2.0 Forces, Momentum, and Work:

In Chapter 1.4, we defined force as a push or pull; an interaction between two bodies where momentum is *exchanged*. Each body receives an equal magnitude *impulse* (Δp) in opposite directions, thus conserving total momentum. We further stated that when a force acts on a body, the body's momentum changes:
$F = dp/dt$;

the body accelerates:
$F = ma$;

and work (W) is done on the body, changing the body's energy:
$F = \Delta E/\Delta x$, or $\Delta E = W$, and we define work as $W = F\Delta x$:

The above equations are for a finite amount of work over a finite displacement. Thus, F represents the *average force* during the process. Similarly, the instantaneous force over an infinitesimal distance is:
$F_{inst} = dE/dx$ or $dE = dW$, and we define the infinitesimal amount of work as $dW = Fdx$.

<u>Exercise 2.0.1</u>: Prove that the unit of energy (*Joules*) is the same as the unit of work: Nm, consistent with Fdx.

Direction matters. You do positive work if the force and direction of movement are in the same direction, you do negative work if they are in opposite directions, and you do no work if force is perpendicular to the change in position. Hence you can see that if a car is moving forward:

- if you push a car forward while it is moving forward, you do positive work on the car, and its kinetic energy increases;
- if you push a car backwards while it is moving forward, you do negative work on the car, and its kinetic energy decreases;
- if you push on a car to the left while it is moving forward, you do zero work on the car, and your efforts do not change the car's kinetic energy.

If you are not sure if work is positive or negative, you can ask yourself how the energy of the body changed. For instance, if you carry a 20 kg mass 100 m across a soccer field, you may feel you've done lots of *work* because you're tired. However, you have not changed the energy of the mass; you have only changed its position. Directions of force and displacement? You have displaced the mass horizontally, but the force you put on it is upward, perpendicular to its change in position, so no work is done. It is different if you carry the mass up a hillside 100 m, increasing your elevation 30 m. Now you know you've done work because the mass has higher gravitational potential energy. If you're not sure, just drop the mass the 30 m downward and see if there is some energy transformation!

<u>Exercise 2.0.2</u>:
What is the force you need to put on the 20 kg mass as you carry it up the hill – so gravity didn't accelerate it downward? Calculate the work you did in the above example. Do you use the 100 m or the 30 m? How can you be sure?

2.1 Power:

Like many other physics terms, *power* has a meaning in everyday use that is not the same as its physics definition. We define Power as the rate of change of energy, or the rate of work:

$P_{ave} = \Delta E/\Delta t = W/\Delta t,$ and the instantaneous power at any given moment is $P_{inst} = dE/dt = dW/dt$. And power would be the <u>*slope*</u> of a graph of *E vs t.* The energy in a vehicle is determined by the amount of fuel in the tank, or the charge held in the battery. The size of the motor determines the power: how fast the energy is converted.

For instance, with the proper pulleys or gearing, a small motor from a toy car could pull a truck up a hill, raising its elevation 100 m, but it would take many days. However, a *powerful* engine changes the truck's potential energy very quickly.

The unit of power is the Watt (W), which is awkward because the symbol, *W* is also the symbol for work, which is energy (Joule, J), not power. The English unit of power is the *horsepower:* 1 hp ~ 746 W.

<u>Exercise 2.1.1</u>: Please show from the definition of power that W = kg(m^2/s^3).

<u>Exercise 2.1.2</u>: In the last chapter, you carried a 20 kg mass 100 m for an elevation gain of 30 m. If it took you 30 s to do that, what was your average power output? Does this seem an appropriate power output for you?

A human being can generate about 1000 W (~ 1.3 hp), but only for a few seconds; and can maintain a mechanical power output of 100 W for an hour or more. Additionally, a person's body produces thermal energy at a rate of about 100 W, just to maintain our daily life processes. That's why a room full of people heats up *even if no one's dancing*. Energy is conserved, so all the energy you produce, you have taken in as *chemical potential energy*, food.

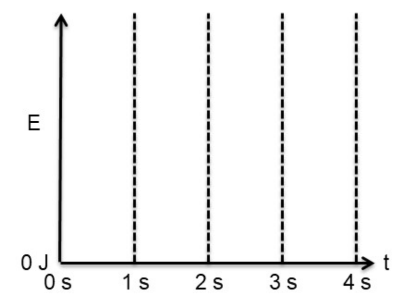
<u>Exercise 2.1.3</u>: A human metabolizes chemical potential energy of food at a rate of about 100 W, resulting mostly in thermal energy. The dietary energy unit is the Calorie (~4200 J), not to be confused with the thermal calorie (with a small "c" of ~4.2 J). Estimate how many calories you consume daily, and see if this corresponds to an average power intake of ~ 100 W.

<u>Exercise 2.1.4</u>: Show that if you are pushing a car up a hill at a constant force and a constant speed, the power you are putting out is:
$P = Fv$. You will want to use the definitions of Power (above) and $W = dE = F\,dx$

Exercise 2.1.5: What is the difference between $P_{ave} = \Delta E/\Delta t = W/\Delta t$, and $P_{inst} = dE/dt = dW/dt$? The instantaneous power could change over time! For instance, a 1000 kg car starts from rest, and accelerates at 5 m/s^2 for 4 s. Let's say we want to know the engine's average power, and the power at the beginning and the power at t = 4 s.

a) Find the total energy put out by the car over the 4 s, and calculate the average power, $P_{ave.}$

b) We know that speed increases linearly with time because there is constant acceleration. Please show that kinetic energy must increase *quadratically* with time. Make a graph of the kinetic energy as a function of time for the first 4 s. Is it a parabola?



c) Estimate the power at t = 0 and t = 4s by taking the slope of the energy vs time graph. Also, please draw a straight line from the origin to the data point a t = 4s, representing if there was constant power during the entire 4 s. What is the slope of this line?

d) In Ex. 2.1.4, we showed that $P = Fv$. We know from dynamics that in order to maintain a constant acceleration, the engine is providing a constant force. Please find the force and the velocity at t = 0 and t = 4 s. And calculate the power. Do your answers agree with that from the graph?

