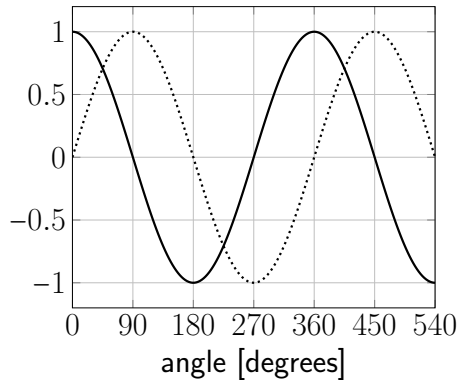


Figure 9.12: Cosine (solid) and sine (dotted) functions have the same shape but are shifted with respect to each other.[1]

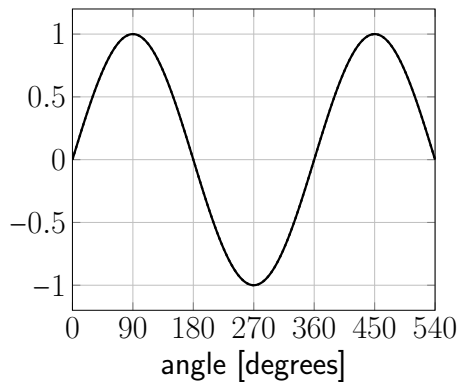


Figure 9.13: Sine of the angle and cosine of the angle minus $90°$ are the same.[1]

Numbers

We can create a mathematical model of simple harmonic motion that works for any starting conditions by introducing an angle ϕ, which is the amount by which the object has to move to reach its maximum positive position. It makes sense to think of it in terms of an angle when the object is following a circular path like Callisto, but for an object like a spring on a block it may be easier to think of it as a shift in starting time to the time that the block reaches the far right position. Then the mathematical model for position during SHM becomes...

$$\boxed{x = \mathcal{A} \cdot \cos(\omega \cdot t - \phi)} \tag{9.4}$$

...where $\mathcal{A}$ is the amplitude of the motion, the maximum distance that the object travels from the equilibrium position. In the case of SHM $\mathcal{A}$ has units of length, but this mathematical model appears in other contexts where $\mathcal{A}$ can represent other physical quantities like an electric field.

If we consider SHM of an object that starts at equilibrium moving in the positive direction, the object travels through $1/4$ of a period before reaching the maximum positive position. One period is $360°$, so ϕ is $1/4$ of a period, or $90°$. Figure 9.13 shows a plot of the cosine of an angle minus $90°$, illustrating that a shift of $90°$ shifts the cosine function to match an object starting at zero (on the vertical axis) and moving in the positive direction as the angle increases. This is completely analogous to an object starting at zero position and moving in the positive direction as time increases.

We have already been introduced to the idea of a "period" as being the time for an object that is following a circular path to complete one revolution. For SHM, the period of the oscillation is the time required for the object to complete one full cycle of motion. That could be the time required for the object to move from the maximum positive position back to the maximum positive position. It is also the time time required to go from the minimum position back to the minimum position.

Looking at Figure 9.14, you can see that during one period T the object passes through the equilibrium position twice, so if you want to measure one period from the equilibrium position then it is important to measure from the time the object passes the equilibrium position to the time when the object again passes through the equilibrium position *in the same direction.*

The maximum distance that the object travels from the equilibrium position is called the amplitude of the motion.

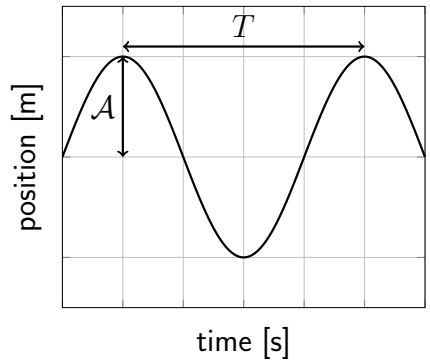


Figure 9.14: Period T and amplitude $\mathcal{A}$. Note that $\mathcal{A}$ is measured only from the equilibrium position, not top-to-bottom.[1]

Equation 9.4 is similar to the expression found for $x_{\parallel}$ in Section 9.3, just replacing r with $\mathcal{A}$ and including ϕ inside the cosine function. Following the same pattern of analysis that was used in Section 9.3, we can find mathematical models for the $\hat{x}$ components of velocity and acceleration for SHM:

$$\boxed{v_x = -\mathcal{A} \cdot \omega \cdot \sin(\omega \cdot t - \phi)} \tag{9.5}$$

. . . and. . .

$$\boxed{a_x = -\mathcal{A} \cdot \omega^2 \cdot \cos(\omega \cdot t - \phi)} \tag{9.6}$$

9.5 A Vertical Spring and Mass

Words

Now we will consider a vertical system with no horizontal forces and no friction-like forces. A spring with a spring constant of 4.5 N/m is hanging from the ceiling. You attach a 0.6 kg mass to the end, holding it so that the end of the spring is in the same position that it was in before you attached the mass, and then you suddenly release the mass. What happens?

Probably you can "see" what will happen when you release it. Because of gravity, the mass will fall, stretching the spring, and then will bounce back to the top, and it will continue repeating this same pattern until friction with the air and within the spring slows it to a stop, but we are assuming no friction so it would just continue to bounce forever.

Can we be more precise about the motion of the mass? If we start by looking at this scenario in terms of energy, the mass initially has no kinetic energy and there is no no potential energy stored in the spring, but it is being held up, so it has gravitational potential energy. As it falls, the mass loses gravitational potential energy but gains kinetic energy, and the spring also gains spring potential energy since it is stretching. When the mass reaches the lowest point in its bounce it stops moving, so at that point it has lost all of its kinetic energy. It has also lost gravitational potential energy, and since we know it will not go lower than this point we can call this height $y = 0$, so it has no gravitational potential energy. All of the energy has changed into spring potential energy.

Graphics

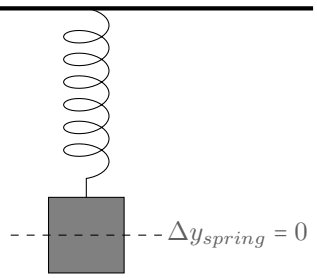


Figure 9.15: A mass is attached to a spring and held in place at the equilibrium position of the empty spring.[1]

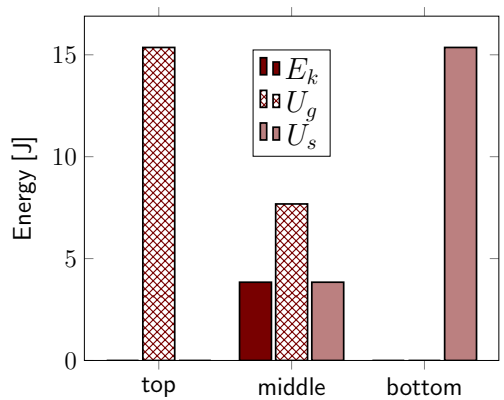


Figure 9.16: Energy bar graphs for the spring and mass.[1]

Numbers

Knowns	Unknowns
$m = 0.6\,\mathrm{kg}$	???
$k_s = 4.5\,\mathrm{N/m}$	
$\vec{p_i} = 0\,\mathrm{kg \cdot m/s}$	
$g = 9.8\,\mathrm{m/s^2}$	

Let's begin with conservation of energy. There are three important heights to consider: the top position, the bottom position, and the point somewhere in the middle where the system is in equilibrium. We can call these y_{top}, y_{bot}, and y_{mid}, and to simplify the math we can make $y_{bot} = 0$. We know more about the energy at the top and bottom, so we will begin with those...

$$E_{bot} = E_{top}$$

$$\cancelto{0}{E_{k,bot}} + \cancelto{0}{U_{g,bot}} + U_{s,bot} = \cancelto{0}{E_{k,top}} + U_{g,top} + \cancelto{0}{U_{s,top}}$$

$$U_{s,bot} = U_{g,top}$$

$$\frac{1}{2}k_s \cdot \left(y_{top} - \cancelto{0}{y_{bot}}\right)^2 = m \cdot g \cdot \left(y_{top} - \cancelto{0}{y_{bot}}\right)$$

Solving for y_{top} gives...

$$y_{top} = \frac{2m \cdot g}{k_s} = 2.61\,\mathrm{m}$$

This means that the bottom position is 2.61 m below the point where the mass was released. The amplitude of the oscillation is therefore...

At the lowest point, the spring begins pulling the mass back up, so we know that the spring force must be larger than the force of gravity. The kinetic energy and the gravitational potential energy both increase as the mass starts to rise, while the spring potential energy decreases. The mass is not moving at the top or the bottom, but has non-zero velocity between the top and bottom, so there must be some point where the speed and the momentum are at a maximum. Momentum increases as long as the force is in the same direction as the velocity, so there must be some point in the path of the mass where the net force drops to zero and then changes direction. In other words, there must be a new equilibrium position along the path followed by the mass.

Since the force of gravity is constant throughout the path followed by the mass and the spring force increases proportionally with the distance the spring is stretched, the net force is also proportional to the distance from the equilibrium position. That means the oscillation of the mass is simple harmonic motion.

