# Physics...

## in Three-Part Harmony

**Words    Graphics    Numbers**

The flaming poi balls follow curved paths mostly because of the tension force applied by the dancer.

$$\vec{x} = \vec{x_0} + \vec{v_0} \cdot t + \frac{1}{2}\vec{a} \cdot t^2$$

$$E_k = \frac{1}{2}m \cdot v^2$$

$$\overrightarrow{F_{\text{foot}\rightarrow\text{board}}} = -\overrightarrow{F_{\text{board}\rightarrow\text{foot}}}$$

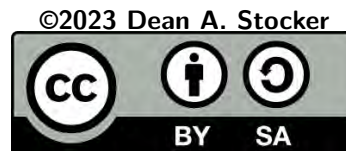Most of the runners are moving at constant velocity, but Usain Bolt continues to accelerate...

**by Dean A Stocker, PhD**

Dean A. Stocker, PhD
University of Cincinnati Blue Ash College
9555 Plainfield Rd
Blue Ash, OH 45236
dean.stocker@uc.edu

**Spiral-bound color copies** of this work can be purchased at https://www.lulu.com/ (search for "stocker physics") for approximately the same cost as printing your own unbound copies.

**The digital version** of this work is freely available for download at
http://sharedcurriculum.peteschwartz.net/parallel-pedagogy/.

**Canvas Quizzes**, **Echo360 polling questions**, and a **Solutions Manual** based upon this work are available to instructors at no cost by emailing dean.stocker@uc.edu and providing a link to their school's website that verifies their role as an instructor.

# Acknowledgments

There are many people who have contributed to this work in many ways. In particular, I would like to thank the following:

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And finally, some design features of this book, like the three-column format, are, as far as I know, unique. For such inspirations I can only thank **my God**.

# Learning Objectives

This textbook was designed for use in conceptual and algebra-based physics courses. The content was chosen in such a way that it aligns with two different standards. The first is the physics content in the "Foundational Concepts" on the Association of American Medical Colleges' Medical College Admissions Test (MCAT). The second is the "Transfer Assurance Guidelines" (TAG) for first-semester algebra-based physics in the State of Ohio.

These two standards each contain dozens of learning objectives, some of which overlap. Appendix F lists each of the learning objectives and shows the section in which each appears for the first time in the textbook. In some cases the learning objectives are developed over the course of several sections; in such cases multiple sections are listed.

The Table of Contents and the upper-right corner of the first page of many sections of the book contain icons that indicate that the new material presented in the section explicitly covers either MCAT foundational concepts or the State of Ohio learning objectives:

&#10133; MCAT foundational concept

&#9632; Ohio TAG learning objective

In a very small number of cases, a foundational concept or learning objective is first introduced in an end-of-chapter question. In those cases, the question includes the appropriate icon.

It should be noted that in fact nearly every chapter contains material that is strongly related to the learning objectives for both MCAT and Ohio TAG. The icons are used only in the places where the section introduces material that is *explicitly* named in the learning objectives. For example, the concept of power is not included in the Ohio TAG requirements for the first semester of algebra-based physics. But the Ohio TAG requirements for the second semester of algebra-based physics include describing the power delivered by batteries and the power dissipated in resistors in simple circuits. The concept of power is usually introduced along with energy, which is a concept developed in first-semester physics. So while the section of this textbook where power is introduced is not marked with the "Ohio TAG" icon since it is not an explicit learning objective, it is prerequisite knowledge for anyone who plans to take the second semester course.

**This book is not endorsed by either the Association of American Medical Colleges or the Ohio Department of Higher Education. The determination of which content is relevant to the MCAT Foundational Concepts and Ohio TAG is based solely on the judgement of the author of this textbook.**

# Contents

11

# Chapter 1

# Our Physics Toolbox

This book approaches physics in a way that is unusual for a physics textbook but is quite natural in our daily lives. If you were an apprentice carpenter, you would expect to quickly start working with an array of different tools. You would also learn to use these tools in a variety of ways, perhaps reading, watching, listening to instruction, and trying them out yourself. Think of yourself now as an apprentice physicist, learning to use the tools of physics.

This chapter introduces two different types of "tool sets." One is a set of conceptual ideas: motion, momentum, energy, and force. It may be helpful to create a concept map to give yourself a framework for how all of these ideas are connected. The template in Figure 1.1 is a good place to start. Make a larger version of it, then add ideas and connections within and between the colored areas. All of these concepts will be built up in parallel over the course of this book.

Figure 1.1: Template for a concept map in physics[1]

The other "tool set" that is introduced in this chapter includes three different but interrelated approaches to help us understand each of these concepts: Words, Graphics, and Numbers. Each of these approaches is presented in parallel in different columns on most pages of this book, to emphasize that these three approaches are interrelated and can be used together to build a more complete understanding of a physical situation.

## 1.1 Motion, Momentum, Energy, and Force ✚

### Words

The left columns of this book generally contain verbal representations of physics.

Physics is an attempt to describe the behavior of the physical universe. But the universe is complicated, so it is necessary to make some simplifications.

The approach of this book is to start out by looking at the behavior of very simple objects, and gradually move to more and more complicated systems. An "object" in physics can be anything that doesn't change shape. A "system" is a collection of one or more objects.

We will start with the area of physics that is called mechanics. There are four main concepts involved in mechanics: motion, momentum, energy, and force.

Pay attention to the color coding. The words, graphics, and numbers are color coded to help show which concept is being considered and how the concepts are interrelated.

Motion is important because it is what we can most directly observe. This includes the object's position; its velocity, which is another way of saying its speed and its direction of motion; and its acceleration, which is the change of an object's velocity over time.

### Graphics

The center columns of this book generally contain graphical representations of physics.

**Photos or drawings**



Figure 1.2: A galaxy. Because every physics textbook should include a picture of a galaxy![2]

**Motion maps**

| 0 s | 1 s | 2 s | 3 s |

Figure 1.3: Example of a motion map[1]

### Numbers

The right columns of this book generally contain mathematical representations of physics.

A letter or symbol in italics is a variable used to describe some physical quantity. The same letter will always represent the same type of quantity. A lower-case $m$, for example, will always represent a mass. Mass is an example of a *scalar* quantity because it has a size (magnitude) but no direction.

A letter or symbol with a half-arrow on top, like $\vec{x}$, represents a vector. With a vector it is important to remember that it has a specific direction, often positive ($+$) or negative ($-$). Subscripts are used to differentiate between several of the same type of variable in a given situation. For example, if a problem includes an adult and a child, their positions could be $\vec{x_{adult}}$ and $\vec{x_{child}}$.

Boxed equations are true except for any limitations described in the accompanying text.

$$\boxed{2 = 2}$$

Unboxed equations are true for a specific example but are not generalizable to all situations.

Motion is described by position $\vec{x}$, velocity $\vec{v}$, and acceleration $\vec{a}$.

Note that all of these physical quantities are vectors.

Momentum is related to the effort that would be needed to stop a moving object. This physics definition overlaps well with the way the word "momentum" is used in our everyday language. If you own a successful business we say that it has momentum, and your competitors will have a hard time stopping you!

In physics, momentum increases with an object's velocity and its mass. Velocity and momentum share the same color in this book because of their close relationship to each other.

**Momentum vs Time Graph**



Figure 1.4: A line graph of momentum over a certain period of time.[1]

The symbol for momentum is $\vec{p}$.

Note that momentum is a vector. It is associated with an object's velocity $\vec{v}$ and its mass $m$, and an object's momentum always points in the same direction as the velocity $\vec{v}$.

Energy is the capacity of an object or system to affect another object or system. It is closely related to the idea of "work," which is a transfer of energy into or out of a system by something external to that system. These physics definitions overlap well with the way the words "energy" and "work" are used in our everyday language. If you have no energy, you can't do any work!

There are several different forms of energy, often associated with an object's velocity, an object's position, or an object's temperature. Energy is able to transform from one form to another through various physical processes.

**Energy Bar Graphs**



Figure 1.5: Example of an energy bar graph[1]

There are two different symbols for energy: $U$ is used to represent potential energy, which depends upon the configuration of a system; and $E$ is used to represent other types of energy that do not depend upon configuration. Examples of potential energy are gravitational potential energy $U_g$ and spring (or elastic) potential energy $U_s$. Examples of other types of energy are kinetic energy $E_k$ and thermal energy $E_{th}$

Note that energy is a scalar.

A force is an interaction between objects, often described as a "push" or a "pull." Forces cause changes in motion, momentum, and energy, so forces are truly the heart of physics. All the grand theories of physics seek to study the forces that are at work in the universe.

**Free-Body Diagrams (FBD's)**



Figure 1.6: Example of a free-body diagram[1]

The symbol for force is $\vec{F}$. Forces cause acceleration. A force acting over time changes momentum. And a force acting over a distance does work, changing energy.

Note that force is a vector.

## 1.2  A Motionless Rock, in the Horizontal Direction

### Words

A 5-kg rock is sitting on a flat place on the ground on a calm day. No wind. No earthquake. You watch for ten seconds. It just sits there in the same place, doing absolutely nothing, for the whole time. We will focus on just one object: the rock.

This may seem like a very simple physical situation already, and to make it as simple as possible, we will only consider the horizontal direction.

For this example, it doesn't matter too much which concept we consider first–we're going to end up with a lot of zeros in any case!

In the last section, we ended with forces, so just for fun this time let's start with forces. Remember, a force is an interaction between objects. Looking at the photo and considering the description above, what are the forces in the horizontal direction? There is nothing pushing or pulling the rock to the left or to the right! That means there are no forces in the horizontal direction.

### Graphics



Figure 1.7: A motionless rock[3]

**FBD of Rock** - **Horizontal Direction**
A Free-Body Diagram (FBD for short) is a simple diagram showing the forces that are acting on an object. The object is represented simply by a rectangle. Then arrows are used to represent the forces acting on the object. In this case, there are no forces acting on the rock in the horizontal direction (which is all that we are considering right now), so our FBD ends up being just a rectangle!

rock

Figure 1.8: Free-body diagram of a rock with no forces acting on it[1]

### Numbers

The only number given in this example is 5. The number by itself is meaningless; it needs to be attached to some physical quantity, and almost always with a specific unit.

The kilogram, abbreviated [kg], is the unit of mass in the Système International (SI) unit system that has been adopted as the official standard by nearly every country in the world.

The other base SI units are the meter [m] for distance and the second [s] for time. This book will use SI units almost exclusively.

When considering forces and how they affect an object, there are often multiple forces acting at one time. It is important to consider the net force that is acting. The net force is defined as the sum of all of the forces…

$$\overrightarrow{F_{net}} \equiv \sum \overrightarrow{F} \qquad (1.1)$$

…where the three horizontal lines ($\equiv$) mean "is defined as," and the Greek letter Sigma ($\sum$) means "the sum of…"

In this particular example, there are no forces at all acting in the horizontal direction, so the net force is zero…

$$\overrightarrow{F_{net}} = 0$$

What can be said about the rock's motion? The rock is not moving, so its position is constant. If it was 2 feet in front of you when you started watching, it was 2 feet in front of you when you stopped (assuming you didn't move).

And no matter where it started, its velocity, which is its change in position over time, is zero. Its acceleration, which is the change in velocity over time, is also zero, because the velocity isn't changing.

**Motion map- motionless rock**
A motion map uses dots to show position at various points in time, as if you were holding a video camera steady and putting a dot at an object's position in each frame. For this scenario, drawing and numbering one dot per second gives...

0–10 s

Figure 1.9: Motion map showing no motion[1]

If the net force on an object is constant, as it is in this situation, position is given by...

$$\vec{x} = \vec{x_0} + \vec{v_0} \cdot t + \frac{1}{2}\vec{a} \cdot t^2 \qquad (1.2)$$

...where the subscript "0" means the value at time zero. In this case, $\vec{v_0} = 0$ and $\vec{a} = 0$, so...

$$\vec{x} = \vec{x_0}$$

...which means that $\vec{x}$ at any time $t$ is equal to whatever $\vec{x}$ was at time $t = 0$.