2.2 Units

Units should be carried along with numbers on every step of a problem:

- A number is meaningless without it – check out the reason that the $125 million Mars Rover burned up in the Martian atmosphere.
- Carrying units *and canceling them* will prevent you from making computational errors, because you'll notice that the units came out wrong and you'll go back to find the problem.
- Carrying units will help you learn physics, because you'll be reviewing the relationships in the cancelling.

You may find this to be annoying, but only the first 50 or so times you do it. Then it will become automatic and much faster. There's an easy way to reduce the amount of work this requires: *don't substitute numbers (and units) until the very end of a computation.*

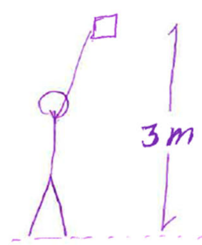
Lastly, you'll have to carry units all the way through a problem in order to receive an "A" on exams.

Exercise 2.2.1: A very tall person drops a 10 kg rock from a height of 3 m. How fast is it moving when it hits the ground?

I'm using an energy lens because

$E_{pg} \Rightarrow E_K$

(gravitational potential)



$E_{initial} = E_f$

$E_{pg_i} + \cancel{E_{K_i}}^{0} = \cancel{E_{pg_f}}^{0} + E_{K_f}$ it starts with no E_K it finishes with no E_{pg}

$\cancel{m} g h_i = \frac{1}{2} \cancel{m} V_f^2$ mass cancels (hurray!)

$V_f^2 = 2gh_i$

$V_f = \sqrt{2gh_i} = (2 \cdot 10\,\text{m/s}^2 \cdot 3\,\text{m})^{\frac{1}{2}}$

$= \left[60\,\frac{\text{m}^2}{\text{s}^2}\right]^{\frac{1}{2}} = \sqrt{60}\,\text{m/s} \approx 8\,\text{m/s}$

I like it because m/s is units of velocity. and $\sqrt{64} = 8$, so the speed is a little less than 8 m/s... a fast running speed ~16 mph... don't drop it on your foot.

2.3 Graphical Analysis of Motion

We want to be able to correctly construct and interpret graphs of position-time (x-t), velocity-time (v-t), and acceleration-time (a-t). Many students understand the concepts involved, but make mistakes. My advice is to practice converting one graph to the other. We remember:

$V = dx/dt$; velocity is the <u>*time derivative*</u> of position. Consequently:

- velocity is the *slope* of the x-t graph.
- Change in position, $dx = v{*}dt$. So, change in position is the <u>area under the v-t curve</u>, or velocity integrated over time.

$a = dv/dt$; acceleration is the <u>*time derivative*</u> of velocity, Consequently:

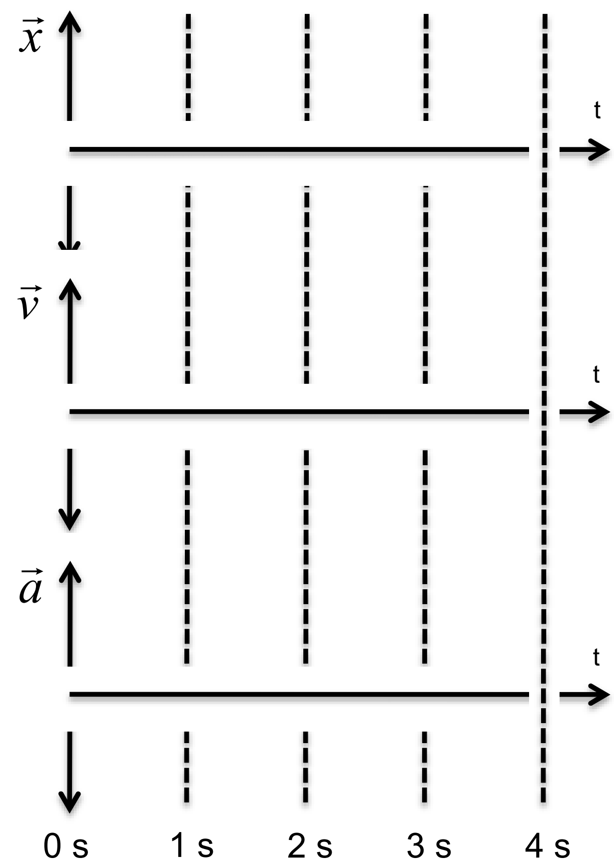
- acceleration is the *slope* of the v-t graph.
- Change in velocity, $dv = a{*}dt$. So, change in velocity is the <u>area under the a-t curve</u>, or acceleration integrated over time.
- $a = d^2x/d^2t$; the second derivative of position. Thus acceleration is the *curvature* of the *x-t graph.*

So, if you start with a v-t graph. You would differentiate it to get a graph of acceleration as a function of time, and you would integrate it to get a graph of position over time, although for the latter, you'd have to know the starting position. Please try:

<u>Exercise 2.3.1</u>: My mass is 70 kg, and the mass of my bike is 10 kg. I'm riding my bike at a constant speed of 10 m/s. At 0 s, my displacement is x = -15 m, I see a car. At t = 1 s, I apply my brakes and smoothly slow to a stop over a period of two seconds.

a) What lens do I use to make these graphs? Why?

b) Please graph my acceleration, velocity, and displacement as a function of time. Label the axes correctly.



c) Please find the force exerted by my brakes;

d) Find the work done by my brakes and the average power.

e) Was energy conserved in this process? How?

f) Was momentum conserved in this process? How?

2.4 Vectors and Direction

Vectors have a magnitude and direction, we represent them as arrows:

- The size (or magnitude) is presented by the length of the arrow.
- The direction of the vector is the direction the arrow is pointing.