Remarkably, the period of the oscillation does not depend on gravity. It also doesn't depend on the distance that the spring is stretched. The period gets longer (so a slower oscillation) as the mass increases, and the period gets shorter (so a faster oscillation) as the strength of the spring (the spring constant) increases.

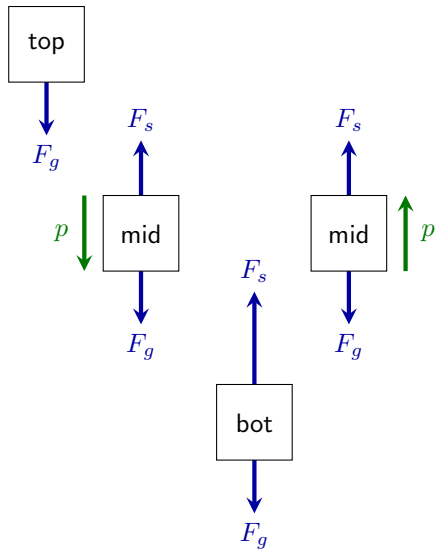


Figure 9.17: FBD of the mass in four different positions as it oscillates, including arrows showing the direction of the momentum when the mass is in the equilibrium position.[1]

$$\mathcal{A} = \frac{y_{top} - y_{bottom}}{2} = \frac{m \cdot g}{k_s} = 1.30\,\mathrm{m}$$

From the perspective of forces, Figure 9.17 shows the two forces that act on the mass. The net force is zero at the equilibrium position and increases linearly with distance from the equilibrium position. The equilibrium position is the point where...

$$F_g = F_s$$
$$m \cdot g = k_s \cdot y_{top} - y_{mid}$$

Solving for y_{mid} gives...

$$y_{mid} = y_{top} - \frac{m \cdot g}{k_s} = 1.30\,\mathrm{m}$$

...at the center of the oscillation.

When the mass is at the top position, the only force affecting it is gravity, so its acceleration is $-g\,\hat{y}$. The mass is also in SHM, so its acceleration is given by Equation 9.6, when the acceleration is at its maximum negative value, so $-\mathcal{A} \cdot \omega^2 = -g$. Solving for ω and using the expression we found earlier for $\mathcal{A}$ gives...

$$\omega = \sqrt{\frac{g}{\mathcal{A}}} = \sqrt{g \cdot \frac{k_s}{m \cdot g}} = \sqrt{\frac{k_s}{m}}$$

The angular velocity does not depend on gravity, so the result is valid for any spring-mass system, horizontal or vertical. It is usually expressed in terms of the period, using Equation 7.10:

$$\boxed{T = 2\pi\sqrt{\frac{m}{k_s}}} \tag{9.7}$$

9.6 A Pendulum

Words

We have considered stability for objects sitting on the ground and objects connected to springs. Now we will consider an object that is hanging from a rope. When the object is hanging straight down and not moving, it is in an equilibrium position. Is it a stable equilibrium?

If we think about it from the perspective of energy, pushing the object away from the equilibrium position in either direction will make it swing upward, increasing its gravitational potential energy. That means the equilibrium position is stable. The object would begin to swing back and forth. Is the oscillation simple harmonic motion? It depends...

We saw with the orbit of Callisto around Jupiter that uniform circular motion looks like SHM when viewed from the side. Let's try looking at this motion the same way, in the horizontal direction. If we start with the mass held out horizontally, that will be the maximum possible displacement. But at that point there are no forces in the horizontal direction, when for SHM the horizontal force would be at a maximum at that position. So a pendulum does not have SHM in the horizontal direction.

Let's try looking at the vertical direction. Here we see that the equilibrium position is at the bottom, but SHM always has the equilibrium position in the center. So no SHM in the vertical direction.

We still have another option–maybe there is SHM in the tangential direction?

Graphics

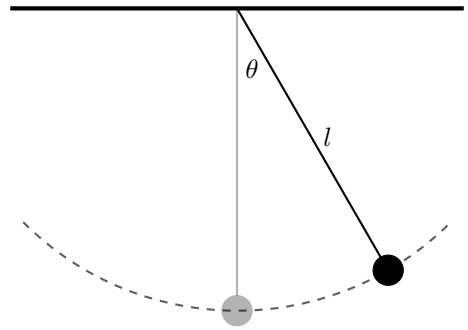


Figure 9.18: A mass hung from a light rope.[1]

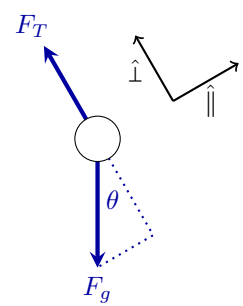


Figure 9.19: FBD of the mass on the string when at the position shown in Figure 9.18[1]

Numbers

Knowns	Unknowns
$m = 3\,\mathrm{kg}$	ω ?
$l = 1.5\,\mathrm{m}$	T ?
$g = 9.8\,\mathrm{m/s^2}$	

When the mass is a path length $s = l \cdot \theta$ (with θ measured in radians) away from the equilibrium position, the force in the tangential ($\hat{\parallel}$) direction as shown in Figure 9.19 would be $F_g \cdot \sin\theta$. So the position is proportional to θ but the force is proportional to $\sin\theta$.

Simple harmonic motion requires that the force be proportional to the displacement, so a pendulum does not display SHM in the tangential direction.

That is unfortunate, because SHM uses such easy mathematical models. This is a good time to remember that many of our mathematical models are really just approximations that work well in some situations. We've already used an approximation for gravitational force when analyzing the pendulum, one that is only valid for a limited range of heights at the surface of the earth.

Let's see if our mathematical models for SHM might also be valid in some limited range of angles for the pendulum.

To be able to work with the numbers, let's make the mass 3 kg, and hang it at the end of a light, 1.5-m-long rope.

The analysis shown in the "Numbers" column demonstrates that in fact there is not SHM in the tangential direction, but for small angles, when the pendulum is not swinging far away from the vertical position, the motion is *almost* the same as SHM, so this "small-angle approximation" is often used for analyzing the motion of the pendulum.

In the small-angle approximation, the period of the pendulum does not depend on the mass, but it does depend on the acceleration of gravity at the earth's surface and the length of the rope. The period increases (so it slows down) as the rope gets longer. If the pendulum were in a location where the acceleration caused by gravity is smaller, for example on the surface of the moon, the period of the pendulum would also increase.

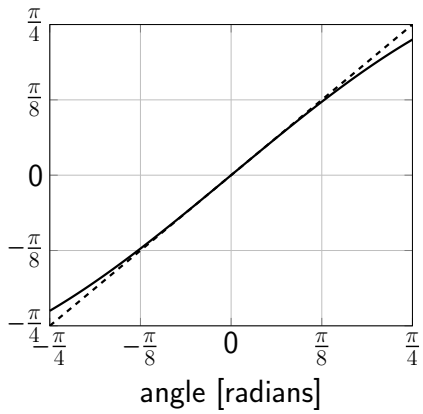


Figure 9.20: Sine of the angle (solid line) compared with the angle itself (dashed line), measured in radians.[1]

Figure 9.20 shows that for small angles (measured in radians), sine of the angle is roughly equal to the angle itself. So if we keep the angle small, the force is roughly proportional to the distance as measured along the path length. That allows us to use the mathematical models for SHM. Converting from $\hat{x}$ directions to tangential directions, the maximum positive value for position along the path length for SHM as given by Equation 9.4 is...

$$s_{max} = \mathcal{A} = l \cdot \theta$$

The maximum negative value for acceleration in the tangential direction, which occurs at the same position, is similarly given by Equation 9.6...

$$-a_{T,max} = -\mathcal{A} \cdot \omega^2 = -(l \cdot \theta) \cdot \omega^2$$

...where the expression found above for $\mathcal{A}$ has been substituted in. We can then use Equation 1.8 to convert from a_T to F_g...

$$-\frac{F_g}{m} \cdot \sin\theta = -\frac{\cancel{m} \cdot g}{\cancel{m}} \cdot \sin\theta = -(l \cdot \theta) \cdot \omega^2$$

Solving for ω and canceling θ with $\sin\theta$ since we are using the small-angle approximation gives...

$$\omega = \sqrt{\frac{g \cdot \sin\theta}{l \cdot \theta}} = \sqrt{\frac{g}{l}}$$

This expression is usually given in terms of the period, and is valid for small angles:

$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}} \tag{9.8}$$

9.7 Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

General

- Simple harmonic motion looks exactly the same as one dimension of uniform circular motion.
- The time needed for an object to go through one complete cycle of simple harmonic motion is called the period.
- The period of a spring-mass system depends only on the mass and the spring constant. As the mass increases the period gets longer, and as the spring constant increases the period gets shorter.
- When the size of the swing is small, the period of a pendulum depends only on its length and the acceleration due to gravity. As the length increases, the period gets longer, and as the acceleration due to gravity increases the period gets shorter.

Forces

- An object is in a stable equilibrium position if disturbing the object causes a force or torque that pushes the object back toward the equilibrium position.
- An object is in an unstable equilibrium position if disturbing the object causes a force or torque that pushes the object even further from the equilibrium position.
- Simple harmonic motion is caused by a net force that is proportional to the distance from the equilibrium position.

Motion

- An oscillation is a back-and-forth motion.
- The maximum distance that an object travels from the equilibrium position during simple harmonic motion is called the amplitude of the motion.