How much effort would be needed to stop this rock? The rock isn't moving, so it would take no effort at all! That means the rock has no momentum.

**Momentum vs Time - motionless rock**
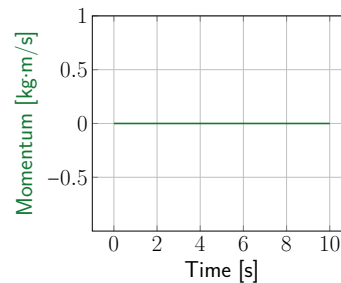


Figure 1.10: Momentum is zero the whole time.[1]

The momentum of an object is equal to its mass times its velocity:

$$\vec{p} = m \cdot \vec{v} \qquad (1.3)$$

As stated above, the velocity $\vec{v} = 0$ in this scenario, so...

$$\vec{p} = m \cdot 0 = 0$$

Remember, energy is a capacity to affect another object or system. This rock can't do anything! So it has no (useful) energy. Technically, the rock does have thermal energy, because it has a non-zero temperature. And Albert Einstein correctly theorized that mass is also a form of energy. But for our purposes, we will consider only mechanical energy, which consists of kinetic, gravitational potential energy, and spring potential energy.

**Energy bar graph - motionless rock**
Kinetic energy, gravitational potential energy, and spring potential energy are all are zero, so a bar graph would have no bars.

No motion, so $E_k = 0$. No spring, so $U_s = 0$. And gravitational potential energy is defined based on vertical position $y$. But we are only considering the horizontal direction, so $U_g = 0$ as well.

## 1.3  A Motionless Rock, in the Vertical Direction

**Words**

A 5-kg rock is sitting on a flat place on the ground on a calm day. No wind. No earthquake. You watch for ten seconds. It just sits there in the same place, doing absolutely nothing, for the whole time. Again we will focus on analyzing just one object: the rock.

Yes, this is the same physical situation that we have already studied, but now we will consider the vertical direction, which adds another level of complexity.

Let's start by considering the motion of the rock. It is doing nothing more in the vertical direction than it did in the horizontal direction. Its vertical position is unchanging; its vertical velocity is zero; and its vertical acceleration is also zero.

What about the rock's vertical momentum? Again, since this rock isn't moving in the vertical direction, it won't take any effort at all in the vertical direction to make it stop! It has no vertical momentum.

**Graphics**



Figure 1.11: A motionless rock[3]

**Motion map- motionless rock**

0–10 s

⬤

Figure 1.12: Motion map of no motion. Again![1]

**Momentum vs Time - motionless rock**

As in the horizontal direction, A momentum vs time graph would be a horizontal line at zero the whole time.

**Numbers**

It is helpful to create lists of known and unknown quantities. Only one "known" is specifically listed: 5 kg. Read carefully for others!

- $m = 5$ kg
- $\vec{v_0} = 0$
- $\vec{a} = 0$

We are not asked to find any specific unknowns; instead, we will use our available tools to find everything we can.

This is exactly the same as the horizontal motion of the rock, but to make it more clear that we are looking only at vertical motion, we can remove the half-arrow and use $y$ in place of $x$...

$$y = y_0$$

As with motion, we can remove the half-arrow and use a $y$ subscript to indicate that we are only considering the $y$ direction, commonly referred to as "$\hat{y}$."

$$p_y = m \cdot v_y = 0$$

This could also have been done in the $\hat{x}$ direction before, with $x$ subscripts instead of $y$.

When we consider forces, there is a fundamental difference between the horizontal and vertical direction. The rock isn't moving at all, but there are two different external forces that are acting on the rock. An "external" force is an interaction with something outside of the system that is being considered. In this case, our system is a single object, the rock.

One external force is gravitational force, an interaction between the rock and the entire mass of the earth. Gravitational force is often referred to as "weight." On the surface of the earth, the gravitational force on an object always points downward, toward the center of the earth.

The other force is a "contact force" that comes from the ground underneath the rock. If the ground were not there, the rock would be falling, so we know that there is a force from the ground that completely opposes the force of gravity. This contact force is called the "normal" force, where the word "normal" means "perpendicular to the surface." In this case, the normal force is pointing directly upward because the ground is flat.

We can look at energy in two different ways. We can consider work done by an external force of gravity or we can consider earth's gravity as a source of potential energy. Work is done by a force acting over a distance, but the rock doesn't move, so no work is done. Gravitational potential energy depends on height–the higher something is, the more gravitational potential energy it has. But in this case the rock is on the ground, so it has no gravitational potential energy.

**FBD of Rock - Vertical Direction**

Gravitational force points downward, and normal force from the ground points upward. Since we know they cancel each other completely, the lengths of the arrows should be the same.
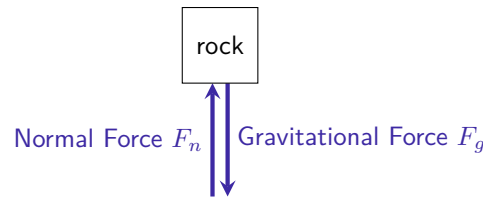


Figure 1.13: Free-body diagram of a rock resting on the ground[1]

Note that we are concerned only with the forces acting on the rock, not about any forces of the rock on something else like the ground. In this case, the forces cancel each other out–another way to describe this is to say that the forces are "balanced," giving zero net force.

**Energy bar graph - motionless rock**

Again, kinetic energy, gravitational potential energy, and spring potential energy are all are zero, so a bar graph would have no bars.

The rock is doing nothing, so net force must be zero.

$$\overrightarrow{F_{net}} \equiv \sum \overrightarrow{F} = \overrightarrow{F_g} + \overrightarrow{F_n} = 0$$

…so…

$$\overrightarrow{F_g} = -\overrightarrow{F_n}.$$

At the surface of the earth, the gravitational force on an object is…

$$\boxed{\overrightarrow{F_g} = -m \cdot g \, \hat{y}} \qquad (1.4)$$

…where $g = 9.8 \,\mathrm{m/s^2}$, the magnitude of the acceleration of gravity at the earth's surface. The magnitude of $\overrightarrow{F_g}$ is given by $(m \cdot g)$ and the direction is given by $-\hat{y}$ (downward). In this case…

$$\overrightarrow{F_g} = -5 \,\mathrm{kg} \cdot 9.8 \,\mathrm{m/s^2} \, \hat{y}$$
$$= -49 \left(\mathrm{kg \cdot m/s^2}\right) \hat{y} = -49 \,\mathrm{N} \, \hat{y}$$

…which makes $\overrightarrow{F_n} = +49 \,\mathrm{N} \, \hat{y}$ in this example. The SI unit for force is the newton [N].

$$1 \,\mathrm{N} = 1 \, \frac{\mathrm{kg \cdot m}}{\mathrm{s^2}}$$

No motion, so $E_k = 0$. No spring, so $U_s = 0$. Gravitational potential energy is based on vertical position:

$$\boxed{U_g = m \cdot g \cdot y} \qquad (1.5)$$

We are free to define the position where $y = 0$, so we can choose our $y = 0$ to make the math as easy as possible. Setting $y = 0$ at ground level gives…

$$U_g = m \cdot g \cdot 0 = 0$$

## 1.4 A Rolling Soccer Ball

### Words

A 0.4-kg soccer ball is rolling to the right at a constant speed of 9 m/s across a level soccer pitch. Friction force and air resistance are very small, so we will ignore them. We will consider only the horizontal direction. We will focus on analyzing just one object: the ball.

This situation is different from a motionless rock, because this time the soccer ball is moving. But in terms of horizontal forces, it is exactly the same. Looking at the photo and considering the description above, what are the forces in the horizontal direction? We are specifically told to ignore any friction forces. And there is nothing that is actively pushing or pulling the soccer ball to the left or to the right. That means there are no forces in the horizontal direction.

If there are no forces in the horizontal direction, how does the soccer ball keep moving? It is the ball's momentum that carries it. Until an outside force tries to make the soccer ball stop, it will just continue going in a straight line with the same momentum. This is called "conservation of momentum;" the momentum of any isolated system remains constant.

An "isolated" system is one that has no interactions with anything outside of the system.

Conservation of momentum is the reason momentum is such a powerful tool for studying physics.

### Graphics



Figure 1.14: A soccer ball rolling to the right[4]

**FBD of Soccer Ball - Horizontal**
Since we are ignoring friction, there are no forces acting on the soccer ball in the horizontal direction.

$$\boxed{\text{ball}}$$

Figure 1.15: FBD of a rolling soccer ball, horizontal direction only[1]
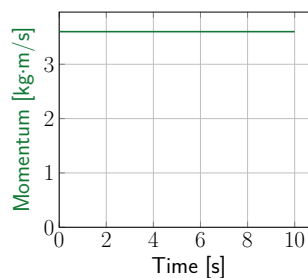


Figure 1.16: Momentum is constant.[1]

### Numbers

Knowns: $m = 0.4$ kg
$v_x = +9\,\text{m/s}$
$a_x = 0$
$F_{net,x} = 0$

In addition to listing "knowns," we should also list our assumptions. For example, in our analysis we will use the convention "to the right" as the positive horizontal direction. We are also assuming that friction and air resistance can be ignored.

We know that $a_x$ is zero because the velocity is constant, 9 m/s to the right.

In this example, there are no forces at all acting in the horizontal direction, so the net force is zero...

$$F_{net,x} = 0$$

...where the subscripts "$net, x$" indicate that we are referring to the net force in the $\hat{x}$ direction.

$$\overrightarrow{p} = m \cdot \overrightarrow{v}$$

...so...

$$p_x = m \cdot v_x = 0.4\,\text{kg} \cdot (+9\,\text{m/s}) = +3.6\,\frac{\text{kg} \cdot \text{m}}{\text{s}}$$

From this equation we find that the units for momentum are $\left[\frac{\text{kg} \cdot \text{m}}{\text{s}}\right]$.

Unlike the motionless rock on the ground, a rolling soccer ball on the ground *does* have energy. It has kinetic energy, or energy of motion. The SI unit for energy is the joule [J].

Any moving object has kinetic energy that increases with the object's mass and its speed. Since a motionless object has zero kinetic energy and mass and speed are both always positive numbers, kinetic energy can never be negative.

Thinking about motion becomes much more interesting when something is moving. Since it is rolling, this soccer ball is definitely moving. What do we mean when we say that an object is moving? That means the object has a non-zero velocity. But when we talk more generally about "motion" in physics, we are considering not only velocity but also position and acceleration. This soccer ball has the simplest motion that we can consider for an object that is actually moving. It has non-zero velocity, but the velocity is not changing, so it has zero acceleration.