We add vectors nose to tail, so that the *resultant vector* (sum of the vectors) stretches from the very beginning to the very end. Forces are vectors. If Mike pushes on a car with 300 N, and Jenny pushes on a car with 500 N, then together they can provide 800 N if they are pushing in the same direction, but only 200 N if they are pushing in opposite directions. This *resultant vector* represents the *net force*, and is proportional to the resulting acceleration, as demonstrated below.

Exercise 2.4.1:

For the two scenarios described above, please find the acceleration of the car if its mass is 1000 kg.

I will use a dynamics lens because forces on the car produce acceleration of the car.

$\Sigma\vec{F} = m\vec{a}$, Identify $\vec{F}$, $\vec{a}$ in a Free Body Diagram

$\vec{F}_J + \vec{F}_M = \Sigma\vec{F} = 800N\hat{x}$ $\qquad$ $\Sigma\vec{F} = 200N\hat{x}$

$\Sigma\vec{F} = m\vec{a}$

$\vec{a} = \frac{\Sigma\vec{F}}{m} = \frac{800N\hat{x}}{1000kg}$ $\qquad$ $\vec{a} = \frac{\Sigma\vec{F}}{m} = \frac{200N\hat{x}}{1000kg}$

$= 0.8 \frac{\cancel{kg}\frac{m}{s^2}}{\cancel{kg}}\hat{x}$ $\qquad$ $= 0.2 \frac{\frac{\cancel{kg}\, m}{s^2}}{\cancel{kg}}$

$= 0.8\ m/s^2\ \hat{x}$ $\qquad$ $= 0.2\ m/s^2\ \hat{x}$

Don't be afraid to draw! Notice how solving a dynamics problem requires you to:

- write $\sum\vec{F} = m\vec{a}$,
- identify the forces and acceleration in a free body diagram,
- add the forces as vectors to produce the resultant force, or *net force*.

Momentum is also a vector and must be added as vectors (please see below).

Exercise 2.4.2:

A 1000 kg car driving at 40 m/s collides with a 3000 kg truck driving at 20 m/s. The vehicles stick together and skid off on the very slippery road. Please find the velocity of the car-truck wreckage immediately after the collision.

a) If the car rear ends the truck driving in the same direction.

b) If the car and truck have a head on collision, driving in opposite directions.

We indicate that a quantity is a vector by putting an arrow over it, hence force should be $\vec{F}$ if direction is relevant. Without the arrow, F just indicates the magnitude (size) of the force.

We will now clarify some more terms we've already been using:

The following are also vectors:

- Position, $\vec{x}$ (where you are),
- Displacement, $\overrightarrow{\Delta x}$ (change in position), "I moved 30 m in the $\hat{x}$ direction," means $\overrightarrow{\Delta x} = 30\ m\ \hat{x}$
- Velocity, $\vec{v}$ (rate of change of position, $\vec{v} = \frac{d\vec{x}}{dt}$, and
- Acceleration, $\vec{a}$ (rate of change of velocity) $\vec{a} = \frac{d\vec{v}}{dt}$.

However, the following are scalars:

- Distance (the length of the path traveled), script lower case L: l
- Speed (the rate of change of distance), symbol is v with no arrow.

Distance and speed are not just the magnitudes of displacement and velocity. They can be very different as demonstrated in the exercise below. Please do it:

Exercise 2.4.3:
Starting on a metric football field at the 30 m line, you run to the 50 m line and then back to the 10 m line…the whole process takes you 10 s. Please draw out your path on a football field and work through the calculations graphically, to show:

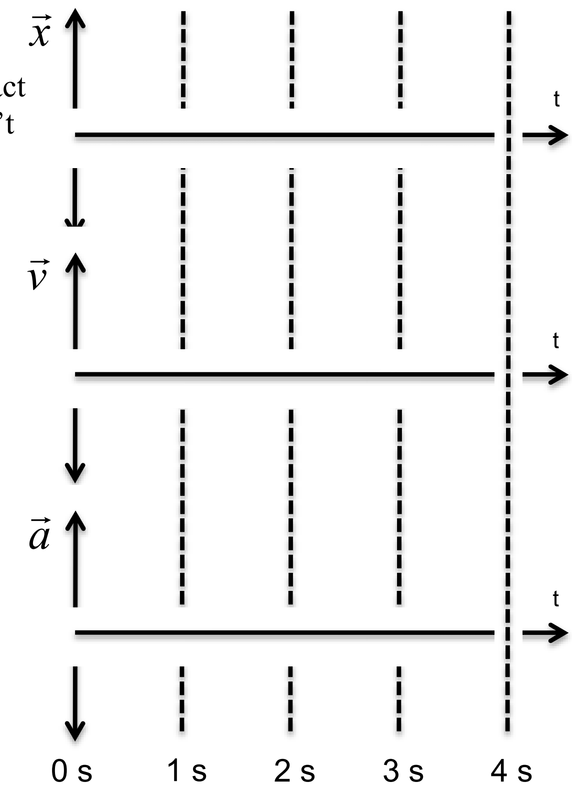
- You displaced yourself a total of: $\overrightarrow{\Delta x} = -20\ m\ \hat{x}$.
- Your final position is $\vec{x} = +10\ m\ \hat{x}$.
- Your total distance (or length) traveled is $l = 60\ m$.
- Your average speed is $v_{ave} = 6\ m/s$
- Your average velocity is $\vec{v}_{ave} = -2\ m/s\ \hat{x}$

Exercise 2.4.4:
Please repeat the above drawing and calculations for the reverse scenario: You start at the 10 m line, run to the 50 m line, and return to the 30 m line.

Acceleration is a vector ($\vec{a}$), but we often just talk about the magnitude. For instance, one might say that the new Tesla has an acceleration that is about equal to a *gravity*, 10 m/s^2 rather than saying "the Tesla's acceleration is about 10 m/s^2 *North* if the car is facing northward."