Momentum

- It is an object's momentum that causes it to continue moving through the equilibrium point during an oscillation.

Energy

- Objects like to go to the position that will give them the smallest possible amount of potential energy.
- An object is in a stable equilibrium position if moving the object away from the equilibrium position causes the object to gain potential energy.
- An object is in an unstable equilibrium position if moving the object away from the equilibrium position causes the object to lose potential energy.

Mathematical Models

equation		restrictions on the validity of the equation
$E_k = \frac{p^2}{2m}$	(9.1)	-none-
$a_c = r \cdot \omega^2$	(9.2)	-none-
$F_c = m \cdot r \cdot \omega^2$	(9.3)	-none-
$x = \mathcal{A} \cdot \cos(\omega \cdot t - \phi)$	(9.4)	Simple harmonic motion (SHM)
$v_x = -\mathcal{A} \cdot \omega \cdot \sin(\omega \cdot t - \phi)$	(9.5)	Simple harmonic motion (SHM)
$a_x = -\mathcal{A} \cdot \omega^2 \cdot \cos(\omega \cdot t - \phi)$	(9.6)	Simple harmonic motion (SHM)
$T = 2\pi\sqrt{\frac{m}{k_s}}$	(9.7)	SHM of a spring-mass system
$T = 2\pi\sqrt{\frac{l}{g}}$	(9.8)	Small-angle approximation, SHM of a pendulum

9.8 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

Level 1 - Remember

9.1 **[W]** What are the factors that affect the period of the simple harmonic motion of a spring-mass system?

9.2 **[W]** What are the factors that affect the period of the simple harmonic motion of a pendulum that is swinging at a small angle?

9.3 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

Level 2 - Understand

9.4 **[W & G]** When an object that is experiencing simple harmonic motion to the left and to the right, in what direction is its velocity when its position is...

(a) ... at the far right?

(b) ... at the far left?

(c) ... at the equilibrium position?

9.5 **[W & G]** When an object that is experiencing simple harmonic motion to the left and to the right, in what direction is its acceleration when its position is...

(a) ... at the far right?

(b) ... at the far left?

(c) ... at the equilibrium position?

Level 3 - Apply

9.6 **[W, & G]** Analyze the static equilibrium position of the acrobat in Figure 9.1:

(a) Create a sketch that includes the main forces acting on the acrobat including an applied torque that could be used to shift the acrobat out of equilibrium, as was done with the pyramid in Section 9.1.

(b) Use the sketch to explain whether the acrobat is in a stable equilibrium position or an unstable equilibrium position.

(c) Create a graph of gravitational potential energy vs angle, as was done with the pyramid in Section 9.1. Without knowing mass or lengths, just draw the shape of the curve for the graph–no scale is needed.

(d) Use the energy graph to explain whether the acrobat is in a stable equilibrium position or an unstable equilibrium position.

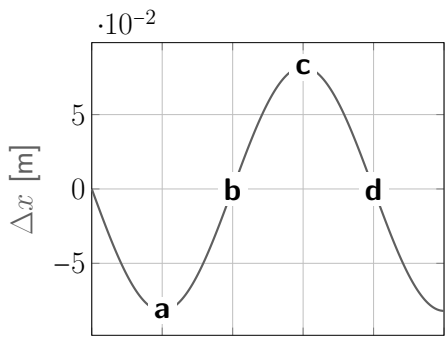
9.7 **[G]** Figure 9.6 shows the block and spring at the time when the applied force suddenly gives momentum to the block. This is moment corresponds to the left side of the graphs in figure 9.8, where the displacement and force from the spring are both zero, and the momentum is at a large negative value. Create your own sketches of the block and spring that correspond to the other times as marked in the displacement graph from figure 9.8 that is reproduced below.

(a) Position "**a**"

(b) Position "**b**"

(c) Position "**c**"

(d) Position "**d**"



A part of the graph from Section 9.2 with positions marked for this question.[1]

9.8 **[G & N]** Figure 9.10 has numbers on all of the axes of each graph, but except at $t = 0$ none of the peaks, valleys or zero crossings occur on any of the gridlines. Calculate the values for...

(a) ...the time when the position first crosses zero.

(b) ...the time when the position first reaches its maximum negative value.

(c) ...the time when the velocity first reaches its maximum positive value.

(d) ...the time when the acceleration first reaches its maximum positive value.

9.9 **[G & N]** Figure 9.10 has numbers on all of the axes of each graph, but except at $t = 0$ none of the peaks, valleys or zero crossings occur on any of the gridlines. Calculate the values for...

(a) ...the maximum positive value of the position.

(b) ...the maximum negative value of the position.

(c) ...the maximum positive value of the velocity.

(d) ...the maximum negative value of the velocity.

(e) ...the maximum positive value of the acceleration.

(f) ...the maximum negative value of the acceleration.

9.10 **[N]** Callisto has a mass of $1 \times 10^{23}\,\mathrm{kg}$. Find the following:

(a) The angular momentum Callisto has due to its orbit around Jupiter.

(b) The magnitude of the force of gravity between Callisto and Jupiter.

(c) The magnitude of the linear momentum Callisto has in its orbit around Jupiter.

9.11 **[N]** Find the period of the simple harmonic motion of the spring and mass from Section 9.5.

9.12 **[N]** The unknowns listed in Section 9.6 were never actually found. Find them.

Level 4 - Analyze

9.13 **[W]** A statement is made in Section 9.1 that objects like to go to the place that will give them the smallest possible amount of potential energy. Does this agree with what you have learned in earlier chapters? Give at least two examples where this is the case. One example should be an object that is held above the ground and then released.

9.14 **[N & G]** Create two more graphs to go with the others in figure 9.8. One should be for velocity of the block and one for acceleration of the block. The vertical scales should be correct for both graphs.

9.15 **[N]** Find the period of the simple harmonic motion of the spring and mass from Section 9.2.

9.16 **[G & N]** Determine the heights of all of the energy bars in Figure 9.16.

9.17 **[N]** How could you change the pendulum in Section 9.6 so that its period would double?

Level 5 - Evaluate

9.18 **[W, G, & N]** The pyramid sitting on its base in Figure 9.2 is said to be in a stable equilibrium position. But that is only true up to a certain point. How far would the pyramid have to be tilted from this position to put it into an unstable equilibrium?

(a) Make a sketch of the pyramid when it reaches this unstable equilibrium position.

(b) Explain in words what it is that makes this position an unstable equilibrium

(c) Find the numerical value for the angle at which this unstable equilibrium occurs.

9.19 **[W, G, & N]** In Figure 9.16 the kinetic energy appears to be the same as the spring potential energy at the middle height. Is there any height for this physical system (whether shown in this energy bar graph or not) where...

(a) ...the kinetic energy is equal to the gravitational potential energy?

(b) ...the spring potential energy is equal to the gravitational potential energy?

If so, find the height. If not, explain why not.

9.20 The "small-angle approximation" doesn't actually say how small the angle needs to be. The critical factor in this approximation is whether $\sin\theta$ is reasonably close to θ when measured in radians. At what angle is $\sin\theta$ different from θ by...

(a) ...0.1%?

(b) ...1%?

(c) ...10%?

(d) ...a factor of 2?

Level 6 - Create

9.21 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Begin a new concept map for stability, oscillations, and waves.

9.22 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

9.23 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

Chapter 10

Solids

Up to this point, almost every object that we have dealt with has been assumed to be completely rigid. In this chapter we will start allowing "solid" objects to bend, stretch, and break. When objects bend and stretch, they can transfer energy in waves. In this chapter we will explore two different types of waves in solids: *transverse waves* and *longitudinal waves.*

Figure 10.1: Transverse waves traveling along battle ropes.[1]

Transverse waves occur when a material bends, as with "battle ropes." A transverse wave occurs when each segment of the object, in this case each small section of rope, moves back and forth perpendicular to the direction that the wave is traveling. The waves travel horizontally along the length of the rope, but each individual section of rope is moving vertically, perpendicular to the direction in which the wave is moving.

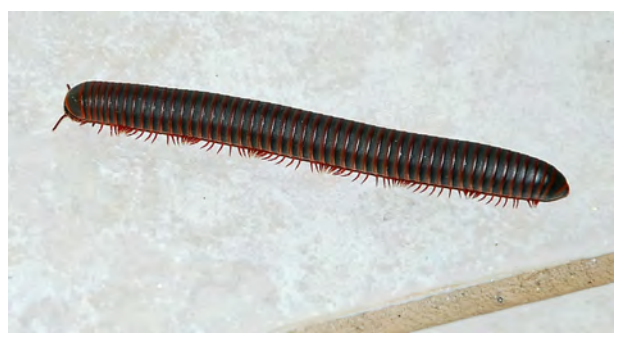


Figure 10.2: A millipede moving its legs in a longitudinal-wave-like fashion.[46]

Longitudinal waves occur when a material stretches and compresses. When a millipede walks, it moves its legs in longitudinal waves. A longitudinal wave occurs when each segment of the object, in this case the legs, moves back and forth parallel to the direction that the wave is traveling. The waves travel horizontally along the length of the millipede's body, and each individual leg also moves forward and back horizontally along the length of the millipede's body. This creates areas of compression, where the legs are closely spaced, and areas of "rarefaction" where the legs are widely spaced.