**Energy bar graph - rolling soccer ball**
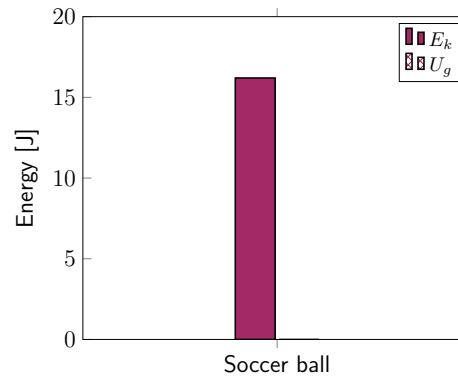


Figure 1.17: Energy bar graph of a rolling soccer ball[1]

**Motion map- rolling soccer ball**

A motion map tells us about more than just position. It can also tell us about the velocity of the object. Notice that all of the arrows between the points are identical. The arrows represent velocity, so the motion map shows that the velocity is constant.
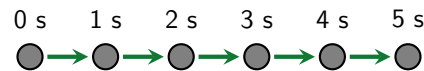


Figure 1.18: Motion map of a soccer ball rolling to the right with a location shown every second for 5 seconds[1]

As with the rock on the ground, $U_g = 0$ and $U_s = 0$. But this time we have kinetic energy. Kinetic energy depends on an object's mass and its speed $v$, which is simply the magnitude of its velocity $|\vec{v}|$,...

$$\boxed{E_k = \frac{1}{2} m \cdot v^2} \tag{1.6}$$

...so in this example the kinetic energy from the linear motion is...

$$E_k = \frac{1}{2} 0.4 \,\text{kg} \cdot (9 \,\text{m/s})^2 = 16.2 \,\text{kg} \cdot \text{m}^2/\text{s}^2 = 16.2 \,\text{J}$$

The SI unit for energy is the joule [J].

$$1 \,\text{J} = 1 \,\frac{\text{kg} \cdot \text{m}^2}{\text{s}^2}$$

For this example, Equation 1.2 becomes ...

$$\vec{x} = \vec{x_0} + \vec{v_0} \cdot t + \frac{1}{2} \vec{a} \cdot t^2 = \vec{x_0} + \vec{v_0} \cdot t + 0$$

...where the subscript "0" is usually pronounced "naught" and means the value at time zero. In this case, $\vec{v_0} = 9 \,\text{m/s} \,\hat{x}$ and $\vec{a} = 0$ , so...

$$\vec{x} = \vec{x_0} + (9 \,\text{m/s} \,\hat{x}) \cdot t$$

...which means that $\vec{x}$ moves 9 meters to the right of its initial position in every second that passes.

## 1.5   A Falling Rock

### Words

A 0.8-kg rock is dropped from a position 2 m above the ground. After 0.5 seconds it is still in the air. Air resistance is very small, so we will ignore it. Describe the behavior of the rock starting from the time just after it was released.

We only need to consider the vertical direction, since there is no motion, momentum, or force in the horizontal direction. The time "just after it was released" is when the hand is no longer touching the rock but the rock hasn't yet started to fall.

At first the rock is not moving, but when it is released gravity creates a unbalanced force that pulls it down. In fact gravity was affecting the rock before it was released, but the force force of gravity was balanced by a force from the hand.

After it is released the rock is in "free-fall," which means the only force affecting the rock is gravity.

Since the rock is initially not moving, it has no initial momentum. But after half a second it is moving downward. The momentum is changing because of the force of gravity. The effect of force on momentum is described by Newton's Second Law of Motion, which defines force as something that changes momentum over time.

The momentum of the rock starts at zero, as it falls downward it gains momentum in the negative direction. This negative momentum gradually increases over the entire time that the rock is falling.

### Graphics



Figure 1.19: A rock falling after being dropped[5]
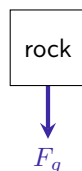


Figure 1.20: FBD of a falling rock[1]
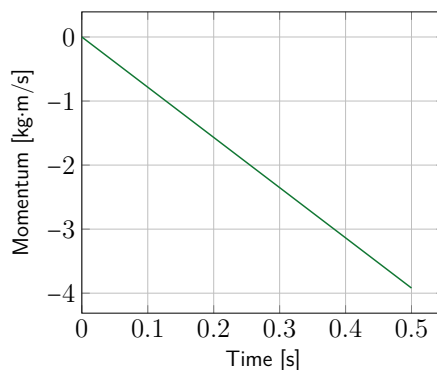


Figure 1.21: Momentum as a function of time[1]

### Numbers

Assumptions:   $+\hat{y}$ is upward

air resistance is negligible

free-fall

Knowns:   $m = 0.8$ kg

$y_0 = 2$ m

$v_{0y} = 0$

Velocity $\vec{v}$ is a vector, but speed $v$ is a scalar that cannot be negative, since it is $|\vec{v}|$. But $v_y$ is the $\hat{y}$ component of $\vec{v}$, so it *can* be negative.

$$\overrightarrow{F_{net}} = \sum \vec{F} = \overrightarrow{F_g}$$

$$\overrightarrow{F_{net}} = -m \cdot g\hat{y} = \left(-0.8\,\text{kg} \cdot 9.8\,\text{m/s}^2\right)\hat{y} = -7.84\,\text{N}\,\hat{y}$$

This is the first situation we have met where the object is not "in equilibrium," meaning that this time the net force on the object is not zero. Net force is given by Newton's Second Law:

$$\boxed{\overrightarrow{F_{net}} \equiv \frac{\overrightarrow{\Delta p}}{\Delta t}} \tag{1.7}$$

"$\Delta$" means "change in…" so $\overrightarrow{\Delta p} = \overrightarrow{p_f} - \overrightarrow{p_i}$, final momentum, with subscript $f$, minus initial momentum, with subscript $i$.

Rearranging, $\overrightarrow{\Delta p} = \overrightarrow{F_{net}} \cdot \Delta t$. The momentum changes linearly from zero to its final value…

$$p_{y,f} = -7.84\,\text{N} \cdot 0.5\,\text{s} = -3.92\,\text{kg} \cdot \text{m/s}$$

Remember, momentum is related to velocity. So if the momentum of the rock is changing, that means its velocity is also changing. Acceleration is a change in velocity over time, so the same force that causes momentum to change also creates an acceleration.

The force of gravity accelerates the rock in the negative (downward) direction, making it fall faster and faster with a constant acceleration that doesn't depend on its mass.

This relationship between force and acceleration is another form of Newton's Second Law.

For this example we will look at energy in terms of gravitational potential energy, not as work done by the force of gravity. The rock is not moving at the moment it is dropped, so it has no kinetic energy but it does have gravitational potential energy since it is elevated. While falling, it is gaining kinetic energy but losing gravitational potential energy.

Energy is a useful concept to use when looking at this situation, because energy is conserved. It can never be created or destroyed; it can only change from one form to another. This is what makes energy such a powerful tool in physics.

If you know how much total energy is in an isolated system at any point in time, you know that same amount of energy is present for all points in time. So if you know, for example, how much gravitational potential energy was lost you can find how much kinetic potential energy was gained.

A motion map can also give information about acceleration. If the arrows are changing, the object is accelerating.
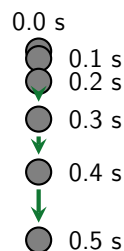


Figure 1.22: Motion map of a falling rock[1]

Since we are considering gravity as a form of energy, conservation of energy tells us that the sum of the heights of all of the bars on the energy bar graph at 0 s is equal to the sum of the heights of all of the bars at 0.5 s.
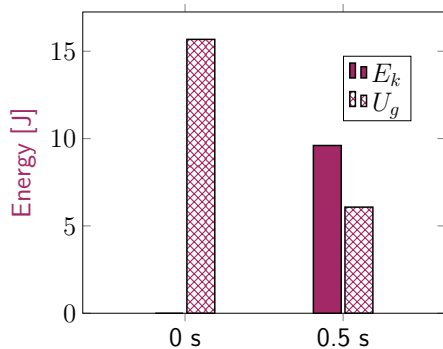


Figure 1.23: Energy bar graph of a falling rock[1]

Force also causes acceleration:

$$\boxed{\overrightarrow{F_{net}} = m \cdot \overrightarrow{a}} \qquad (1.8)$$

Rearranging, we find...

$$\overrightarrow{a} = \frac{\overrightarrow{F_{net}}}{m} = \frac{-7.84\,\text{N}\,\hat{y}}{0.8\,\text{kg}} = -9.8\,\text{m/s}^2\,\hat{y}$$

In fact, $\overrightarrow{a}$ for anything in free-fall is $-9.8\,\text{m/s}^2\,\hat{y}$.

$$y = y_0 + v_{0y} \cdot t + \frac{1}{2} a_y \cdot t^2 = 2\,\text{m} + 0 \cdot t - 4.9\text{m/s}^2 \cdot t^2$$

...so after 0.5 s, $y = \left(2 - 4.9 \cdot 0.5^2\right)\,\text{m} = 0.775\,\text{m}$

There is no spring, so $U_s = 0$. Using 0 seconds for the initial time and 0.5 s for the final time...

$$E_{0,total} = E_{f,total}$$

$$E_{0k} + U_{0g} = E_{k,f} + U_{g,f}$$

$$0 + m \cdot g \cdot y_0 = \frac{1}{2} m \cdot v_f^2 + m \cdot g \cdot y_f$$

Everything in the last equation was given in the knowns or has already been found, except for the final speed, so we can solve for that:

$$v_f = \sqrt{2 \cdot \frac{m \cdot g \cdot y_0 - m \cdot g \cdot y_f}{m}}$$

Note that mass cancels out, so when air resistance is negligible the final speed of a falling object is not dependent upon its mass.

## 1.6 Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

### General

- An "object" can be anything that doesn't change shape.

- A "system" is a collection of one or more objects.

- An "isolated system" is one that has no interactions with anything outside of the system.

- The kilogram [kg] is the SI unit of mass.

- The second [s] is the SI unit of time.

### Forces

- The newton [N] is the SI unit of force. $1\,\text{N} = 1\frac{\text{kg} \cdot \text{m}}{\text{s}^2}$

- A force is an interaction between objects, often described as a "push" or "pull."

- Gravitational force is the same as weight, and is pointed down toward the earth.

- Normal force is a contact force that points directly out of a surface.

- Forces cause acceleration.

- Forces change momentum.

- An object that is affected only by gravity is said to be in "free-fall."

- An object whose net force is zero is said to be in equilibrium.

- Forces can be shown graphically using a "Free-body diagram," or "FBD," which has a box representing the object and arrows representing the external forces affecting the object. Arrows in a FBD should be drawn in the correct directions, with lengths corresponding to the magnitudes of the forces.



Sample Free-Body Diagram (FBD)[1]
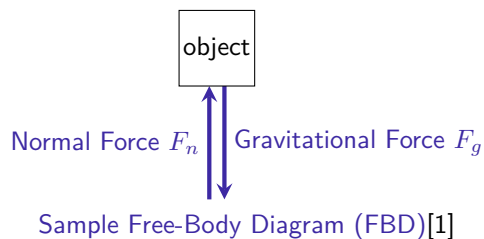
### Motion

- The meter [m] is the SI unit of distance.

- An object's motion is described by its position, velocity, and acceleration.

- Velocity is change in position over time.

- Acceleration is change in velocity over time.

- Velocity includes both speed (which is always positive) and direction, so velocity can be negative.

- An object moves at a constant velocity if and only if the net force on the object is zero.

- Motion can be shown graphically using a "Motion map," which has a series of dots representing the position of the object at equally-spaced intervals of time. Arrows are drawn between the dots to indicate the object's velocity, and acceleration appears as changes in the arrows.



Sample motion map[1]
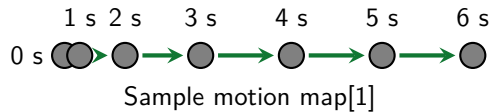
## Momentum

- $[\mathrm{kg \cdot m/s}]$ is the SI unit of momentum.

- The momentum of an object is zero if the object is not moving.

- Momentum increases with an object's mass and an object's velocity.

- The momentum of any isolated system is conserved.

## Energy

- The joule [J] is the SI unit of energy. $1\,\mathrm{J} = 1\frac{\mathrm{kg \cdot m^2}}{\mathrm{s^2}}$

- Energy is a capacity of an object or system to affect another object or system.

- Work is a transfer of energy into or out of a system by something external to that system.

- Energy of an isolated system is conserved; it cannot be created or destroyed, but it can change form.

- Kinetic energy is energy of motion.

- Kinetic energy can be positive or zero, but never negative.

- Gravitational potential energy is related to an elevated position.

- Spring potential energy and thermal energy are other types of energy that will be dealt with later in this book.

- Mechanical energy consists of kinetic energy, gravitational potential energy, and spring potential energy.