Another example: You start at rest at t = 0 s, accelerate forward for 2 s, maintain constant velocity from t = 2 s to t = 3 s and then over 1 s, slow to a stop at t = 4 s: The instantaneous acceleration may be $\vec{a} = 4\frac{m}{s^2}\hat{x}$ at some time, t = 1s, then zero as you run at a constant speed, and then $\vec{a} = -8\frac{m}{s^2}\hat{x}$, as you skid to a stop. On average, $\vec{a} = 0$, because $\overrightarrow{\Delta v} = 0$ over the entire time period: the forward and reverse accelerations cancel out. However, the average magnitude of your body's acceleration could be about $a_{ave} = 4\frac{m}{s^2}$.

Exercise 2.4.5:
For the above scenario, please make three graphs: $\vec{a} \Leftrightarrow t$, $\vec{v} \Leftrightarrow t$, and $\vec{x} \Leftrightarrow t$, assume you start at $\vec{x} = 10\ m\ \hat{x}$. Your exact numbers will be different for different people because I haven't exactly defined the motion.



Vectors and Kinetic Energy
Where momentum is a vector, kinetic energy is not. For instance, we've shown that if two carts with equal and opposite momenta collide and stick to each other, the final wreckage has no momentum, and therefore the total momentum of the system must have been zero to begin with. How about the energy? We find that in this scenario, the thermal energy generated is identically equal to the kinetic energy "lost" in the collision. Kinetic energy can only be positive.

Work and Vectors
Work done on a body is the body's change of energy which is a scalar. It has no direction, but can be positive or negative, because you can increase or decrease the energy of a body. If you push a car in the same direction it is moving, then the work is positive (increasing E_K), negative if the force is in the opposite direction of the motion (decreasing E_K), and zero if the force is perpendicular to motion. Accordingly, work is the scalar *dot product* of the two vectors:
$W = \vec{F} \cdot \overrightarrow{\Delta x}$

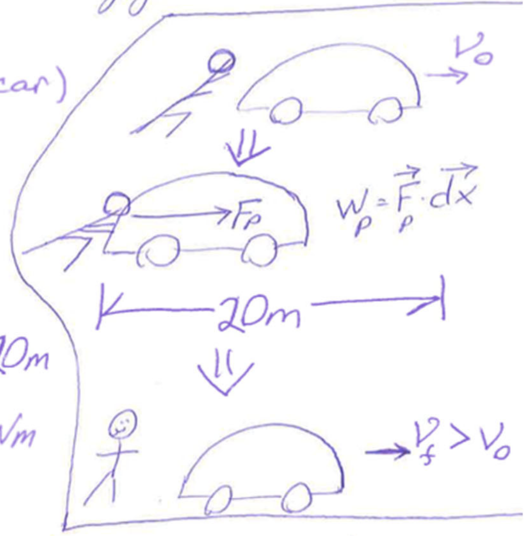
Exercise 2.4.6:
You are pushing a car on flat level ground. The mass is 1000 kg, and you are pushing with a force of 500 N. If the car starts with a speed of 5 m/s and you push it 20 m in the same direction,

a) what is the final speed of the car?
b) How about if you push the car in the direction opposite its velocity for 10 meters?

You might be inclined to use a dynamics lens because we have force causing acceleration, and then use a kinematics lens to calculate the new speed. You could do this, but we would have to figure out how much time you were pushing. The energy lens provides a simpler way:

a) I'm going to use an energy lens because

$W_{Pete} = \vec{F}_p \cdot \Delta\vec{x}_p = \Delta E_{K(car)}$



$E_{Kf} = E_{Ko} + W_p$

$\frac{1}{2} m v_f^2 = \frac{1}{2} m v_o^2 + \vec{F}_p \cdot \Delta\vec{x}_p$

$= \frac{1}{2}(1000\,kg)(5\,\frac{m}{s})^2 + 500N \cdot 20m$

$= 12{,}500\,\frac{kg\,m^2}{s^2} + 10000\,Nm$

E_o $W_p = \Delta E$

$\frac{1}{2} m v_f^2 = 22{,}500 J \Rightarrow v_f = \left(\frac{2 \cdot 22{,}500\,J}{1000\,kg}\right)^{\frac{1}{2}}$

$v_f = \left(45\,\frac{\cancel{kg}\,m^2/s^2}{\cancel{kg}}\right)^{\frac{1}{2}} \approx$ Pretty close to 7 m/s.

Notice how I justify the energy lens by describing the energy flow or conversion. I also notice that 7 m/s is getting pretty close to as fast as I could run, so I couldn't have pushed the car much further with this force.

c) Please find the final velocity if I pushed the car in the opposite direction while it moved 10 meters!

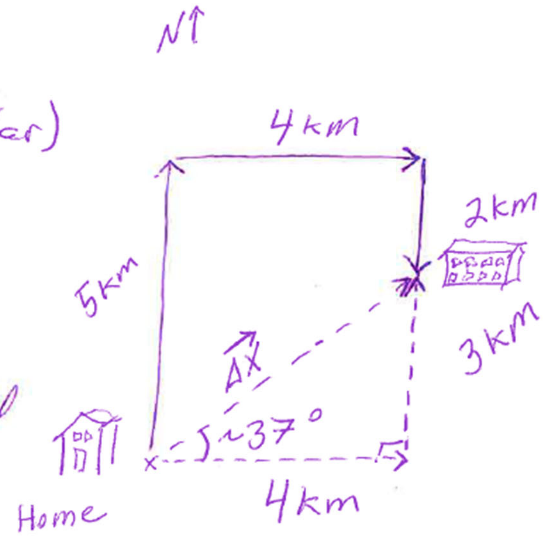
Spoiler Alert! We add vectors nose to tail even when they are not parallel. For instance:

Exercise 2.4.7: starting at home you walk a long way to school: 5 km *North*, then 4 km *East*, then 2 km *South.*

a) What is the total distance that you walked?
b) What is the straight-line distance between your house and the school?
c) What is the *displacement* from your home to your school? Displacement is a vector, so please include direction.

a) Total distance walked =
5 km + 4 km + 2 km = 11 km (Scalar)

b) Straight line distance =
$\sqrt{(4km)^2+(3km)^2} = \sqrt{25\,km^2} = 5\,km$
(scalar)



c) Displacement can be expressed in cartesian coordinates:
$\vec{\Delta x} = 4km\,\hat{E} + 3km\,\hat{N}$
or in Polar coordinates:
$\vec{\Delta x}$ = 5 km, ~37° North of East, or 5 km in This Direction

Exercise 2.4.8:
A 100 kg block is floating in a canal when two people pull on ropes connected to it: One person pulls with a force of 80 N *North*, and another person pulls with 60 N *West.* Please draw picture and identify lens.

a) Which way will the block accelerate?
b) What is the magnitude of the acceleration?