10.1 Modeling a Rope

Words

In the "Numbers" column we have been using "mathematical models" to describe physics with equations. These models are useful tools for describing physical situations, but often they are really only approximations that work under certain conditions. We can also create other types of models to help us simplify physical scenarios to make them easier to understand. Now that we are allowing solid objects to bend and stretch, it can be helpful to think of them not as single objects but as a system of many small, interconnected objects.

For example, a rope can be modeled as a line of small pieces of rope that are all held together. We want the rope to be able to bend and stretch, so we can think of all of the small pieces as being held together with springs. Ropes don't stretch very much, but they bend easily, so for our model of a rope we can assume that each individual piece in Figure 10.3 can move up and down, but not left or right.

What would happen if we started moving the rightmost piece of rope up and down with a period of 0.5 s, like the people are doing in Figure 10.1? Let's assume that the rope has a length of 6 m, a mass of 12 kg, and is being pulled horizontally with a tension of 150 N.

In our model of a rope, each piece of rope is connected the the pieces on either side of it, so they can pull on each other.

Graphics

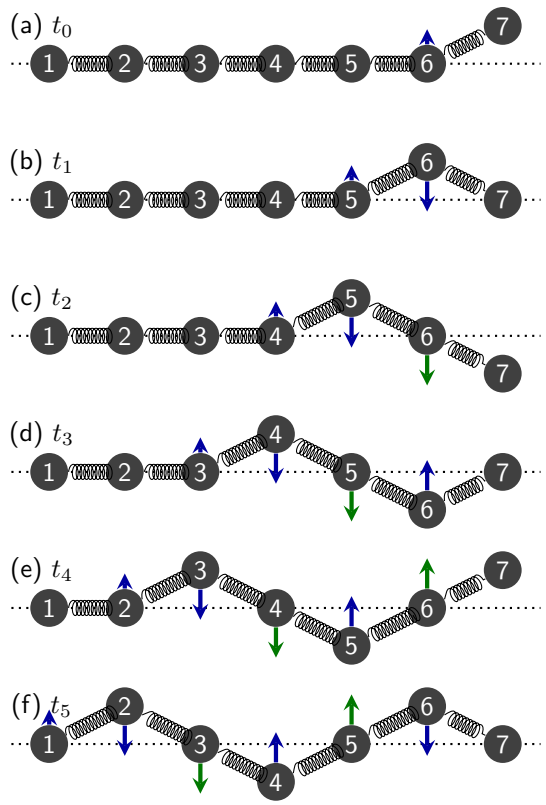


Figure 10.3: Parts (a) through (f) are diagrams of the same pieces of rope at different times. If section "7" of the rope is moved up and down, it creates a chain reaction of forces and momentum traveling left through the rope.[1]

Numbers

Knowns	Unknowns
$m = 12\,\mathrm{kg}$	???
$l = 6\,\mathrm{m}$	
$T = 0.5\,\mathrm{s}$	
$F_T = 150\,\mathrm{N}$	

Each successive part in Figure 10.3 is a fixed time interval after the previous part. Close examination shows that the various "pieces" of the rope are all affected in the same way, but at slightly different times. Note that in part (b) of the figure, section 6 of the rope has no momentum (no green arrow) but is experiencing a large downward force (blue arrow). In part (c), section 5 of the rope has no momentum but it is experiencing a large downward force. In part (d) it is section 4 that is in the same condition, and this trend continues to the left at each successive time interval. There is a transverse wave moving to the left.

As long as the transverse displacement of the wave is small compared to the length of the wave, the speed of a transverse wave on a rope is given by:

$$\boxed{v = \sqrt{\frac{F_T}{\mu_m}}} \tag{10.1}$$

...where μ_m is the linear mass density of the rope, which is defined as...

$$\boxed{\mu_m \equiv \frac{m}{l}} \tag{10.2}$$

These forces and the resulting momenta of the pieces of rope are shown at different times in Figure 10.3. The force shown is the net vertical force caused by with the two pieces on either side of each piece. The forces on piece 7 aren't shown because there are unknown forces coming from an external source–only the forces that are within the rope are shown. Each of the other pieces 1-6 are affected in exactly the same way, just at different points in time. Notice that whatever happens to piece 4, for example, next happens to piece 3. This creates a wave moving to the left, as shown in Figure 10.4.

The speed of the wave depends only on the tension in the rope and the "linear mass density" of the rope, where linear mass density is the mass of the rope per unit of length.

We can see a sine-wave-like pattern developing in the rope. We saw patterns like this in Chapter 9, but before it was normally in a graph with time as the horizontal axis. This time we are seeing the same pattern appearing not in time but in physical space. The time between successive peaks was called the period. The distance between successive peaks in space is called the wavelength.

Wavelength and period are related, because a period is the time required for one wavelength to pass by a given point in space. This can be seen by looking at piece 6 in Figure 10.4. It goes through one complete period from (b) t_1 to (f) t_5, during the time that one wavelength passes by at speed v.

Frequency, measured in Hertz (Hz), is the inverse of the period, or the number of times a wavelength passes a given point per second.

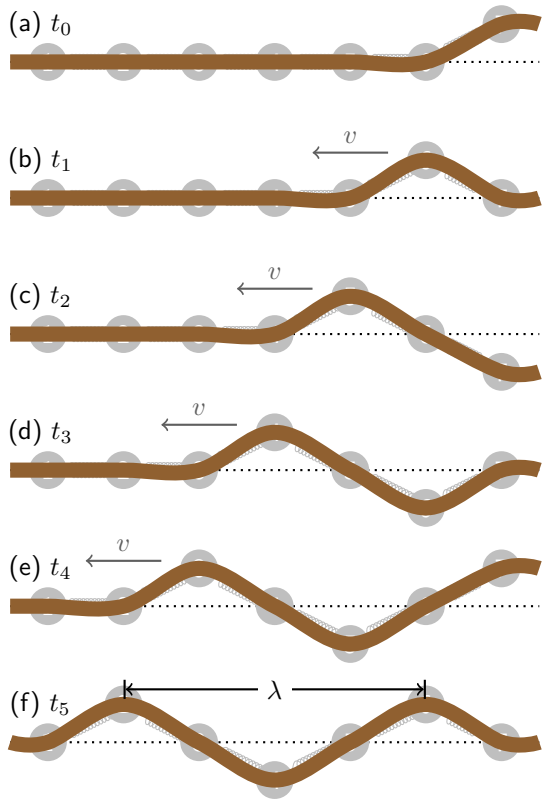


Figure 10.4: This figure is based on Figure 10.3, with the rope itself drawn over our model of the rope. The up-and-down motion of the pieces of rope result in a wave that moves to the left. The length of the wave is called the wavelength λ, and can be measured between successive peaks of the wave.[31]

So this wave travels to the left at a speed of...

$$v = \sqrt{\frac{150\,\mathrm{N} \cdot 6\,\mathrm{m}}{12\,\mathrm{kg}}} = 8.66\,\mathrm{m/s}$$

Note that the speed of the wave doesn't depend on the transverse speed of each individual piece!

Wavelength λ is the physical distance between successive crests in a wave, and it is related to the speed and either the period or the frequency f of the wave:

$$\boxed{\lambda = v \cdot T = \frac{v}{f}} \tag{10.3}$$

...where...

$$\boxed{f = \frac{1}{T}} \tag{10.4}$$

So this wave has a wavelength of...

$$\lambda = 8.66\,\mathrm{m/s} \cdot 0.5\,\mathrm{s} = 4.33\,\mathrm{m}$$

...and a frequency of...

$$f = \frac{1}{0.5\,\mathrm{s}} = 2\,\mathrm{Hz}$$

10.2 The End of the Rope

Words

We have seen that a wave can travel along the length of a rope, but what happens when the rope ends? It depends. First we will consider what happens if the end of the rope is free to move.

Figure 10.5 uses the same model that we used before, but this time we are sending a single pulse to the right instead of constantly shaking one end up and down, and we are letting the end of the rope ("piece 7") move freely up and down, so it is only affected by the forces from "piece 6."

Notice what happens in the figure. Piece 7 actually goes much higher than the height of the original pulse, and as it comes down (because of the force from piece 6), it sends a pulse back to the left. Notice that no piece of the rope ever crosses the dotted equilibrium position line where the rope would be if there were no waves on it at all.

The wave pulse that was sent in actually bounces back from the free end of the rope.