- Energy can be shown graphically using an "Energy bar graph." If the system is isolated, the total height of all energy bars for the system at any point in time is the same as the total height of all energy bars at any other point in time.
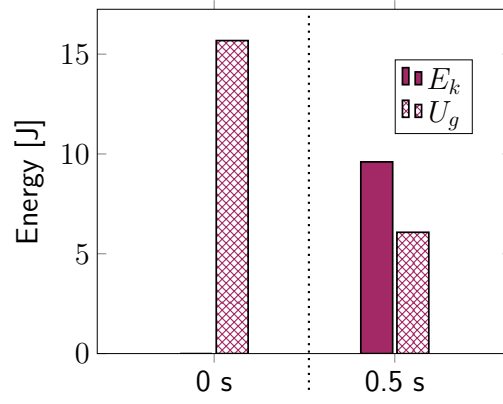
Sample energy bar graph[1]

## Mathematical Models

| equation | | restrictions on the validity of the equation |
|---|---|---|
| $\overrightarrow{F_{net}} \equiv \sum \vec{F}$ | (1.1) | -none- |
| $\vec{x} = \vec{x_0} + \vec{v_0} \cdot t + \frac{1}{2}\vec{a} \cdot t^2$ | (1.2) | only valid when the net force is constant |
| $\vec{p} = m \cdot \vec{v}$ | (1.3) | -none- |
| $\overrightarrow{F_g} = -m \cdot g\,\hat{y}$ | (1.4) | on the surface of the earth, with $+\hat{y}$ defined as "up" |
| $U_g = m \cdot g \cdot y$ | (1.5) | on the surface of the earth, with $+\hat{y}$ defined as "up" |
| $E_k = \frac{1}{2}m \cdot v^2$ | (1.6) | -none- |
| $\overrightarrow{F_{net}} \equiv \frac{\overrightarrow{\Delta p}}{\Delta t}$  (1.7)<br><br>"Newton's Second Law" | | only valid when the net force is constant |
| $\overrightarrow{F_{net}} = m \cdot \vec{a}$  (1.8)<br><br>"Newton's Second Law" | | -none- |

## 1.7 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**.

### Level 1 - Remember

1.1 **[W]** How are forces defined in this book?

1.2 **[W]** How is energy defined in this book?

1.3 **[W]** How is acceleration defined in this book?

1.4 **[N]** What does a half-arrow over a symbol mean?

1.5 **[N]** What does it mean in this book if an equation has a box around it?

1.6 **[N]** What symbol is used to represent momentum?

1.7 **[G]** What do the arrows represent on a motion map?

1.8 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents. For example, Equation 1.1 should be labeled "net force."

### Level 2 - Understand

1.9 **[W]** What information is included in an object's velocity that is not included in its speed?

1.10 **[N]** Give an example of a vector quantity in physics.

1.11 **[N]** Give an example of a scalar quantity in physics.

1.12 **[W]** The four main building blocks of physics as described in this book are forces, motion, momentum, and energy. These are foundational physical concepts because...

   (a) _____ describes some of the most easily observed characteristics of an object, like its position and speed.

   (b) _____ and _____ are both conserved for an isolated system, so if we can find their values at any point in time we know that the same values are also valid at any other point in time.

   (c) _____ can cause changes in momentum and energy and they can also cause acceleration, changing motion.

### Level 3 - Apply

1.13 **[G]** Draw a motion map for an object moving to the left at constant speed.

## Level 4 - Analyze

1.14 **[G]** Consider the free-body diagram in Figure 1.8, where only the horizontal direction is being considered. Should that figure change if the mass of the rock were doubled? If so, in what way?

1.15 **[G]** Consider the free-body diagram in Figure 1.13, where only the vertical direction is being considered. Should that figure change if the mass of the rock were doubled? If so, in what way?

1.16 **[W & N]** What is the weight of an 80-kg person? Remember to include direction.

1.17 **[N]** One thing that we should have been able to find in Section 1.5 was the time needed for the rock to reach the ground when dropped from a height of 2 m. Find that time for the moment when the rock hits the ground, that is, when $y = 0$.

1.18 **[W & G]** Explain how the free-body diagram for the motionless rock in Figure 1.8 can be the same as that for a moving soccer ball in Figure 1.15.


## Level 5 - Evaluate

1.19 **[W]** In the analysis of a motionless rock in the vertical direction in Section 1.3, it is stated that there is no wind. If there were a horizontal wind, would it have affected our analysis of the vertical direction? Explain your answer.

1.20 **[W]** In the analysis of a rolling soccer ball in Section 1.4, we ignored friction to make the analysis simpler, and found that the soccer ball would continue rolling in a straight line indefinitely. Was this a realistic simplification to make? If we had included friction, what would the soccer ball have done?

1.21 **[G & N]** Estimate the slope of the line in Figure 1.21. Compare it to the net force found in the column to the right of the figure. Explain why they are the same, or why they are different.

1.22 **[W, G, & N]** Figure 1.23 shows the energies of a rock when it is released from a height of 2 m and when it has fallen for 0.5 s.

   (a) Reproduce this graph, with correct heights for each bar, and add a third set of bars for the moment just before the rock hits the ground. You can label the last set of bars "before hitting."

   (b) Find the speed of the rock just before it hits the ground.

   (c) Combine the speed of the rock with other information you have about the direction of the rock's motion to find the velocity of the rock just before it hits the ground.


## Level 6 - Create

1.23 **[W, G, & N]** At the beginning of this chapter in Figure 1.1 was a template for a concept map. Hand-draw your own large version of it, adding in the main ideas from this chapter. Leave plenty of space to add things from other chapters! Here are some examples to help you start:
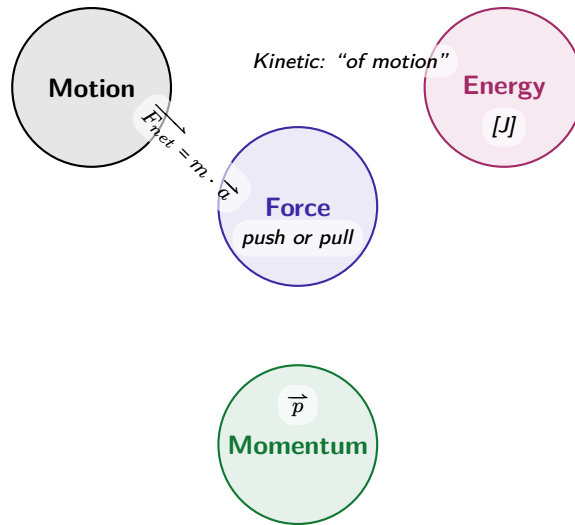
Partially completed concept map[1]

1.24 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

1.25 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

# Chapter 2

# Working in One Dimension

We have now been introduced to most of the concepts and most of the tools that we will use in this entire book to study mechanics. We will continue to look at more and more complicated situations. In this chapter we will continue to restrict ourselves to single objects in one dimension, either horizontal or vertical. We will also continue to consider only constant forces.
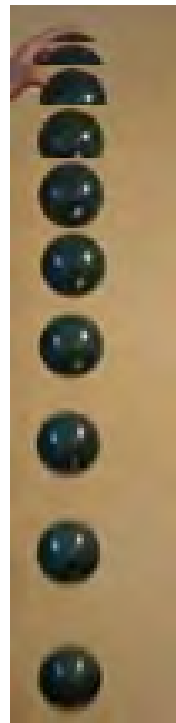
Figure 2.1: Images of a falling ball, taken by a stationary camera at a rate of 20 frames per second[1]

## 2.1   Units

### Words

If you live in the United States or one of a handful of other countries, you may be bothered by the fact that this book is using unfamiliar units. There are good reasons for this.

One reason is that the vast majority of the world has adopted SI units as the official system of measurement.

A second reason is that SI units are based on factors of 10 and universal physical quantities like the speed of light, while most other systems of measurement, including U.S. Customary units, are based upon arbitrary numbers and measurements like the distance between a person's fingertips and elbow.

A third reason is that attempting to use U.S. Customary units to do physics is difficult and confusing even for those who use those units on a daily basis. For example, mass is an important physical property, but in U.S. Customary units people assume that mass is measured in pounds. In fact, pounds are a unit of force, and the correct unit of mass should be slugs, which nobody anywhere in the world uses, even in the U.S.

For all of these reasons, SI units are the standard units used in science throughout the world, and also in this book.

### Graphics



Figure 2.2: Rulers often show inches across the top and centimeters (cm) across the bottom.[1]

The U.S. Customary unit for length is the foot, and the SI unit for length is the meter. Conversion factors can be found in the appendices of this book.



Figure 2.3: Weights used in a gym are often labeled in pounds (lb) and kilograms (kg).[1]

On earth, a 10-kg object experiences a gravitational force of 98 N, and 98 N corresponds to 22 pounds of force. On the moon, a 10-kg object experiences a gravitational force of 16.2 N, or 3.65 pounds.

So mass is not dependent on gravity, but weight is. This is why we more often refer to an object's mass than its weight in physics. They are related, but they are different.

### Numbers

**Note: These calculations are wrong! This is why we avoid U.S. Customary Units!**

Consider a 50-pound child. What is the child's mass? 50 pounds, right? The acceleration of gravity at the surface of the earth is $32\,\mathrm{ft/s^2}$. Weight is the gravitational force on an object, so a 50-pound child experiences a gravitational force of 50 pounds on earth. Using Newton's second law…

$$F_y = m \cdot a_y = m \cdot (-32\,\mathrm{ft/s^2})$$

$$-50\,\mathrm{pounds} = (50\,\mathrm{pounds}) \cdot (-32\,\mathrm{ft/s^2})$$

Dividing both sides by -50 pounds gives us …

$$1 = 32\,\mathrm{ft/s^2}$$

1 = 32 !? This cannot be correct. Most people who use U.S. Customary units don't know that the correct unit for mass is the slug. A slug is the mass of an object that weighs 32 pounds at the surface of the earth. That's why our calculation is off by a factor of 32—we used the wrong units.

To do physics using U.S. Customary Units we must introduce units like the slug that are not in regular use anywhere in the world. Rather than taking this step backwards, we step forward into using SI units.

**Note: The above calculations are wrong! This is why we avoid U.S. Customary Units!**

Failing to pay attention to units can be a costly mistake, as was famously demonstrated in 1999 when NASA lost its $125-million space probe that was supposed to have gone into orbit around Mars. The engineers who designed the rocket system used U.S. Customary Units, but the engineers who designed the guidance system used SI units. When it came time to enter orbit, the probe instead skipped off of the Martian atmosphere and was never heard from again.

This NASA error was simply a matter of not converting all units into the same system. But there is another type of error that is common for students who are learning physics. That is having answers with the wrong "dimensions" or doing calculations that are dimensionally impossible.

For example, if someone says a baby weighs six pounds, nine ounces, that makes sense. It is dimensionally correct, even though it is not in the SI units that are preferred in physics. If we want SI units, we can convert to find the answer we wanted.

But, if someone says that a baby weighs fifteen inches, that does not make sense. Weight is a force, and inches are a measure of length, so these two things have different "dimensions." In physics, a "dimension" doesn't have to refer to length. It can be any physical quantity: energy, momentum, acceleration, etc.

**Graphing position over time**
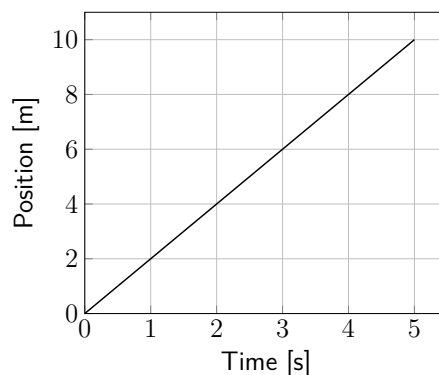


Figure 2.4: Position as a function of time[1]

The slope of a graph often provides useful information. In Figure 2.4, position is changing in time, which means that the object must be moving. The slope of a line is the "rise over the run," how much the vertical value changes divided by how much the horizontal value changes. In this case …

$$\text{slope} = \frac{10\,\text{m} - 0\,\text{m}}{5\,\text{s} - 0\,\text{s}} = 2\,\text{m/s}$$

Keeping the units in the slope calculation gives a hint about the meaning of the slope. The unit [m/s] is a velocity, and in fact the slope of a position vs time graph gives velocity.