Exercise 2.4.9:
I come to a river that is flowing southward with a velocity of 4 m/s *South.* I wish to cross the river from *East* to *West*. I point my canoe due *West* and paddle at 3 m/s. My friend, Nick, is watching from shore. What does he see as my velocity?

2.5 The Dynamics Protocol

"The Protocol" (as we call it) for dynamics is a method for understanding and solving dynamics problems. You are not required to follow this protocol completely, but your protocol should include the basic steps in some sensible order.

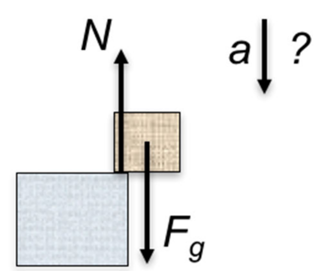
Step 0: Identification
Before using any lens, you have to identify the most effective lens or lenses and provide motivation for it. Start by becoming familiar with the scenario you're exploring. Close your eyes and imagine it happening. Consider if you have any experience with anything like it. What was the outcome? Draw a good picture and indicate any relevant parameters. What's going on?

If the problem involves forces causing acceleration or a change in momentum, then dynamics is often a good lens to look at the problem. State, "I'll use a dynamics lens because forces cause acceleration."

Step 1: Write $\sum \vec{F} = m\vec{a}$,
Now identify the parts:
- What is the object that is being accelerated by the forces?
- What are the forces *acting on this object* (and what directions)?
- What do you know about the acceleration of this object? Direction?

How do we best represent this? We make a Free Body Diagram (below).

Step 2: Make a Free Body Diagram, and indicate acceleration.
Make a simple diagram of the chosen body showing all the forces acting on that body ONLY. That is, don't include forces the body exerts on something else. Make the forces start from the point they are acting from. For instance, in the FBD at right, we examine the small mass. We can draw the force of gravity coming from the center of mass, but the normal force is drawn from the place where the two blocks are in contact, *where the normal force is being applied*. Indicate the direction of acceleration, but do this *off to one side, because the acceleration is **NOT** one of the forces.* The acceleration *is **caused by** the resultant force.*

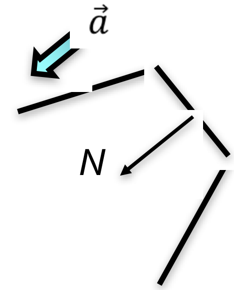


Step 3: Add the forces to get a resultant vector, the net force
Add the forces "nose to tail" like vectors and make a *resultant vector* from the very beginning to the very end. If you know the direction of the acceleration, this resultant vector must be in the same direction of the acceleration. If the body is in equilibrium, $\vec{a} = 0$ and the point of the last vector should be at the very beginning, or "the snake bites its tail."

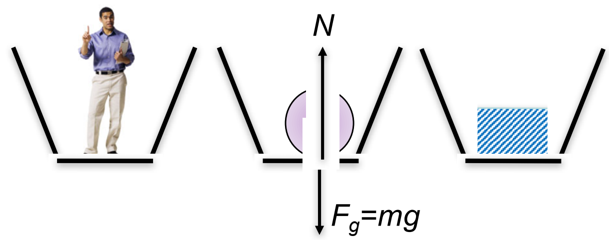
Step 4: Define the positive direction, and use $\sum \vec{F} = m\vec{a}$ if you need to find a numerical answer.

Exercise 2.5.1:
Please grade my solution for example 2.4.1. How well did I follow the protocol?

Gravity and Another Force:



If we were floating in outer space, we would notice no force of gravity. How would we keep water inside a bucket, so that the water is touching the bottom of the bucket? “Touching” the bottom of the bucket means that there is a nonzero normal force; that is $N > 0$. Given that this normal force is the only force, it would mean that the bucket would be accelerating in the direction of the normal force. So, you could imagine accelerating the bucket back and forth or up and down while rotating it to maintain a positive normal force. In Lacrosse, players make use of the motion as they cradle the ball in order to keep it in the pocket.

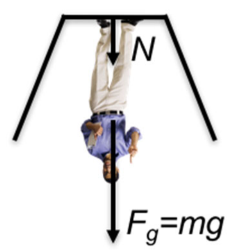


Noting that $\sum \vec{F} = m\vec{a}$, consider the questions below as you keep a man, a ball, or water in a bucket as shown at right.

Exercise 2.5.2:
If the object is kept in the bucket, what do we know? Identify all correct statements. Answers at chapter end.

a) $N > F_g$.
b) $F_g > N > 0$.
c) $N > 0$.
d) acceleration must be upward or zero only.
e) acceleration can be zero or any value upward or downward.
f) acceleration cannot be downward with a magnitude greater than gravity.

Exercise 2.5.3:



How about if we see someone in a bucket upside down, what do we know? Identify all correct statements. Answers at chapter end.

a) $N > F_g$.
b) $F_g > N > 0$.
c) $N > 0$.
d) acceleration must be downward with a magnitude greater than gravity.
e) acceleration can be any value, but must be downward.
f) acceleration can be zero or any value upward or downward.
g) acceleration cannot be downward with a magnitude greater than gravity.

Exercise 2.5.4:
You are in an elevator that is accelerating upwards at 2 m/s^2. Using your own mass, how much force is the elevator putting on you? Please follow the protocol:

- Lens
- Write down the operative equation
- FBD
- Add the forces
- Pick a positive direction and find the answer.