Graphics

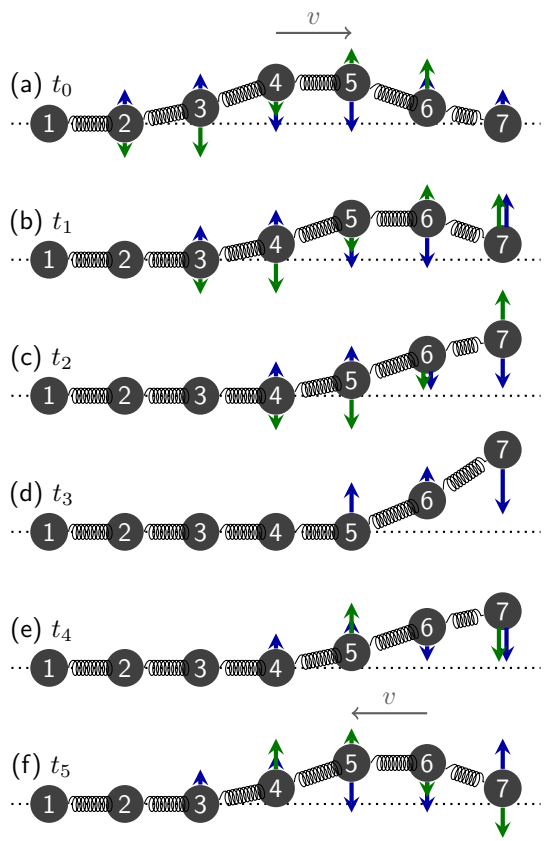


Figure 10.5: If piece "7" at the end of the rope is allowed to move freely up and down, a wave pulse moving to the right reflects back to the left when it reaches the end.[1]

Numbers

Now we will consider what happens if a wave pulse reaches the end of a rope that is held firmly in place.

Figure 10.6 again uses the same model that we used before, we are holding the end of the rope ("piece 7") firmly in place, so whatever force comes from piece 6 will be counteracted by whatever is holding the rope. Since this end of the rope doesn't move at all, we can say that it is fixed in place.

Notice what happens in the figure. Piece 7 stays fixed in place, and at t_1 and t_4 it is pulling very hard on the rope. In fact, it pulls so hard on the rope that the wave flips over when it reaches the end.

The wave pulse that was sent in flips over and bounces back from the fixed end of the rope. Notice that before it reaches the end of the rope, the wave pulse is completely above the equilibrium position, and when it bounces back it is completely below the equilibrium position.

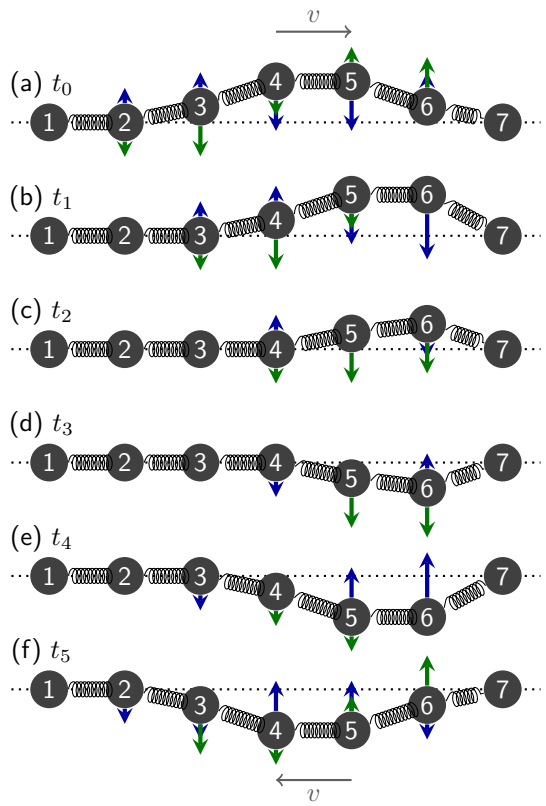


Figure 10.6: If piece "7" at the end of the rope is held firmly in place, a wave pulse moving to the right flips over and reflects back to the left when it reaches the end.[1]

10.3 Adding Waves

Words

In Section 10.2 we saw waves that were reflected back from the end of a rope. During the time that the wave is reflecting, the pieces near the end of the rope are actually feeling the effects of both the wave that is traveling to the right and the wave that is bouncing off to the left, at the same time. Interesting things can happen when waves are traveling on a rope in two different directions.

Figure 10.7 shows two waves with equal amplitudes and a sine-wave shape moving in opposite directions on the same rope. On the left and right, in the regions where the waves aren't interfering with each other yet, each piece of the rope moves up and down just as before. But notice what happens in the center area once the waves meet.

The wave from the left is shown as a dashed line and the wave from the right is a dotted line. When they meet, they interfere with each other. At t_3, t_5, and t_7 the two waves are aligned with each other, so there is "constructive interference." The position of the rope is the sum of the positions of the two waves, so in this case it is a sine wave with twice the amplitude as each individual wave.

At t_4 and t_6 the two waves are completely misaligned with each other, so one is at a positive maximum at the same place that the other is at a negative maximum. This causes "destructive interference." The position of the rope is still the sum of the positions of the two waves, which is the equilibrium position at every point on the rope!

Graphics

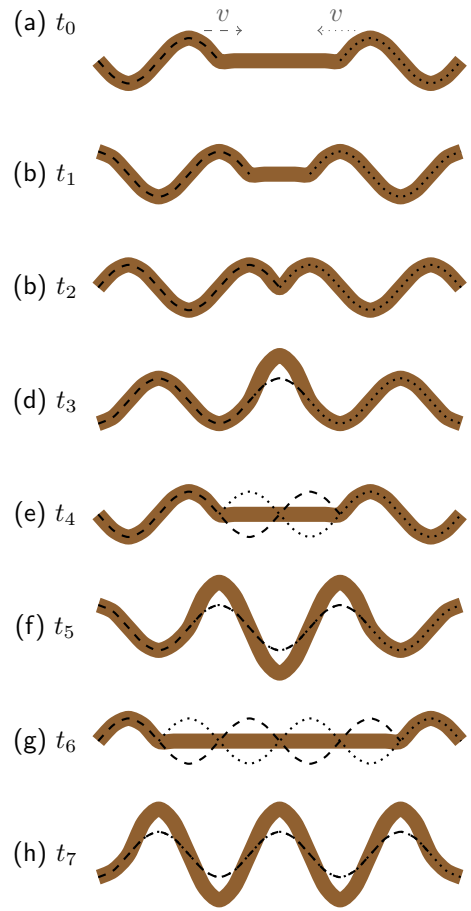


Figure 10.7: The two waves interfere with each other in the region where they overlap, creating a standing wave.[1]

Numbers

We can do an example involving a musical instrument. The highest "E" string on a steel-string acoustic guitar has a linear mass density of 4×10^{-4} kg/m and a length of 0.65 m. How much tension is needed for the string to be correctly tuned with a first harmonic frequency of 330 Hz? What is the frequency of the second harmonic?

Knowns	Unknowns
$\mu_m = 4 \times 10^{-4}\,\mathrm{kg/m}$	F_T
$l = 0.65\,\mathrm{m}$	f_2
$f_1 = 330\,\mathrm{Hz}$	

Figure 10.9 shows that for the first harmonic the length of the string is half of one wavelength, so $\lambda_1 = 2l$. Since the harmonic is created by traveling waves reflecting back and forth from the fixed ends of the string, we can use the same mathematical models that we used for traveling waves. Equation 10.3 relates the wavelength to the frequency and the speed of the wave, and Equation 10.1 relates the speed of the wave to the tension and linear mass density. Combining those mathematical models gives...

$$\lambda = \frac{v}{f} = \frac{1}{f} \cdot \sqrt{\frac{F_T}{\mu_m}}$$

...which can be rearranged to solve for the tension force:

$$F_T = (\lambda \cdot f)^2 \cdot \mu_m$$

The interference of these two identical waves coming from opposite directions creates some points on the rope, called "nodes," that don't move at all. The interference also creates some points on the rope, called "antinodes," where the amplitude of oscillation is very large. The nodes are equally spaced at intervals of half of the wavelength of the traveling waves.

Stringed instruments use this effect. If the ends of a string are held tightly in place and then the string is plucked, reflections of the resulting waves on the string create "harmonics" in the string. The first harmonic in such a string, shown at the top of Figure 10.9, has a node at each end (since the ends are held in place) and an antinode in the center. The wavelength of the first harmonic is twice the length of the string.

There are also other harmonics that are allowed on such a string. The second harmonic still has a node at each end, but also has a node in the middle. The wavelength of the second harmonic is equal to the length of the string. The third harmonic has two nodes in the middle; the fourth harmonic has three; and so forth. Higher harmonics have shorter wavelengths and faster frequencies.

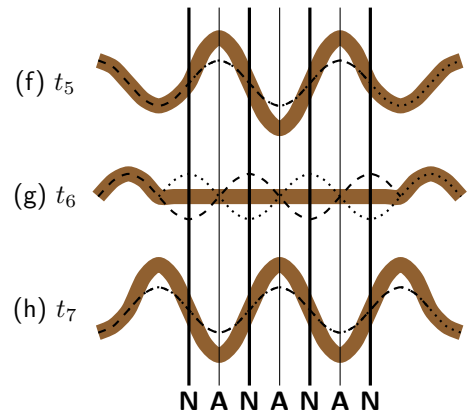


Figure 10.8: Looking at the parts of Figure 10.7 where the waves are interfering with each other, in places called nodes (thick lines) the rope doesn't move at all and in places called antinodes (thin lines) the motion of the rope is large.[1]

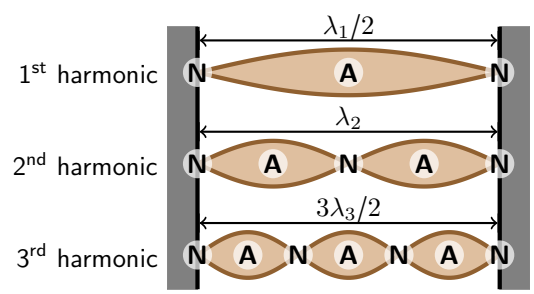


Figure 10.9: The first three harmonics on a string that is tightly anhored at each end. "**N**" indicates a node and "**A**" indicates an antinode.[1]

Substituting in for the first harmonic...

$$F_T = (\lambda_1 \cdot f_1)^2 \cdot \mu_m = (2l \cdot f_1)^2 \cdot \mu_m = 73\,\mathrm{N}$$

We can again use Figure 10.9 to find information about other harmonics. Notice the pattern relating λ to the length of the string when the string is held tightly at each end:

$$\boxed{\lambda_n = \frac{2l}{n}} \tag{10.5}$$

...for the nth harmonic.