If you walk at a constant speed of 4 miles per hour, what distance will you travel in 30 minutes? *Figure out the answer to that question before continuing.*

Hopefully you came to an answer of 2 miles. Whether you realize it or not, you went through all of these steps, possibly in a different order:

(1) Compare units, finding both hours and minutes.

(2) Unit conversion to make time consistent:

$$30\,\text{min} \cdot \left(\frac{1\,\text{hr}}{60\,\text{min}}\right) = 0.5\,\text{hr}$$

(3) Combine given information in such a way that you find an answer with units of distance:

$$\left(\frac{4\,\text{mi}}{1\,\text{hr}}\right) \cdot 0.5\,\text{hr} = 2\,\text{mi}$$

Notice that units cancel just like variables and numbers. For unit conversion, multiply by a fraction using the appropriate conversion factor (see the appendices), in such a way that units cancel to give the answer required.

This way of canceling only works with multiplication and division. Two numbers cannot be added or subtracted unless they have the same units.

The mathematical model used above is…

$$v_{avg} = \frac{s}{\Delta t} \tag{2.1}$$

…the average speed $v_{avg}$ is the path length $s$ divided by the time elapsed $\Delta t$. Since the speed was constant in this example, the average speed is the speed traveled the entire time.

## 2.2 Sliding to the Left

### Words

A 170-g hockey puck is sliding to the left for four seconds across a smooth sheet of ice at a constant speed of 24 m/s. Frictional force and air resistance are very small, so we will ignore them. We will consider only the horizontal direction.

We have looked at a similar situation with a rolling soccer ball, so some of the analysis will be left for the end-of-chapter exercises. Here we will try a few new approaches. This time there is no picture, and since images are helpful for understanding, we will start with a sketch. The sketch should include as much of the given information as possible.

The net force on an object causes a change in velocity over time, i.e. an acceleration. Since the velocity of this puck is constant, we know that the net force is zero.

A sliding hockey puck is capable of affecting another object (for example, if it hits an egg it can break the egg). That means it has energy, in this case kinetic energy. And since the puck is traveling at a constant speed, the kinetic energy would also be constant.

The hockey puck doesn't have any other type of mechanical energy, because there is no spring and it is on the ground.

### Graphics
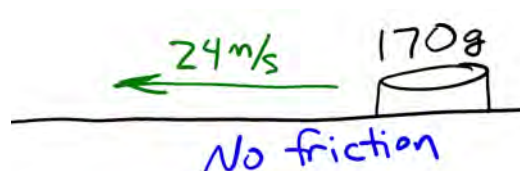
Figure 2.5: Rough sketch of the given physical situation[1]

The sketch does not need to be beautiful, although you can feel free to let your artistic skills shine. But the sketch does need to get across the essential information about the situation.

Figure 2.6: FBD of a puck with no forces[1]
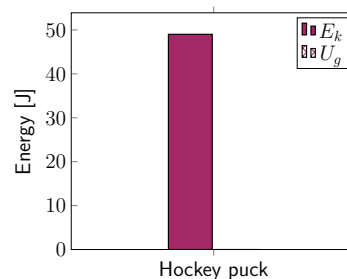
Figure 2.7: Energy bar graph of a sliding hockey puck[1]

### Numbers

Assumptions: $+\hat{x}$ is to the right
friction is negligible

Knowns: $m = 170$ g $= 0.17$ kg

$t = 4$ s

$v_x = -24$ m/s

$a_x = 0$

$F_{f,x} = 0$

Mass needs to be converted to the SI unit kg:

$$170 \, \text{g} \cdot \frac{1 \, \text{kg}}{1000 \, \text{g}} = 0.17 \, \text{kg}$$

$v_x$ is negative because conventionally right is positive. $a_x$ is zero because velocity is constant. $F_{f,x}$ is the force of friction in the x direction.

Newton's Second Law says that $\overrightarrow{F_{net}} = m \cdot \overrightarrow{a}$, and since $a_x = 0$, it follows that $F_{net,x} = 0$.

Kinetic energy is given by...

$$E_k = \frac{1}{2} m \cdot v^2$$

...so in this case ...

$$E_k = \frac{1}{2} \left( 0.17 \, \text{kg} \right) \cdot \left( -24 \, \text{m/s} \right)^2 = 49 \, \text{J}$$

Like the rolling soccer ball that we considered earlier, the hockey puck is moving, so it has momentum. This time, the puck is moving in the opposite direction from the soccer ball we considered earlier, so its momentum is also in the opposite direction, since momentum is a vector. It is important to remember that momentum also includes direction.

The hockey puck is moving to the left with constant speed, so its position is changing throughout the four seconds. We are not given the initial position of the puck, so we can only describe how much the position changes, not whether after 4 seconds it will reach the goal. The change in the position of an object has a special name, "displacement."

From the other columns on this page, we can see that the puck travels -96 m in 4 seconds. The right direction is conventionally considered postive, so left is negative. That means the displacement is 96 m to the left. The total distance it moves during this same time is 96 m. Distance is always positive–you would never say that your home is negative five miles from your workplace, right?

In a situation like this, when the object is moving only in one direction, the displacement is the same as the distance that an object moves, but if the object changes direction at any point in its motion, the total distance it moves will be larger than its displacement. We will encounter this type of situation later in this chapter.
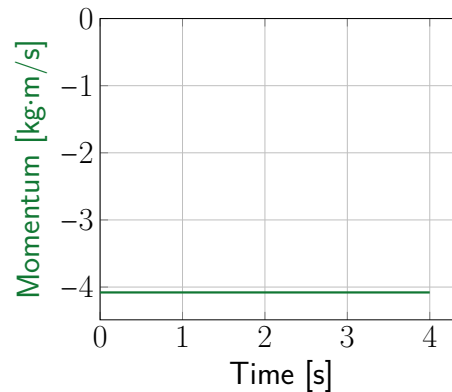


Figure 2.8: Momentum is negative.[1]

Motion Map
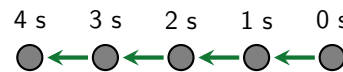


Figure 2.9: A hockey puck sliding to the left.[1]
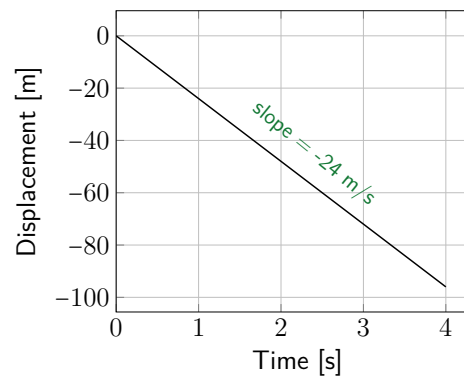


Figure 2.10: Puck displacement over time[1]

Momentum is given by...

$$\vec{p} = m \cdot \vec{v}$$

...so in this case...

$$p_x = m \cdot v_x = 0.17\,\mathrm{kg} \cdot (-24\,\mathrm{m/s}) = -4.08\,\mathrm{kg} \cdot \mathrm{m/s}$$

If the net force on an object is constant, as it is in this situation, position is given by...

$$\vec{x} = \vec{x_0} + \vec{v_0} \cdot t + \frac{1}{2}\vec{a} \cdot t^2$$

...and in this case we are only concerned with the $x$ direction...

$$x = x_0 + v_{0x} \cdot t + \frac{1}{2}a_x \cdot t^2$$

$$x = x_0 - 24\,\mathrm{m/s} \cdot t$$

We are not given $x_0$, but if we subtract it from both sides we have found the change in $x$, also called displacement.

$$x - x_0 = \Delta x = -24\,\mathrm{m/s} \cdot t$$

This allows us to find the displacement for any time $t$, including the final time 4 s.

$$\Delta x_f = -96\,\mathrm{m}$$

## 2.3 Falling to the Ground

### Words

A 2.4-kg ball is first dropped from a height 1.3 m above the ground. Air resistance is very small, so we will ignore it. We will consider only the vertical direction. Then the same ball is dropped again, this time from a height 2.6 m above the ground.

We should be able to find the amount of time that is needed to reach the ground in each situation and also the kinetic energy, velocity, and momentum of the ball just before it hits the ground.

When considering the motion of the ball, we should note that it is in free-fall, which means that it is affected only by the force of gravity, and has a constant acceleration that doesn't depend on the mass of the ball.

The first "unknown" mentioned in the description above is the time it takes the ball to fall to the ground. At first guess, one might think that the time needed to drop 2.6 m should be twice as much as the time needed to drop 1.3 m, but this is not correct. Notice in the images and motion map provided that in every interval of time the ball moves a greater distance than in the interval before.

From the images, we can see that the ball fell roughly the same distance in the first 0.3 seconds as it did in the next 0.15 seconds. This is because the longer it falls, the faster it is moving.

### Graphics



Figure 2.11: Images and corresponding motion map of a falling ball. Images were taken every 0.05 seconds.[1]

Comparing with the calculations on the right, we see that this ball fell slightly less than 1.3 m while it was being recorded by the camera.

### Numbers

**Assumptions:** $+\hat{y}$ is upward; no air resistance

| Knowns | Unknowns |
|---|---|
| $m = 2.4$ kg | $t_f$ |
| $y_0 = 1.3$ m or 2.6 m | $E_{k,f}$ |
| $y_f = 0$ | $v_{y,f}$ |
| $v_{0y} = 0$ | $p_{y,f}$ |
| $a_y = -g$ | |

We can find the time needed to hit the ground by considering the motion of the ball. Since the acceleration is constant...

$$\overrightarrow{x} = \overrightarrow{x_0} + \overrightarrow{v_0} \cdot t + \frac{1}{2}\overrightarrow{a} \cdot t^2$$

...and since we only need the y direction...

$$y = y_0 + v_{0y} \cdot t + \frac{1}{2}a_y \cdot t^2$$

Putting in everything we already know for the moment just before the ball hits the ground ($t_f$)...

$$y_f = 0 = y_0 - \frac{1}{2}g \cdot t_f^2$$

Solving for $t_f$, we find ...

$$t_f = \sqrt{\frac{2\,y_0}{g}}$$

$t_f$ is 0.515 s when dropped from 1.3 m and 0.728 s when dropped from 2.6 m.

The total energy of an object or system of objects can only change if something external to the system does work is done on the object(s). This is called the "Work-Energy Theorem." So the total gravitational potential energy and kinetic energy of the ball just before it is released is equal to the total energy of the ball just before it hits the ground. This is called conservation of energy.

We will not yet start to consider spring or thermal energy. Kinetic energy is energy of motion, and since the ball isn't moving at the moment it is released, it doesn't have any kinetic energy. But we can think of it has having gravitational potential energy since it is elevated above the ground.

Just before it hits the ground, it will not have any gravitational potential energy, so all of the gravitational potential energy has changed into kinetic energy, and the ball dropped from 2.6 m will have twice as much kinetic energy as the same ball dropped from 1.3 m.

When the ball is dropped from twice as high, it hits the ground at a higher speed, but not quite double. That is because the acceleration is constant the whole time, and acceleration changes velocity over time. Not over distance. Since the time is not double, neither is the final velocity.