Exercise 2.5.5:
You are in an elevator and your mass is 50 kg. However, you find yourself standing upside down on the ceiling with the scale (between you and the ceiling) reading 100 N. What's the direction and magnitude of your acceleration? Please **follow the protocol** to convince me of your answer.

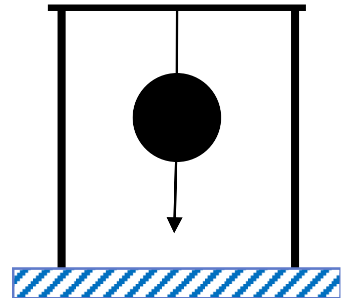
Strings: The special thing about strings: if the applied force exceeds the breaking force, they snap and the tension immediately becomes zero. Because it only takes an instant to snap a string, the brief tension of the string on a body produces a tiny impulse ($\Delta\vec{p} = \vec{F}\Delta t$), very little change in velocity, and a negligible displacement in the very small period of time.

*Assume that the strings in exercises 6-10 have a **breaking tension of 300 N**.*

Exercise 2.5.6:
Imagine a 10 kg iron sphere with a string tied to it in outer space. You pull on it with increasing force until it breaks.

- Make a graph of the force on the sphere as a function of time until after the string breaks.
- Make graphs of acceleration, velocity, and displacement of the sphere as a function of time until after the string breaks.
- Imagine if you are able to snap the string very quickly with an immediate tension > 300 N. Describe what you will see while watching the sphere.

Exercise 2.5.7:



Imagine a 10 kg iron sphere hanging on a string as shown at right. The same kind of string is also connected to the bottom of the sphere for you to pull on. Pull downward on the string with a force that increases very slowly until one string breaks. Which string breaks? What is the tension on the other string at this point? What is the acceleration of the ball before the string breaks? Immediately after the string breaks? In order to answer these questions:

- Assert that this is a dynamics lens, and support why this is the case.
- Write $\sum \vec{F} = m\vec{a}$, then examine the forces and consider acceleration with a good FBD.
- Show how the forces add to provide the net force on the ball before and after the string breaks.

Exercise 2.5.8:

Assume the same set up as in Ex 2.5.7 above. However, this time, pull on the lower string with an immediate force greater than 300 N. Which string breaks? What is the tension on the other string at this point? What is the acceleration of the ball immediately before the string breaks? In order to answer these questions, follow either the protocol outlined in Ex. 2.5.7, or at the beginning of this chapter.

Exercise 2.5.9:

Assume the same set up as in Ex. 2.5.7. How could you break the top string without pulling on the bottom string at all? What is the acceleration of the ball before and after the string breaks? In order to answer these questions, follow either the protocol outlined in Ex. 2.5.7, or at the beginning of this chapter.

Exercise 2.5.10:

Assume the same set up as in Ex. 2.5.7. However, this time, there is nothing pulling upward on the ball – only you pulling downward on a string attached to the ball. Can you still break the string pulling downward? If so, what would the acceleration of the ball be before and after the string breaks? In order to answer these questions, follow either the protocol outlined in Ex. 2.5.7, or at the beginning of this chapter.

2.6 Springs

Springs have the ability to provide a force and store energy. We can see this as:

- We have to push/pull a spring to compress it or expand it. Big springs hold up our cars.
- Kids' toy guns require that you do work to compress the spring, storing the energy as spring potential energy (*E_{PS}* or just *E_S*). Later this energy is released as the kinetic energy of the Nerf bullet.

Hooke's law is named after Robert Hooke, who documented that when you pull or push on a spring, the spring stretches a displacement that is proportional to the force *applied to the spring*. So, we can write: $\vec{F} \propto \overrightarrow{\Delta x}$, where $\overrightarrow{\Delta x}$ is the *displacement* of the end of the spring from the neutral position *not the length of the spring*.

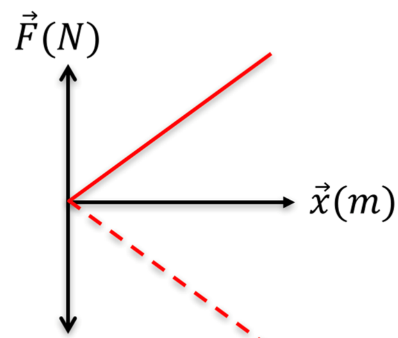
The ratio between the force you put on a spring and the spring's displacement depends on the spring constant (rigidity) so we can write this as an equation, called ***Hooke's Law***:

$\vec{F} = k\vec{x}$, where k is the *spring constant,* and x is *really* Δx, how far the spring is stretched.

What are the units of k? Consider: "how much force does that spring apply to me?" If you have to pull with 5 N to get an extension of 50 cm, so you conclude that you have to pull *10 N per meter* of extension, or k = 10 N/m. Because of the *linear* relation between force and displacement, we could graph the displacement of the end of the spring versus the force applied.

At right, you see Hook's Law on a $\vec{F} \leftrightarrow \vec{x}$ graph.

The solid red line represents the force you have to put on the spring to displace it a certain amount. The dotted line is the force *the spring puts on you*. These forces are the same interaction acting *in opposite directions* on your finger and the spring. The steeper the line, the more force you need for the same displacement, meaning a greater spring constant.



Exercise 2.6.1:

Please prove to yourself that the slope of the (solid) line is the spring constant, k.

The energy stored in a spring, is $E_s = ½kx^2$.

Again, x is *really* Δx, the elongation or compression of the spring. This potential energy must be equal to the work you did stretching the spring: $W = \vec{F}_{ave} \cdot \overrightarrow{\Delta x}$. The force increases as we stretch the spring. Where does the ½ come from? The ½ is explained in the exercises below.

Exercise 2.6.2:

Prove to yourself that the work you do is the area under the $\vec{F} \leftrightarrow \vec{x}$ graph. Show using simple geometry that our formula for the stored potential energy in a spring is correct.