Rearranging the first mathematical model in this section to solve for f and using Equation 10.5 for λ gives the frequency for the nth harmonic:

$$\boxed{f_n = \frac{n}{2l} \cdot \sqrt{\frac{F_T}{\mu_m}}} \tag{10.6}$$

Putting in the values we already know gives a second harmonic frequency $f_2 = 660\,\mathrm{Hz}$.

10.4 Longitudinal Waves

Words

Longitudinal waves create areas of compression and rarefaction in a material. Figure 10.10 shows a compression wave pulse initially traveling to the right and then reflecting back to the left.

With longitudinal waves, monitoring the maximum and minimum locations of the pieces of material was a good way to find the location of the wave. But notice that in the figure the positions of the individual pieces do not indication the location of the wave. Piece 2 shifts to the right when the wave passes, and then it stays in the same position until the wave passes that location again.

A better way to find the location of a longitudinal wave is to look for the places where the spacing between the pieces is different, either a compression or a rarefaction. In Figure 10.10, the location of the compressed spring is clearly moving to the right from t_0 to t_3 and moving to the left from t_4 to t_6.

The location of the pressure wave is also where the pieces of the rope are affected by the largest net force. In the case of a compression wave pulse, the net force on each piece is outward from the compression.

The speed of a longitudinal wave in a solid depends on the density and the elastic properties of the material. A higher density causes a lower speed, and a higher Young's modulus (a measure of elasticity) causes a higher speed.

Graphics

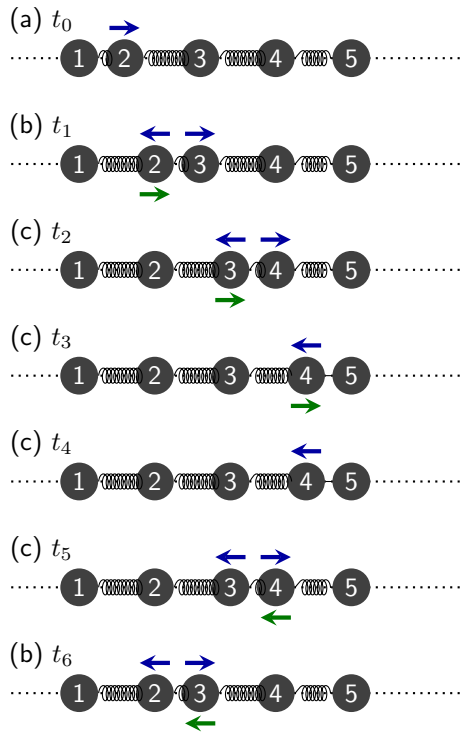


Figure 10.10: If piece "5" is held firmly in place, a compression wave pulse moving to the right reflects back to the left when it reaches the end.[1]

Numbers

To find the speed of a longitudinal wave through a material, we need to know the Young's modulus Y and the density ρ_m of the material. The m subscript is to make it clear that we are using the *mass* density. The following table gives these values for several different materials. Note that Y varies by a factor of more than $100\,000$ across these different materials.

Material	ρ_m [kg/m^3]	Y [N/m^2]
Rubber	$1\,500$	5×10^6
Tendon	$1\,200$	5×10^8
Nylon	$1\,200$	3×10^9
Bone	$4\,000$	2×10^{10}
Concrete	$2\,400$	3×10^{10}
Steel	$8\,000$	2×10^{11}
Diamond	$3\,500$	1×10^{12}

The speed of a longitudinal wave in a solid material is given by...

$$\boxed{v = \sqrt{\frac{Y}{\rho_m}}} \tag{10.7}$$

Using the same guitar string example that we used in Section 10.3, we can find the speed of a longitudinal wave traveling along the length of the string:

$$v = \sqrt{\frac{Y}{\rho_m}} = \sqrt{\frac{2 \times 10^{11}\ \mathrm{N/m^2}}{8\,000\ \mathrm{kg/m^3}}} = 5\,000\ \mathrm{m/s}$$

Figure 10.10 considered a compression wave pulse reflecting off of a fixed end of a piece of material. Figure 10.11 shows what happens when the same pulse reaches a free end of the material. With nothing pushing back on piece 5, it moves outward from its original position, pulling on piece 4 as it goes. This sends a *rarefaction* wave pulse back to the left.

With longitudinal waves, the wave reflected off of the end of the rope would be either flipped over or not depending on the conditions at the end of the rope. Longitudinal waves behave in a similar way–compression bounces back as compression if the end is firmly held in place but compression bounces back as rarefaction if the end is free to move.

Rarefaction in a solid material creates regions of tension, as can be seen by the stretched strings in our model and the resulting net force arrows that pull neighboring pieces of material toward each other. compression creates areas of pressure in the material, where the springs in our model are compressed and the resulting net forces push neighboring pieces of material away from each other.

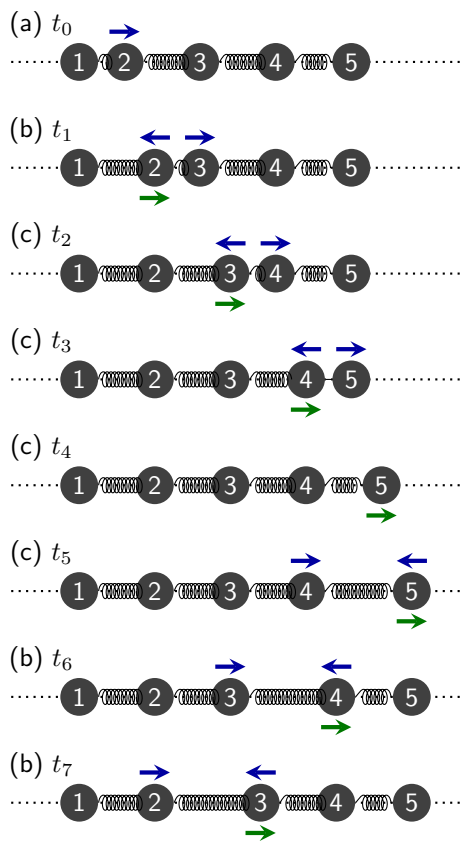


Figure 10.11: If piece "5" is allowed to move freely, a compression wave pulse moving to the right creates a rarefaction pulse that reflects back to the left.[1]

10.5 Stretching and Breaking

Words

The Young's modulus tells us more about a material than the speed of waves. It is a measure of how much the material stretches or compresses when a force is applied. The Young's modulus of a material is determined by its molecular structure–how tightly bound the atoms are to each other and also the configuration of bonds in the material. There is also a limit to how much force can be applied to a given material before it breaks. Depending upon the complexity of the bonds, the Young's modulus and other measures of the material properties could be dependent upon direction or vary depending on whether the material is under tension or compression.

Some materials have structure in them that is not related to the molecules themselves but to larger variations in the material. Wood is a good example of this, because wood has a "grain," and wood is much stronger in the direction perpendicular to the grain than it is along the grain.

Materials can also have vastly different properties when they are under compression than when they are under tension. Concrete, for example, is very strong under compression but easily breaks under tension. That is why iron "rebar" is often placed inside concrete slabs in places where the concrete could be under tension. Iron is very strong under tension, so it holds the concrete together.

Graphics

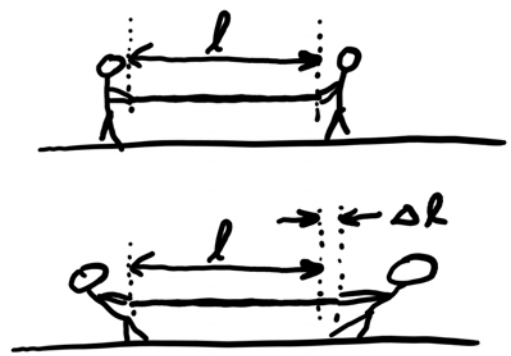


Figure 10.12: A rope of length l stretch to a length $l + \Delta l$ when a force is applied.[1]

Figure 10.13: The board broke vertically into two pieces, in line with the grains visible in the board, when the black belt hit it with her foot. [47]

Numbers

By how much does a 5-m-long nylon rope with a diameter of 2 cm stretch when it is pulled by a force of 700 N?

Knowns	Unknowns
$r = 0.01\,\mathrm{m}$	Δl
$l = 5\,\mathrm{m}$	
$F_T = 150\,\mathrm{N}$	
nylon	

The amount by which the length of a solid changes when a force is applied is given by...

$$\boxed{\Delta l = \frac{l \cdot F}{Y \cdot A}} \tag{10.8}$$

...where l is the original length, F is the force applied parallel to the length, Y is Young's Modulus for the material, and A is the cross-sectional area of the material.