The ball started with no momentum, but the force of gravity acting on it while it was in free-fall gave the ball momentum in the downward direction, and as with energy and velocity, when it is dropped from a higher position it gains more momentum before reaching the ground.
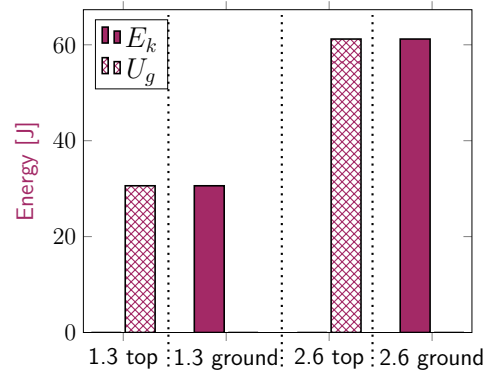


Figure 2.12: Energy bar graph of a ball falling from two different heights[1]

The first set of bars, labeled "1.3 top," (one of which has zero height) represents the kinetic and gravitational potential energies of a ball at the instant it is dropped from 1.3 m. The second two bars, "1.3 ground," (again, one of which has zero height) represent the same ball just before it hits the ground. The third and fourth sets of bars are similar, but for a ball dropped from 2.6 m.

Work $W$ is a transfer of energy into or out of a system.

$$W = E_{tot,f} - E_{tot,i} \qquad (2.2)$$

…where the subscript $tot$ means "total." In this case, we are considering gravity to be a source of potential energy, not an external force doing work on the ball, and we the only types of energy we have are $U_g$ and $E_k$, so…

$$U_{g,i} + E_{k,i} = U_{g,f} + E_{k,f}$$

…or…

$$m \cdot g \cdot y_i + \frac{1}{2} m \cdot v_i^2 = m \cdot g \cdot y_f + \frac{1}{2} m \cdot v_f^2$$

…so in this case…

$$m \cdot g \cdot y_0 + 0 = 0 + \frac{1}{2} m \cdot v_f^2$$

The term on the right is $E_{k,f}$, which is equal to $U_{g,i}$. That is 30.6 J when dropped from 1.3 m and 61.2 J when dropped from 2.6 m.

We can find the final velocities by solving the equation above for $v_f$:

$$v_f = \sqrt{2 \cdot g \cdot y_0}$$

Adding in the direction, $v_{y,f} = -5.05\,\mathrm{m/s}$ for 1.3 m and $v_{y,f} = -7.14\,\mathrm{m/s}$ for 2.6 m

We can use $m$ and $v_{y,f}$ to find $p_{y,f}$:

$$p_{y,f} = m \cdot v_{y,f}$$

That is, -12.1 $\frac{\mathrm{kg \cdot m}}{\mathrm{s}}$ for 1.3 m and -17.1 $\frac{\mathrm{kg \cdot m}}{\mathrm{s}}$ for 2.6 m

## 2.4 Being Thrown to the Ground

### Words

A 2.4-kg ball is thrown straight downward with a speed of 5.05 m/s from a height 1.3 m above the ground. Air resistance is very small, so we will ignore it. We will consider only the vertical direction, and only after the ball has been released.

We should be able to find the amount of time that is needed to reach the ground and also the kinetic energy, velocity, and momentum of the ball just before it hits the ground.

The ball is in free-fall, which tells us that it is accelerating downward because of gravity. The ball is being thrown downward, which also indicates that a force was used to throw the ball, but this situation doesn't include the actual throwing. This situation describes the ball after it is released, so there is NO "force of throwing" in this situation. If we looked at how the ball was thrown, then there would be a normal force from a hand or something similar to consider. But in this situation the ball has already been thrown, so there is no hand to consider.

There are many different correct ways to go about finding all of the unknown quantities. To demonstrate this, since we know the net force on the ball, the calculations in the right column can be done for final momentum after finding the final time when the ball reaches the ground. In the previous section of this book, momentum was the last quantity found.

### Graphics

If we are not provided with an image of some kind, again we should make a sketch of the situation.
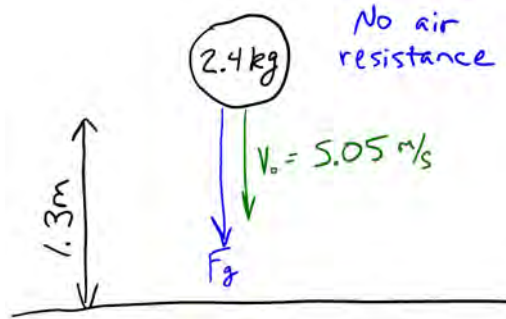


Figure 2.13: Rough sketch of a ball that was thrown downward[1]

Making a sketch like this helps us to see and remember important information, for example in this case the force and the initial velocity are in the same direction.



Figure 2.14: FBD of a ball in free-fall[1]

### Numbers

**Assumptions:** $+\hat{y}$ is upward; no air resistance; start after ball has been released

| Knowns | Unknowns |
|---|---|
| $m = 2.4$ kg | $t_f$ |
| $y_0 = 1.3$ m | $E_{k,f}$ |
| $y_f = 0$ | $v_{y,f}$ |
| $v_{0y} = -5.05$ m/s | $p_{y,f}$ |
| $a_y = -g$ | |

We can find $t_f$ by considering the motion of the ball. Since the acceleration is constant...

$$y_f = y_0 + v_{0y} \cdot t_f + \frac{1}{2} a_y \cdot t_f^2$$

...which can be rearranged to...

$$\frac{1}{2} a_y \cdot t_f^2 + v_{0y} \cdot t_f + (y_0 - y_f) = 0$$

$t_f$ and $t_f^2$ both appear in this equation, so we can use the quadratic formula to get two solutions: 0.213 s and -1.24 s. The negative solution is mathematically correct but cannot physically be the solution, so $t_f = 0.213$ s.

Since in this situation $\overrightarrow{F_{net}} = \overrightarrow{F_g}$, we can use Newton's Second Law to find the final momentum.

The momentum of the ball just before it hits the ground is directed downward, and is actually equal to the final momentum in the last section, for a similar ball that was dropped from a higher initial position. Since the balls have the same mass and the same final momentum, they also have the same final velocity.

We can again use conservation of energy to find the final kinetic energy, but this time we need to remember that the ball also starts with kinetic energy. As it falls, its kinetic energy increases as its gravitational potential energy decreases, so that its total energy stays constant.

Just before hitting the ground, all of the ball's gravitational potential energy has transformed into kinetic energy. For this situation, since the ball has the same mass and the same final velocity as the ball dropped from the higher position in the last section, the two balls also have the same final kinetic energy.

The final condition of the ball described in this section is the same as that of the ball dropped from a higher position in section 2.3. That is because when the ball dropped from the higher position reaches the same height as the ball described in this section, it has the same velocity as the initial velocity of the ball described in this section. It appears that only the time is different, but the time found in this section is equal to the time needed for the ball in section 2.3 to travel the *last* 1.3 m of its fall.
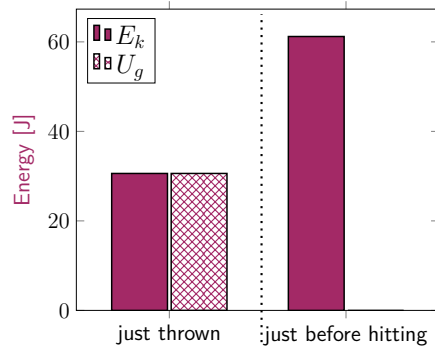


Figure 2.15: Energy bar graph for a ball being thrown to the ground, showing energy when it is just thrown and just before it hits the ground[1]
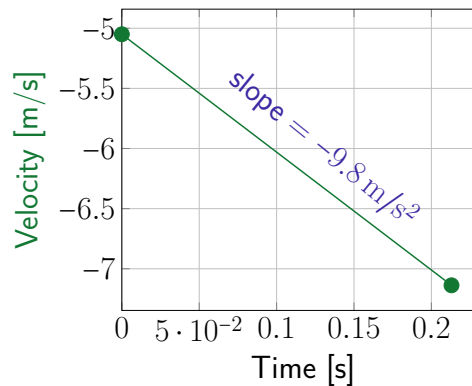


Figure 2.16: Velocity during free-fall[1]

$$\frac{\Delta p_y}{\Delta t} = \frac{p_{y,f} - p_{y,i}}{\Delta t} = -m \cdot g$$

Rearranging,

$$p_{y,f} = -m \cdot g \cdot \Delta t + p_{0y}$$
$$= -m \cdot g \cdot \Delta t + m \cdot v_{0y} = -17.1 \, \text{kg} \cdot \text{m/s}$$

Now we can find the final velocity of the ball.

$$p_{y,f} = m \cdot v_{y,f}, \quad \text{so} \ldots$$

$$v_{y,f} = \frac{p_{y,f}}{m} = -7.14 \, \text{m/s}$$

We can use work and energy, Equation 2.2, to find $E_{k,f}$, and since no external work $W$ is being done…

$$U_{g,i} + E_{k,i} = U_{g,f} + E_{k,f}$$
$$m \cdot g \cdot y_0 + \frac{1}{2} m \cdot v_0^2 = m \cdot g \cdot y_f + E_{k,f}$$

…which gives $E_{k,f} = 61.2 \, \text{J}$.

We could also have found the final velocities using the knowledge that acceleration is the change in velocity over time…

$$\boxed{\vec{a} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v_f} - \vec{v_i}}{\Delta t}} \qquad (2.3)$$

The y components of this equation can be rearranged to solve for $v_{y,f}$:

$$v_{y,f} = v_{y,i} + a_y \cdot \Delta t$$

## 2.5 Being Thrown Upward

### Words

A 2.4-kg ball is thrown straight *upward* with a speed of 5.05 m/s from a height 1.3 m above the ground. Air resistance is very small, so we will ignore it. We will consider only the vertical direction, and only after the ball has been released.

We should be able to find the height at the peak of the ball's flight, the amount of time that is needed to reach the peak, and also the kinetic energy, gravitational potential energy, velocity, and momentum of the ball at the peak.

The ball is in free-fall, which tells us that it is accelerating downward because of gravity. This is *almost* exactly the same as the situation considered in the last section–the only difference is that now the ball has been thrown upward, opposite the force of gravity. Again, there is no force from the hand that threw it (or of whatever else might have thrown it), because the question begins with the ball already having been thrown.

In every situation up until this one, a force has increased both the speed and the magnitude of the momentum of the object we are considering. This time the force does the opposite. That is because the initial velocity is *opposite* the direction of the net force.

The ball still accelerates in the direction of the net force, but the acceleration *slows down* the ball instead of speeding it up!

### Graphics

Again we should make a sketch. This is the same as the sketch from section 2.4 except that one arrow changed direction.
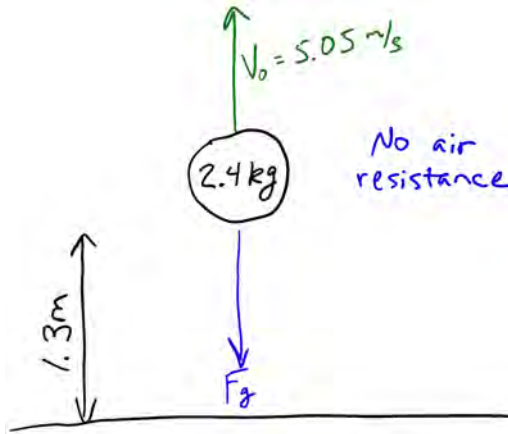


Figure 2.17: Rough sketch of a ball that was thrown *upward*[1]



Figure 2.18: FBD of a ball in free-fall[1]

A FBD shows downward force with no indication of upward velocity, exactly as in section 2.4.

### Numbers

**Assumptions:** $+\hat{y}$ is upward; no air resistance; start after ball has been released

| Knowns | Unknowns |
| --- | --- |
| $m = 2.4$ kg | $y_f$ |
| $y_0 = 1.3$ m | $t_f$ |
| $v_{0y} = 5.05$ m/s | $E_{k,f}$ |
| $a_y = -g$ | $U_{g,f}$ |
| | $v_{y,f}$ |
| | $p_{y,f}$ |

Note that the only changes to the "knowns" from section 2.4 are that there is no "-" on $v_{0y}$ and $y_f$ is now unknown. $U_{g,f}$ has also been added as an unknown.