Exercise 2.6.3:
Show using calculus that when force changes with x, you must change the product to an integral, adding up all the little bits of work you do as you stretch the spring:
$E_s = W = \sum dW = \sum \vec{F} \cdot \overrightarrow{\Delta x}$,
$E_s = W = \int dW = \int \vec{F} \cdot \overrightarrow{dx}$, integrated over the full extension, $\Delta x = x_f - x_i$.
Please make the substitution of $\vec{F} = k\vec{x}$ into the integral and find the correct expression for spring potential energy: $\frac{1}{2}k(\Delta x)^2$.

Exercise 2.6.4: I compress a spring 2 cm with a force of 50 N and lock it in place. I put a 100 g mass on the spring and then let it go, straight up!

- How fast is the mass moving immediately after release?
- How high did it get?
- Where you mindful of which lens you're using? Did you draw a good picture?

Exercise 2.6.5:
You may notice that you can compress and release a spring many many times and it doesn't get hot.

- What does this tell you about a spring's ability to store energy?
- Was it reasonable of us to use the equation above equating the stored potential energy in a spring to the work done on the spring or do we need to consider conversion to thermal energy?

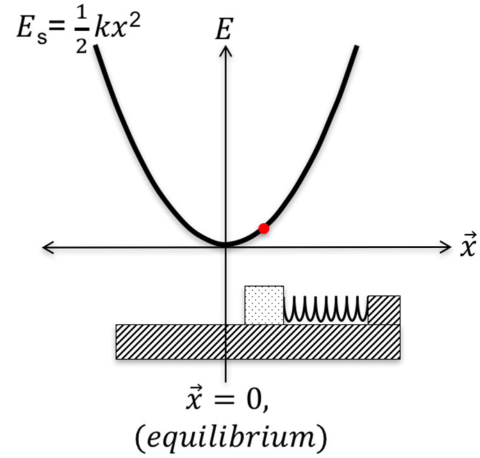
Exercise 2.6.6:
Imagine that a ball has a perfectly elastic collision with a rigid wall and bounces back with its original speed in the opposite direction.

- Momentum is a vector. Did its momentum change? If so, how is momentum conserved?
- Energy is a scalar. Did its energy change from beginning to end? Was there an energy transition anywhere in the process (like when the ball turned around)?

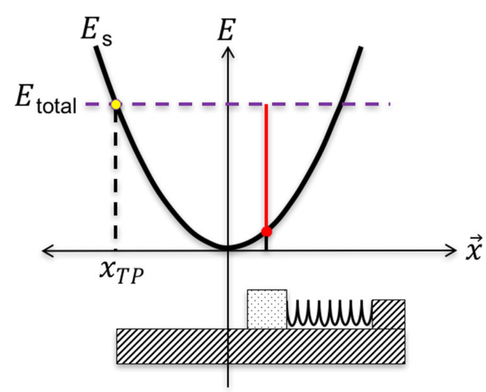
2.7 Potential Energy Graphs

Sometimes, an object's potential energy depends explicitly on position, such as the two potential energies we've explored so far:

- $E_s = \frac{1}{2}kx^2$, the potential energy in a spring scales *quadratically* with displacement.
- $E_g = mgh$, the gravitational potential energy of a body scales *linearly* with elevation.



It is handy to make maps of the potential energy of an object as a function of position. Please see that there is a block connected to a wall via a spring. The block is free to slide on the surface without friction. The spring potential energy is shown above the drawing. When the spring is at its relaxed length (we call $x = 0$), the system has its lowest energy and there is no force on the block in either direction, so the system is in equilibrium. The red dot on the graph indicates the present energy stored in the spring because the block is displaced to the right of the equilibrium point. As the block moves around, the total energy of the system won't change, but loss of potential energy in the spring will change to kinetic energy of the block. Hence, if the block slides to the left, it will lose potential energy, and gain kinetic energy; and if it moving to the right, it will slow down as it gains potential energy at the expense of losing kinetic energy. We also see that the system "wants" to go to lowest energy. That is, a displaced spring will provide a force toward a lower energy configuration.



Let's say we give the block in the above diagram a good impulse to the left, imparting some kinetic energy to the block. Initially, it will relax the spring, speeding the block, but then the spring will stretch, slowing the block. When all the kinetic energy is gone, the block will stop with the total energy being spring potential energy. We call this point the turning point, x_{TP}, because *it's where the object turns around.* The spring will pull the system back to the equilibrium point, transforming potential energy back into kinetic energy, with the total energy remaining constant. The total energy is indicated by the horizontal dotted line, $E_{S(max)} = E_{total}$, when all the energy is spring potential energy and kinetic energy = 0. We see that outside of the turning points, the stored energy in the spring would be *greater* than the total energy… indicating a *negative kinetic energy*… which is impossible. So, the body will not go beyond the turning points.[1] If we give the mass more kinetic energy, the turning points will move to a higher energy point on the graph.

Exercise 2.7.1: Let's say after you hit the block, it travels to the left, turns around, and returns to its original position as shown in the graph.

- Which line segment represents the kinetic energy of the block at this point? The red, the black, or the full red black segment?

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[1] You may one day learn that very small particles like electrons enter this *classically forbidden region* with a probability that decreases with increasing amount of energy deficit.

- At which point is the body moving the fastest? How do you know?

- What happens to the position of the turning points if you give more energy to the system by for instance kicking the block the same way it is moving? Why?

- Which lens are you using to answer these questions?

Forces in the Energy Diagrams

We learned force can be thought of as the *gradient of the energy*. Let's see why this should be. Imagine that you do work on an object, you put a force on it and move it a distance of *dx*. We know:

$W = dE = \vec{F} \cdot d\vec{x}$, or

$W = dE = F_x dx$, if we just keep things along the $\hat{x}$ direction, or

$F_x = dE/dx$, the gradient of the energy of the object.