So for this example...

$$\begin{aligned}\Delta l &= \frac{5\,\mathrm{m} \cdot 150\,\mathrm{N}}{3 \times 10^9\,\mathrm{N/m^2} \cdot \pi \cdot (0.01\,\mathrm{m})^2} \\ &= 8 \times 10^{-4}\,\mathrm{m}\end{aligned}$$

Even for a relatively stretchy material like nylon, the change in length is quite small. Most solids are very difficult to stretch or compress.

The breaking point of a material is strongly affected by local factors. It is often not in the place where the largest force is applied, but in the place where the largest pressure is applied. Pressure is defined as force per area, and it has units of Pascals [Pa]. Often tools that are made for breaking or cutting have sharp edges, so the force will be applied over a very small area, creating large pressure.

If force is applied evenly instead of being concentrated at a single location then breaks will usually start at an inside corner or a place where the material has a defect. Think of a chain, which is only as strong as its weakest link.

Glass workers take advantage of breaks starting at defects by intentionally scratching one side of a plane of glass in the place where they want it to break. This makes the glass break cleanly along the scratch when the glass is flexed in such a way that the scratched surface is under tension.

Figure 10.14: Rebar being added to cement.[48]

Figure 10.15: A thumbtack creates very little pressure on the thumb, but enough pressure at the bottom to punch into wood.[49]

Figure 10.16: Flexing upward creates tension along the top and compression along the bottom.[1]

Consider the thumbtack in Figure 10.15. The top surface has a diameter of 6 mm and the point at the bottom narrows to 0.1 mm. How much pressure is applied to the thumb and to the wood if you push down on the thumbtack with a force of 40 N?

Knowns	**Unknowns**
$r_{thumb} = 0.003\,\mathrm{m}$	P_{thumb}
$r_{point} = 5 \times 10^{-5}\,\mathrm{m}$	P_{point}
$F_{thumb} = 40\,\mathrm{N}$	

Pressure is defined as force per area:

$$\boxed{P \equiv \frac{F}{A}} \tag{10.9}$$

. . . where the force is applied parallel to the normal vector; that is, perpendicular to the surface of A.

If we neglect the gravitational force and assume that this is a static situation then the net force on the thumbtack will be zero, so the force between the thumb and the thumbtack is equal to the force between the thumbtack and the wood. This allows us to find the pressures:

$$P_{thumb} = \frac{F_{thumb}}{A_{thumb}} = 1.4 \times 10^{6}\,\mathrm{Pa}$$

$$P_{point} = \frac{F_{thumb}}{A_{point}} = 5 \times 10^{9}\,\mathrm{Pa}$$

The pressure on the wood is more than 1000 times larger than the pressure on the thumb!

10.6 Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

General

- In a transverse wave, each piece of an object is moving in a direction perpendicular to the direction in which the wave is moving.
- In a longitudinal wave, each piece of an object is moving in a direction parallel to the direction in which the wave is moving.
- Compression refers to a state in which pieces of a material are pressed more closely together than normal.
- Rarefaction refers to a state in which pieces of a material are separated more widely than normal.
- We can model a solid object as many small, interconnected objects to study how the object bends and stretches.
- Wavelength is the physical distance between successive peaks in a wave.
- Frequency is the number of times an object goes through an oscillation (or also the number of times a wavelength passes a given point) in a unit of time.
- Constructive interference occurs where two waves are completely aligned with each other, so their amplitudes add.
- Destructive interference occurs where two waves are completely misaligned with each other, so their amplitudes subtract.

Forces

- Rarefaction in a solid material creates regions of tension, and compression in a solid creates areas of pressure.
- Pressure is defined as force per area, and it has units of Pascals [Pa].
- Materials often break not in the area with the largest force but in the area with the largest pressure.

Motion

- The speed of a transverse wave depends upon tension and linear mass density.
- If the end of a rope is allowed to move freely, transverse waves bounce back from the end of the rope.
- If the end of a rope is held firmly in place, transverse waves flip over and bounce back from the end of the rope.
- When two waves pass by each other, they interfere with each other so that the position of the object is given by the sum of the two waves.
- A node is a place on a vibrating object that does not move.
- An antinode is a place on a vibrating object where the motion is at a maximum.

- A rope that is held tightly at each end can vibrate at harmonic frequencies where the wavelength is some multiple of half the length of the rope.
- If the end of a rope is held firmly in place, longitudinal waves bounce back from the end of the rope.
- If the end of a rope is allowed to move freely, longitudinal waves change from compression to rarefaction (or rarefaction to compression) and bounce back from the end of the rope.
- The speed of longitudinal waves depends on density and elastic properties of the material.

Momentum

- (Nothing!)

Energy

- Waves can transfer energy.

Mathematical Models

equation		restrictions on the validity of the equation
$v = \sqrt{\frac{F_T}{\mu_m}}$	(10.1)	Transverse displacement is small compared to wavelength
$\mu_m \equiv \frac{m}{l}$	(10.2)	-none-
$\lambda = v \cdot T = \frac{v}{f}$	(10.3)	-none-
$f = \frac{1}{T}$	(10.4)	-none-
$\lambda_n = \frac{2l}{n}$	(10.5)	Vibrating string held firmly at each end
$f_n = \frac{n}{2l} \cdot \sqrt{\frac{F_T}{\mu_m}}$	(10.6)	Vibrating string held firmly at each end
$v = \sqrt{\frac{Y}{\rho_m}}$	(10.7)	-none-
$P \equiv \frac{F}{A}$	(10.9)	-none-

10.7 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

Level 1 - Remember

10.1 **[W]** What is the name of a wave in which the pieces of the material don't actually move in the direction that the wave is moving?

10.2 **[W]** What is the name of a wave in which the pieces of the material move parallel to the direction that the wave is moving?

10.3 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

Level 2 - Understand

10.4 **[W]** Can transverse waves travel through a solid? If so, give an example.

10.5 **[W]** Can longitudinal waves travel through a solid? If so, give an example.

10.6 **[W & G]** Can wavelength be measured from trough to trough instead of from peak to peak?

10.7 **[W & N]** How could you the amount of force to increase the pressure on an object?

10.8 **[W & N]** How could you the area over which a force is applied to increase the pressure on an object?

Level 3 - Apply

10.9 **[W & N]** How would doubling the mass of the rope while keeping its length the same affect the speed of a longitudinal wave?

10.10 **[N]** Frequency is a useful concept not just for waves but also for circular motion and oscillations. What is the frequency of Callisto's orbit around Jupiter, which we considered in Section 9.3?

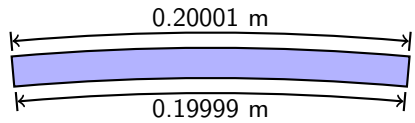
10.11 **[W & N]** The image below shows the end of the neck of a guitar. There are six strings, all made from steel, that are various thicknesses. Each of the strings is connected to a knob at the top that can be used to change the tension in the string. There are also horizontal metal "frets" along the neck of the guitar. The string can be held down tightly against these frets to effectively shorten the length of the string. Explain how each of these three things affects the frequency of the first harmonic of each string:

(a) The thicknesses of the strings?

(b) The knobs that change tension?

(c) The use of the frets?

10.12 **[G & N]** The caption in Figure 10.16 states that the top surface is under tension and the bottom surface is under compression. Bending a material slightly increases the length of the outer surface and slightly decreases the length of the inner surface. Use Young's modulus for glass and the figure below to determine the amount of tension in the upper surface and the amount of compression in the lower surface of the glass. The original length of the glass is 0.20000 m.



A bent piece of glass.[1]

Level 4 - Analyze

10.13 **[W & G]** In Figure 10.4, in what direction is the wave moving and in what direction is the piece of rope moving at piece 5...

(a) ...at time t_2?

(b) ...at time t_3?

(c) ...at time t_4?

(d) ...at time t_5?

10.14 **[W & G]** Compare the rope at time t_1 and t_5 in Figure 10.5. What do you notice about...

(a) ...the positions of the pieces of the rope?

(b) ...the forces on the pieces of the rope?

(c) ...the momenta of the pieces of the rope?

Explain why there are similarities or differences in the forces and momenta.

10.15 **[W & G]** Compare the rope at time t_0 and t_5 in Figure 10.6. What do you notice about...

(a) ...the positions of the pieces of the rope?

(b) ...the forces on the pieces of the rope?

(c) ...the momenta of the pieces of the rope?

Explain why there are similarities or differences in the forces and momenta.

10.16 **[N]** Find the speed of transverse waves on the guitar string studied in Section 10.3. Compare this to the speed of longitudinal waves on the same string studied in Section 10.4.

10.17 **[W, G, & N]** Explain using words, graphics, or numbers why lying down on thin ice is safer than standing on thin ice.

Level 5 - Evaluate

10.18 **[W & G]** In the introduction to this chapter it mentions that waves can carry energy. Consider Figures 10.3 & 10.4. Is energy being carried by this wave? What kind of energy or energies are involved? If energy is being, in what direction is it moving on average?