Three of the six unknowns are easy to find, once we realize that at the peak the ball is not moving. That makes $E_{k,f} = 0$, $v_{y,f} = 0$, and $p_{y,f} = 0$.

In regular English, it is normal to refer to "slowing down" as "decelerating," and "speeding up" as "accelerating." That is not the case in physics. We will always use the word "acceleration" to indicate that an object's velocity is changing. Sometimes the magnitude of the velocity (speed) may be increasing; sometimes it may be decreasing; and in some cases the speed may be staying constant but the direction may be changing. To avoid confusion, we will refer to all of these situations as acceleration.

We can again use conservation of energy to find the final gravitational potential energy. At the beginning, the ball has both gravitational potential energy, since it is elevated above the ground, and also kinetic energy, since it is moving. As it goes upward, it slows down, ultimately stopping when it reaches the peak. That means it is losing kinetic energy, and all of the kinetic energy it loses is transforming into gravitational potential energy. So at the peak, all of its initial energy has changed into gravitational potential energy.

When the ball reaches the peak, its height and its gravitational potential energy are at a maximum, while its speed and kinetic energy are at a minimum. The momentum was positive as the ball was going up, since "upward" is usually considered a positive direction. At the top, the momentum is zero, and unless something intervenes, a short time later the ball will be moving downward, so the momentum will be negative.
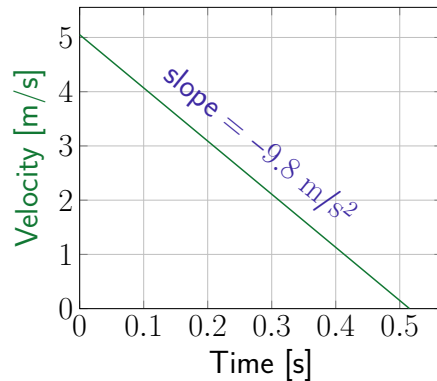


Figure 2.19: Velocity during free-fall[1]
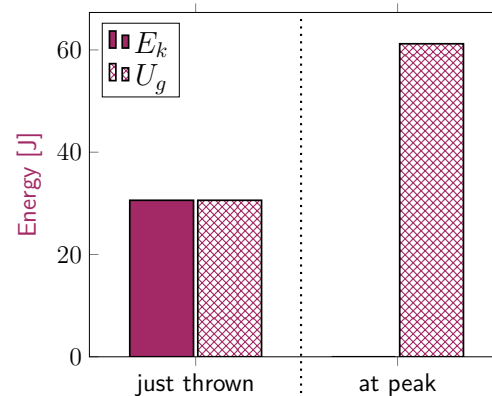


Figure 2.20: Energy bar graph for a ball being thrown upward, showing energy when it is just thrown and at the peak of its flight[1]

We know the acceleration and the initial and final velocity of the ball, so we can rearrange the $\hat{y}$ components of Equation 2.3 to find $t_f$:

$$\Delta t = t_f - t_0 = \frac{v_{y,f} - v_{0y}}{a_y}$$

...giving $t_f = 0.515\,\text{s}$, using $t_0 = 0$.

We can again use conservation of energy with no external work, Equation 2.2, to find $U_{g,f}$.

$$U_{g,i} + E_{k,i} = U_{g,f} + E_{k,f}$$

$$m \cdot g \cdot y_0 + \frac{1}{2} m \cdot v_0^2 = U_{g,f} + 0$$

...which gives $U_{g,f} = 61.2\,\text{J}$.

The final height can be found from the gravitational potential energy.

$$U_{g,f} = m \cdot g \cdot y_f$$

This gives $y_f = 2.6\,\text{m}$.

41

## 2.6  Up and Back Down

✚ ◆

### Words

A 2.4-kg ball is thrown straight *upward* with a speed of 5.05 m/s from a height 1.3 m above the ground. Air resistance is very small, so we will ignore it. We will consider only the vertical direction, and only after the ball has been released.

In Section 2.5 we considered this exact same situation, but stopped when the ball reached the peak of its flight. This time we will consider the entire flight of the ball until the moment just before it hits the ground. Let's focus on the initial condition of the ball when it has just been thrown, the peak of its flight, the time when it passes the same height from which it was thrown, and the moment just before it hits the ground.

Again the ball is in free-fall, which tells us that it has a constant downward acceleration because of gravity. That acceleration is valid for the *entire time* that the ball is in the air. Some people are surprised by this, thinking that the acceleration should be zero at the peak of the ball's flight, but if that were true then the ball would just stay there instead of coming back down.

If we look back at the previous sections of this chapter, we have already found all of the pieces of information for the ball as it travels along this entire path. We just need to put it all together.

### Graphics

An initial sketch if this situation would look exactly the same as the sketch from Section 2.5. So instead let's try to sketch a motion map for the ball.
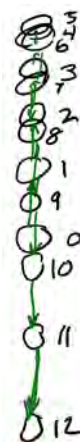


Figure 2.21: An attempt at a motion map. The ball's path turns back on itself, making a motion map difficult to read. The numbers give the order of the points, but the time scale is not given.[1]

A motion map is not the best way to show the motion of an object that retraces its path.

### Numbers

**Assumptions:** $+\hat{y}$ is upward; no air resistance; start after ball has been released

| Knowns | Unknowns |
|---|---|
| $m = 2.4$ kg | $y_{peak}, y_{down}$ |
| $y_0 = 1.3$ m | $t_{peak}, t_{down}, t_{bottom}$ |
| $y_{bottom} = 0$ | $E_{k,peak}, E_{k,down}, E_{k,bottom}$ |
| $v_{0y} = 5.05$ m/s | $U_{g,peak}, U_{g,down}, U_{g,bottom}$ |
| $a_y = -g$ | $v_{y,peak}, v_{y,down}, v_{y,bottom}$ |
| | $p_{y,peak}, p_{y,down}, p_{y,bottom}$ |

"Peak" is for the peak of the ball's flight; "down" is when it passes the same height from which it was thrown; "bottom" is just before it hits the ground. We have already found the time required to go from each of these positions to the next:

| Description | Interval | Where found |
|---|---|---|
| $t = 0$ to $t_{peak}$ | 0.515 s | Section 2.5 |
| $t_{peak}$ to $t_{down}$ | 0.515 s | Section 2.3 |
| $t_{down}$ to $t_{bottom}$ | 0.213 s | Section 2.4 |

Section 2.3 gives the time for $t_{peak}$ to $t_{down}$ if we realize that the time required for a ball to fall 1.3 m does not depend on the height from which it was dropped. So the ball that was dropped from a height of 1.3 m reaches the ground in the same amount of time that a ball dropped from a height of 2.6 m falls 1.3 m.

The ball starts out moving upward, steadily slowing down until it reaches the peak of its flight, so that its velocity is zero at the peak. It doesn't stay at the peak, but turns around and falls downward, steadily speeding up until it hits the ground. When the ball is falling and passes the same point from which it was originally thrown, it will have the same speed that it started with, but the direction will have changed from upward to downward.

If you graph the position as a function of time, the graph forms a parabola shape with its opening pointed downward.

If you graph the velocity as a function of time, the graph forms a straight line, because the acceleration is the same during the entire flight: the acceleration caused by gravity in the downward direction.

Momentum follows the same pattern as velocity. Initially it is upward, and it decreases as the ball goes upward. At the peak, the ball has no momentum, and on the way down its momentum increases, this time in the downward direction.

Initially, the ball has some gravitational potential energy, since it is above ground level, and it also has some kinetic energy, since it is moving. At the peak of its flight the ball is not moving, so it has no kinetic energy. All of the kinetic energy has changed to gravitational potential energy at that point. As the ball falls, the gravitational energy transforms into kinetic energy until by the time it is about to hit the ground it has only kinetic energy.
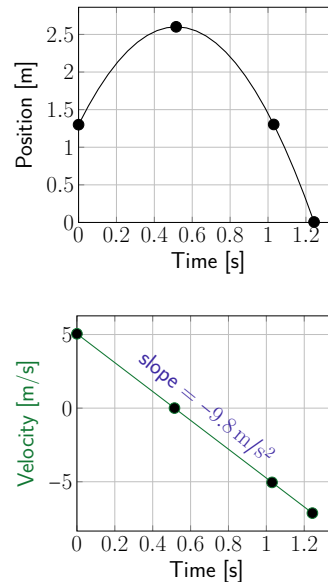


Figure 2.22: Position & velocity for ball in free-fall[1]
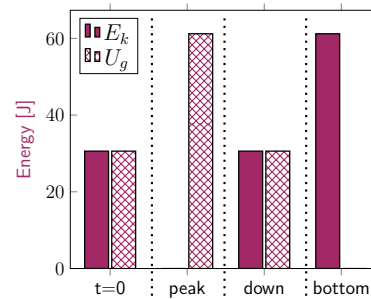


Figure 2.23: Energy bars for ball in free-fall.[1]

Since the acceleration is constant and we are considering only the vertical direction…

$$y = y_0 + v_{0y} \cdot t + \frac{1}{2} a_y \cdot t^2$$

In this situation…

$$y = 1.3\,\text{m} + 5.05\,\text{m/s} \cdot t - \frac{1}{2}\left(9.8\,\text{m/s}^2\right) \cdot t^2$$

And for the velocity we can rearrange Equation 2.3 to find the velocity in the $\hat{y}$ direction at any time $t$:

$$v_y = v_{0y} + a_y \cdot t$$
$$v_y = 5.05\,\text{m/s} - 9.8\,\text{m/s}^2 \cdot t$$

Momentum is mass times velocity, so…

$$p_y = m \cdot v_y = m \cdot \left(v_{0y} + a_y \cdot t\right)$$
$$= 2.4\,\text{kg} \cdot \left(5.05\,\text{m/s}\right) - 2.4\,\text{kg} \cdot 9.8\,\text{m/s}^2 \cdot t$$

Gravitational potential energy is given by $U_g = m \cdot g \cdot y$, so…

$$U_g = 2.4\,\text{kg} \cdot 9.8\,\text{m/s}^2 \cdot y$$

Kinetic energy is given by $E_k = \frac{1}{2} m \cdot v^2$, so…

$$E_k = \frac{1}{2}\left(2.4\,\text{kg}\right) \cdot v_y{}^2$$

43

## 2.7   Accelerating in a Car

### Words

Up to this point, we have only worked with acceleration due to the force of gravity. But other forces can cause acceleration. Now we will consider a car sitting motionless, accelerating to a given velocity, and maintaining that velocity.

A Ferrari Enzo with a mass of 1500 kg can go from zero to 60 mph in 3 seconds. Assume that it accelerates uniformly for 3 s and then maintains its speed for another 3 s. Describe the net force on the car, the energy of the car, the displacement of the car, and the momentum of the car during these time intervals.

Let's start by just imagining ourselves in the Ferrari. What would it feel like? We would feel the car accelerating as we started, rapidly gaining speed. It is interesting to note that it actually feels like our body is shoved back against the seat as the seat tries to accelerate forward through us. Then it would feel different after the first three seconds because we would just be moving at constant speed. In fact, if you are moving at constant speed it feels almost like you are not moving at all, unless you are watching the scenery go past outside the window.

Since it is an unbalanced force that causes acceleration, there must be a large net force for the first three seconds pushing us forward, but then the net force drops to zero after three seconds. That doesn't necessarily mean that there are no forces after three seconds but the forces are balanced.

### Graphics



Figure 2.24: A Ferrari Enzo.[6]

**Motion Map**
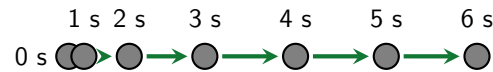


Figure 2.25: A car accelerating for 3 seconds and then moving at constant velocity for 3 seconds[1]

Acceleration on a motion map appears as a change in the arrows from one dot to the next. For the first three seconds, the arrows are getting longer to the right, showing that there is acceleration to the right, and thus a net force to the right. After three seconds the arrows are all the same length and the same direction, showing that there is no acceleration, so zero net force.