However, when we look at the force exerted by the environment (the spring in this case), it's the opposite of the force that you would be putting on a body to hold it there, so we make this correction to state that the force exerted by environment = the *negative* gradient of the potential energy, or

$F_x = -dE_P/dx$,

Exercise 2.7.2: Does the above formula make sense to us?:

- Does potential energy (gravity or spring) push things toward lower or higher potential energy?

- Does the force of a potential energy depend on how high the potential energy is; or on how *steep* is the potential energy slope? For instance, does the force of gravity pushing you depend on your elevation, or how much the land is sloped?

Stable or *Unstable* Equilibrium?

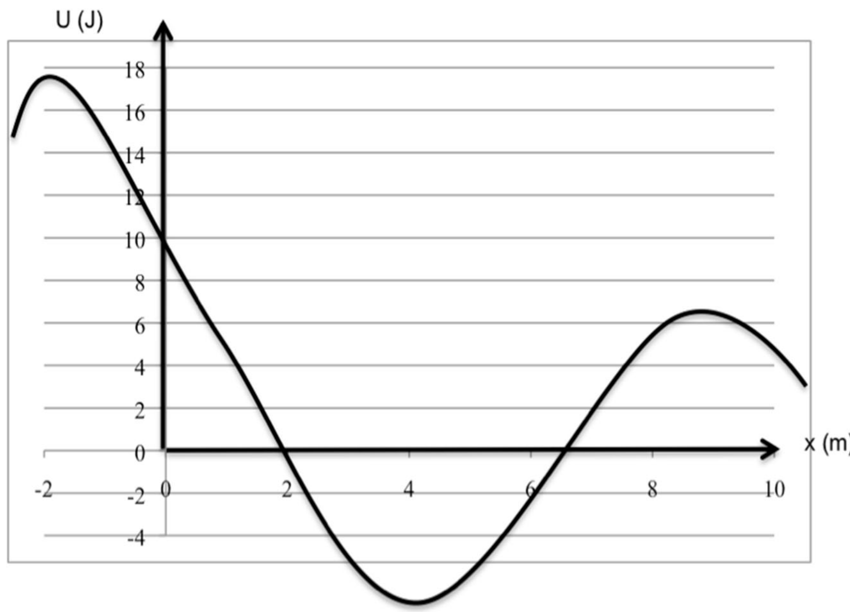
We have defined the equilibrium point as where $F = 0$, or where $a = 0$. This is where the potential energy graph is flat, and where if you place a body, there will be no net force on it. However, if you put something there and give it a little push is it *stable* or *unstable*? That is, will the potential energy push it back or push it further away? A spring system is stable because if you push the body away from equilibrium, it will come back. Therefore, when the spring is in its relaxed state, the body is in a *stable equilibrium.* If you put a marble on top of a large sphere, it is also in equilibrium. However, if you leave it there, it will roll off the top, lose potential energy, and gain kinetic energy, and it won't come back to the top. Thus, the marble on top of the sphere would be in an *unstable equilibrium*.

Exercise 2.7.3: In a potential energy diagram, how can you tell the difference between stable and unstable equilibrium points? In particular, what can be said about the *curvature* of the graph for stable vs unstable equilibrium points?

Exercise 2.7.4: Imagine that you are looking at a body that you lift or drop vertically… from a cliff. How would you draw an energy diagram of this system: The gravitational potential energy versus the height? This is a little awkward because you have to graph the vertical displacement on the $\hat{x}$ axis.

Exercise 2.7.5: You see below a potential energy diagram for a **2 kg block**, as a function of displacement. (positive x is to the right). The block **starts out at $x = 0$ moving at 2 m/s** to the left. *There may be more than one correct answer. In this case, list all correct answers.* *

a) Label stable equilibria with "S"
b) Label unstable equilibria with "U"
c) Label any turning points with "T"
d) Where does the block attain its highest speed, and what is this v_{max}?



e) What is the approximate acceleration of the block at x = 6m? (What two lenses are necessary for this?) *Include direction in your answer, with a unit vector or an arrow.*

* How can something have *negative* potential energy? It's like being in a well, or below sea level. In fact, *absolute energy is arbitrary, it's only the change in energy that affects a body.* If you raise the curve in all places by 10 J, it would have no effect on the behavior of the body subject to the potential. For instance, moving an entire physical system 10 meters higher will not affect how it behaves.

Recognize that the cart is not *moving* up and down on the y axis. The movement is in the x direction only. The y axis is the energy, which could be the result of some electric field, magnets, springs, rubber bands, etc.

2.8 Applying the Four Lenses: Update from 1.8

Adding what we've learned with vectors, we update the "how to" use a lens summary.

The lens method:

- Identify a lens
- Motivate why you chose that lens
- Identify how to apply the lens

Momentum:

- Motivation: Forces change momentum. The change in momentum, or impulse, $d\vec{p} = \vec{F}dt$
- Make a drawing
- If there are no outside forces on a system, then momentum is constant. Typically, this is true in collisions where $F_{collision} >> F_{external}$. Make a drawing and conserve momentum: apply $\sum \vec{p}_i = \sum \vec{p}_f$
- If there is an outside force, then the impulse, change in momentum, $d\vec{p} = \vec{F}dt$

Energy Lens:

- Motivation: If there is a transformation of energy or external work is turned into energy.
- Make a drawing, and conserve energy: identify all energy and work terms, and apply: $\sum E_i + W_{in} = \sum E_f + W_{out}$
- Caution: total energy is always conserved, but *kinetic* energy is not conserved as it can turn into heat; for instance, in an inelastic collision, or friction (section 3.2)

Dynamics Lens:

- Motivation: When forces cause acceleration (or we know that $a = 0$).
- (1) $\sum \vec{F} = m\vec{a}$, (2) draw a free body diagram, label forces, and identify direction of acceleration, (3) add forces like vectors, (4) establish a positive direction, solve.

Kinematics Lens:

- Motivation: Information related to motion as an explicit function of time.
- Make a drawing, make a velocity ⇔ time graph.

Answers: Question 2.5.2: c)f). Question 2.5.3: c)d)