10.19 **[W & G]** Explain how the forces shown by blue arrows affect the momentum shown by green arrows in Figure 10.3 for piece 6.

10.20 **[G]** Make a sketch showing what the rope would look like in Figure 10.5 at time t_6.

10.21 **[G]** Make a sketch showing what the rope would look like in Figure 10.6 halfway between times t_2 and t_3. Include arrows showing the forces and momenta associated with each of the "pieces" of the rope at that time.

10.22 **[W & N]** The force of gravity was neglected when considering the pressures involved when using a thumbtack in section 10.5. Determine whether it was reasonable to neglect the force in this situation by determining what the mass of the thumbtack would have to be to create a 10% change in one of the calculated pressures.

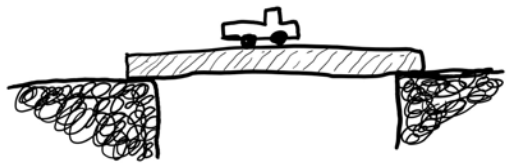
10.23 **[W & G]** The image below shows a truck sitting on a bridge that is made from a simple slab of concrete, shown with light diagonal markings. Answer the following questions using words or drawings.

(a) What parts of the concrete are under tension?

(b) What parts of the concrete are under compression?

(c) Where is the concrete most likely to break, considering that it is weaker under tension than it is under compression?

(d) If you could add iron rebar to only one part of the concrete, where should you put it?



A truck on a bridge.[1]

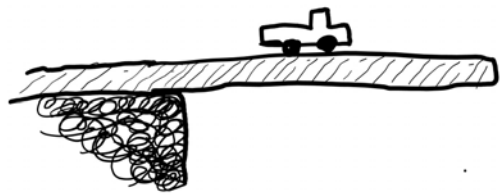
10.24 **[W & G]** The image below shows a truck sitting on a cantilever bridge that is made from a simple slab of concrete, shown with light diagonal markings. Answer the following questions using words or drawings.

(a) What parts of the concrete are under tension?

(b) What parts of the concrete are under compression?

(c) Where is the concrete most likely to break, considering that it is weaker under tension than it is under compression?

(d) If you could add iron rebar to only one part of the concrete, where should you put it?



A truck on a cantilever bridge.[1]

Level 6 - Create

10.25 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Add the main ideas from this chapter to the similar concept map that you began for the question at the end of Chapter 9.

10.26 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

10.27 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

Appendix A

Symbols, Subscripts, & Abbreviations

Symbol	Quantity	SI Unit	
a	acceleration (magnitude or component)	meters per second squared	$[\mathrm{m/s^2}]$
$\vec{a}$	acceleration (vector)	meters per second squared	$[\mathrm{m/s^2}]$
A	area	square meters	$[\mathrm{m^2}]$
$\mathcal{A}$	amplitude	(whatever is being measured)	(any)
E	energy	Joules	$[\mathrm{J}]$
$\hat{E}$	the East direction	-none-	
f	frequency	Hertz	$[\mathrm{Hz}]$
F	force (magnitude or component)	Newtons	$[\mathrm{N}]$
$\vec{F}$	force (vector)	Newtons	$[\mathrm{N}]$
g	acceleration of gravity at earth's surface	(this is a constant)	$9.8\,\mathrm{m/s^2}$
h	height	meters	$[\mathrm{m}]$
I	moment of inertia	kilogram meters squared	$[\mathrm{kg \cdot m^2}]$
k_s	spring constant	Newtons per meter	$[\mathrm{N/m}]$
l	length	meters	$[\mathrm{m}]$
L	angular momentum	kilogram meters squared per second	$[\mathrm{kg \cdot m^2/s}]$
m	mass	kilograms	$[\mathrm{kg}]$
n	an ordinal number (like 2 for 2nd)	-none-	
N	a large number	-none-	
$\hat{N}$	the North direction	-none-	
p	momentum (magnitude or component)	kilogram-meters per second	$[\mathrm{kg \cdot m/s}]$
$\vec{p}$	momentum (vector)	kilogram-meters per second	$[\mathrm{kg \cdot m/s}]$
P	power	Watts	$[\mathrm{W}]$

Symbol	Quantity	SI Unit	
P	pressure	Pascals	$[\mathrm{Pa}]$
r	radius or distance from center	meters	$[\mathrm{m}]$
R	range	meters	$[\mathrm{m}]$
s	path length	meters	$[\mathrm{m}]$
t	time	seconds	$[\mathrm{s}]$
T	period	seconds	$[\mathrm{s}]$
U	potential energy	Joules	$[\mathrm{J}]$
v	speed or component of velocity	meters per second	$[\mathrm{m/s}]$
$\vec{v}$	velocity	meters per second	$[\mathrm{m/s}]$
W	work	Joules	$[\mathrm{J}]$
x	horizontal position or component of $\vec{x}$	meters	$[\mathrm{m}]$
$\vec{x}$	position (vector)	meters	$[\mathrm{m}]$
y	vertical position	meters	$[\mathrm{m}]$
Y	Young's modulus	Newtons per meter	$[\mathrm{N/m}]$
α	angular acceleration	radians per second squared	$[\mathrm{rad/s^2}]$
θ	angle	degrees or radians	[° or rad]
λ	wavelength	meters	$[\mathrm{m}]$
μ_k or μ_s	coefficient of friction	-none-	
μ_m	linear mass density	kilograms per meter	$[\mathrm{kg/m}]$
ρ_m	mass density	kilograms per cubic meter	$[\mathrm{kg/m^3}]$
τ	torque	Newton meters	$[\mathrm{N \cdot m}]$
ω	angular velocity	radians per second	$[\mathrm{rad/s}]$
Δ	"Change in ..."	-none-	
$\overrightarrow{\Delta x}$	displacement	meters	$[\mathrm{m}]$
$\sum$	"Sum of ..."	-none-	
$\equiv$	"is defined as..."	-none-	
$\hat{}$	"in the ... direction"	-none-	
$\rightharpoonup$	(Indicates a vector quantity)	-none-	

Subscript	Meaning
0	at time $t = 0$
$1, 2, etc.$	of object #1, #2, *etc.*
avg	average
c	centripetal
com	Center of Mass
E	East
f	friction (for force)
f	final (for everything except force)
g	gravitational
i	initial
k	kinetic
m	mass
max	maximum
$mech$	mechanical
n	normal (for force)
n	nth (f_3 is for the 3rd harmonic, for example)
N	North
net	net (total)
r	rotational
s	static (for frictional force)
s	spring
t	tension
T	tangential
th	thermal
tot	total
x	in the $\hat{x}$ direction
y	in the $\hat{y}$ direction
$\parallel$	in the parallel direction
$\perp$	in the perpendicular direction
$\rightarrow$, for example "$1 \rightarrow 2$"	of the first object acting on the second
$\leftarrow$, for example "$1 \leftarrow 2$"	of the first object as seen by the second

Abbreviation	Meaning
COM (or com)	Center of Mass
FBD	Free Body Diagram
SHM	Simple Harmonic Motion

Appendix B

Conversion Factors & Metric Prefixes

	SI Unit = US Customary Unit
Length	1 m = 3.28 feet
	1609 m = 1 mile
Mass	14.6 kg = 1 slug
	1 kg = the mass of an object that weighs 2.2 lb on earth
Speed	1 m/s = 2.24 mph
Force	1 N = 0.225 lb
Energy	4.186 J = 1 cal
	4186 J = 1 Cal (or 1 kcal)
	3.6×10^6 J = 1 kWh
	1055 J = 1 BTU
Power	746 W = 1 hp
Angle	π rad = $180°$
Angular speed	$\frac{\pi}{30}$ rad/s = 1 RPM

Metric prefix	Abbreviation	Meaning
Tera-	T	10^{12}
Giga-	G	10^9
Mega-	M	10^6
kilo-	k	1000
centi-	c	0.01
milli-	m	0.001
micro-	μ	10^{-6}
nano-	n	10^{-9}
pico-	p	10^{-12}
femto-	f	10^{-15}

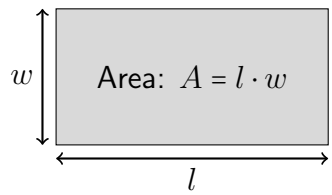
Appendix C

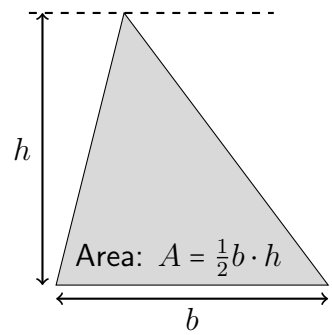
Physical Constants

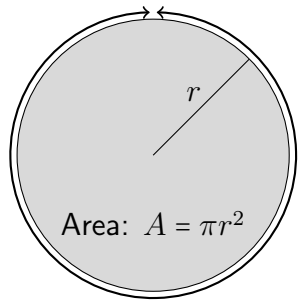
Symbol	Description	Approximate value
g	acceleration of gravity at the earth's surface	$9.81\,\mathrm{m/s^2}$
G	Newton's Universal Gravitation Constant	$6.67 \times 10^{-11}\,\mathrm{N \cdot m^2/kg^2}$

Appendix D

Geometrical Shapes







Circumference: $C = 2\pi r$

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