### Numbers

This situation has two distinct parts that need to be separated mathematically. Acceleration is constant in the first 3 s, and constant in the second 3 s. But since they are different, we will have one set of mathematical models from $t = 0$ to $t = 3\,\text{s}$ and another from $t = 3\,\text{s}$ to $t = 6\,\text{s}$.

| Knowns | Unknowns |
|---|---|
| $m = 1500\,\text{kg}$ | $\overrightarrow{F_{net,0-3}}$ ; $\overrightarrow{F_{net,3-6}}$ |
| $v_{0x} = 0$ | $E$ for the whole time |
| $v_{x,3-6} = 60\,\text{mph}$ | $\Delta x_{0-3}$ ; $\Delta x_{3-6}$ |
| | $\overrightarrow{p}$ for the whole time |

Let's begin by finding the net force. Using Newton's Second Law, we find…

$$\overrightarrow{F_{net}} \equiv \frac{\overrightarrow{\Delta p}}{\Delta t} = \frac{\left(\overrightarrow{p_f} - \overrightarrow{p_i}\right)}{\Delta t} = \frac{m \cdot \left(\overrightarrow{v_f} - \overrightarrow{v_i}\right)}{\Delta t}$$

Velocity is not changing from 3 to 6 s, so $\overrightarrow{F_{net,3-6}} = 0$. And from 0 to 3 s…

$$\overrightarrow{F_{net}} = \frac{1500\,\text{kg} \cdot (60\,\text{mph}\,\hat{x} - 0)}{3\,\text{s}}$$

We still need to convert the units, so…

$$\overrightarrow{F_{net}} = \left(\frac{1500\,\text{kg} \cdot 60\,\text{mph}}{3\,\text{s}}\right)\left(\frac{1609\,\text{m}}{1\,\text{mile}}\right)\left(\frac{1\,\text{hour}}{3600\,\text{s}}\right)\hat{x}$$

$$\overrightarrow{F_{net,0-3}} = 1.34 \times 10^4\,\text{N}\,\hat{x}$$

What about the energy of the car? If we are on flat ground, the height is not changing, so we don't need to worry about gravitational potential energy. We can focus on just the horizontal direction in this situation.

Initially the car is not moving, so there is no kinetic energy. But over the first three seconds the speed of the car is increasing, so the kinetic energy is also increasing. After the first three seconds the speed is constant, so for the last three seconds the kinetic energy remains constant at whatever value it had after the first three seconds.

Energy is conserved in a closed system, so if the kinetic energy is increasing then energy is coming from somewhere. This Ferrari burns gasoline, so the source of energy is the chemical potential energy stored in the gasoline. Tracing the energy transformations in the engine and drive train is complicated, but the final result is that the car uses an external frictional force between the tires and the road to push itself forward. This force does work on the car as it moves, increasing its kinetic energy.

The car's momentum will change in the same way that its velocity changes. So initially it has no momentum; then momentum steadily increases for the first three seconds while a constant net force is applied; and finally the momentum remains constant after three seconds when there is no net force acting on the car.
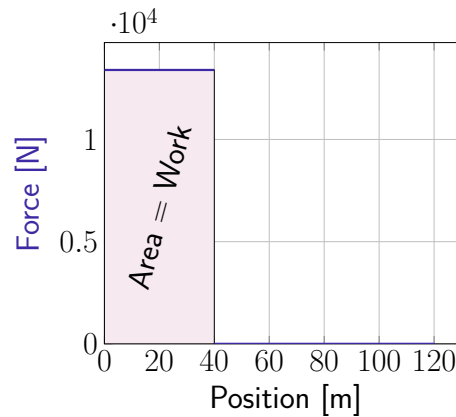


Figure 2.26: A force in the direction of motion does an amount of work equal to the area under the curve in a Force-vs-Position graph.[1]
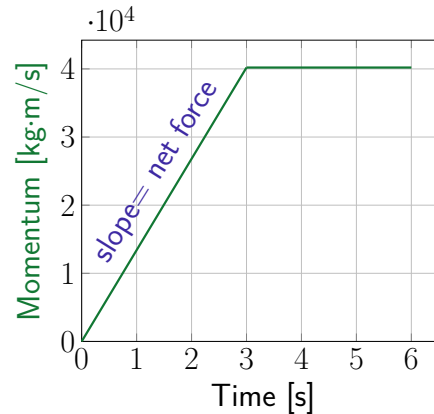


Figure 2.27: Momentum changes while there is a net force, then remains constant when the net force is zero.[1]

The net force acting on the car does work on the car, giving it kinetic energy. The amount of work done by the force depends on the displacement:

$$\boxed{W_{net} = \overrightarrow{F_{net}} \cdot \overrightarrow{\Delta x} = F_{net} \cdot \Delta x \cdot \cos(\theta)} \quad (2.4)$$

...where $\theta$ is the angle between $\overrightarrow{F_{net}}$ and $\overrightarrow{\Delta x}$.

This work will often be used in conjunction with Equation 2.2 to find the change in energy of a system. In this example, we are only working with kinetic energy. We know the mass of the car and its velocity at $t = 0$ and $t = 3\,\text{s}$, so that gives us the kinetic energy at each of these times. Using Equation 2.4 and Equation 2.2 we can find the displacement of the car during that time.

$$\Delta E_k = W_{net} = \overrightarrow{F_{net}} \cdot \overrightarrow{\Delta x} = F_{net} \cdot \Delta x \cdot \cos(0)$$

$\theta$ is zero because $\overrightarrow{F_{net}}$ and $\overrightarrow{\Delta x}$ are in the same direction. Rearranging to solve for $\Delta x$ gives...

$$\Delta x_{0-3} = \frac{\Delta E_{k,0-3}}{F_{net,0-3}} = \frac{5.39 \times 10^5\,\text{J}}{1.34 \times 10^4\,\text{N}} = 40\,\text{m}$$

Now we can find $\Delta x_{3-6}$:

$$x = x_0 + v_{0x} \cdot t + \frac{1}{2}a_x \cdot t^2$$

With $a_x = 0$ and using 3 s as our "$t = 0$", ...

$$\Delta x_{3-6} = (x - x_0) = v_{0x} \cdot t + 0 = 60\,\text{mph} \cdot 3\,\text{s} = 80\,\text{m}$$

## 2.8 Braking in a Car

### Words

A Ferrari Enzo with a mass of 1500 kg is traveling at 26.8 m/s and suddenly applies its brakes, giving it an acceleration of $15\,\mathrm{m/s^2}$ in the direction opposite its motion. Describe the net force on the car, the car's momentum, the time required to stop, and the stopping distance.

If the car had been traveling at twice that speed and then braked with the same acceleration, by how much would the net force, the momentum, the time required to stop, and the stopping distance change?

When we are in a car that is braking, its speed is decreasing. That means the acceleration is opposite the direction of motion, so if your car is moving forward then the acceleration is backwards.

Again, since it is force that causes acceleration, the net force must be pointing in the direction opposite the car's motion. The acceleration described above, $15\,\mathrm{m/s^2}$, is larger than the acceleration caused by gravity on the surface of the earth, so this car must be braking very hard. We should expect it to stop quickly.

The car's momentum changes along with velocity, starting large and steadily decreasing until it reaches zero when the car is stopped.

### Graphics



Figure 2.28: A Ferrari Enzo.[6]

**Free Body Diagram**



Figure 2.29: FBD of a car that is traveling to the right and braking, horizontal direction only[1]

### Numbers

**Assumptions:** initial velocity is in the $+\hat{x}$ direction

| Knowns | Unknowns |
|---|---|
| $m = 1500$ kg | $\overrightarrow{F_{net}}$ |
| $v_{0x} = 26.8\,\mathrm{m/s}$ | $t_f$ |
| $a_x = -15\,\mathrm{m/s^2}$ | $\Delta x$ |
| $v_f = 0$ | |

Note that the acceleration is negative, while the initial velocity is positive. This is equivalent to the statement that acceleration is opposite the direction of motion. The stopping distance would be the change in position, in other words, the displacement.

Let's begin by finding the net force. Using Newton's Second Law…

$$\overrightarrow{F_{net}} = m \cdot \overrightarrow{a} = 1500\,\mathrm{kg} \cdot \left(-15\,\mathrm{m/s^2}\,\hat{x}\right)$$
$$= -2.25 \times 10^4\,\mathrm{N}\,\hat{x}$$

Since the acceleration is constant until the car stops moving, this force is also constant until the car stops moving.

Since we have the mass and the initial and final velocities, we can find the initial and final momentum of the car:

46

What would be different if the car had been traveling at twice the speed, and had braked with the same acceleration?

Since force is directly related to acceleration and not directly related to velocity, the frictional force of braking would be the same.

The final momentum would still be zero, but the initial momentum would double along with the velocity.

Remember that acceleration is a change in velocity over time. Since we are more familiar with speed and distance, let's use that as an example. Speed is a distance over time. If you travel at the same speed but need to go double the distance, the time would have to double, right? It is the same with acceleration and velocity:

- Speed is a distance over time; double the distance means double the time.

- Acceleration is velocity over time; double the velocity means double the time.

The distance traveled before stopping would certainly be longer if we started with double the speed and braked with the same acceleration. In fact the distance increases by much more than double. If you start at double the speed, it takes half of the braking time just to get down to the original speed; in the first half of the braking time you would move a much farther distance than if you had been traveling at the original speed that whole time. And you still wouldn't have stopped!
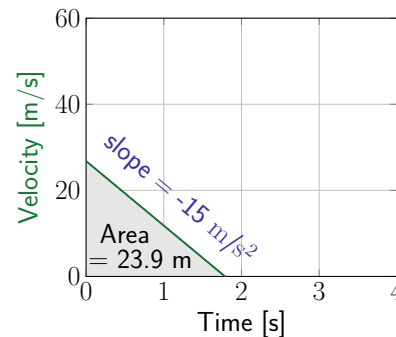


Figure 2.30: Velocity vs time, braking.[1]

The area under the curve of a velocity vs time graph is equal to the displacement. In this case, the maximum velocity is 26.8 m/s and the maximum time is 1.79 s. The area of a triangle is:

$$A_{triangle} = \frac{1}{2} b \cdot h$$

...where $b$ is the triangle's base and $h$ is its height.
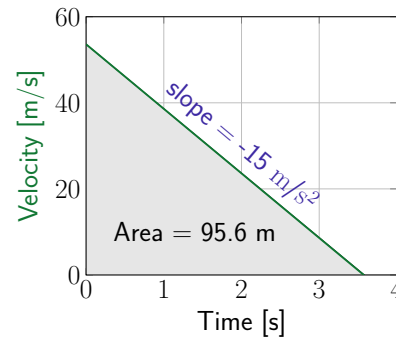


Figure 2.31: Velocity vs time, braking.[1]

$$\vec{p_0} = 1500 \text{ kg} \cdot 26.8 \text{ m/s} \, \hat{x} = 4.02 \times 10^4 \text{ kg} \cdot \text{m/s} \, \hat{x}$$

$$\vec{p_f} = m \cdot \vec{v_f} = 0$$

The time required to stop can be found using Equation 2.3:

$$\vec{a} = \frac{\overrightarrow{\Delta v}}{\Delta t}$$

If we set $t_i = 0$, this can be rearranged to...

$$t = \frac{\vec{v_f} - \vec{v_0}}{\vec{a}} = \frac{0 - 26.8 \text{ m/s} \, \hat{x}}{-15 \text{ m/s}^2 \, \hat{x}} = 1.79 \text{ s}$$

The displacement can be found by considering energy. Initially the car has kinetic energy, and the braking force acting over a distance does *negative* work on the car, removing all of its kinetic energy:

$$\Delta E_k = E_{k,f} - E_{k,i} = 0 - \frac{1}{2} m \cdot v_{0x}^2$$

And according to Equation 2.4 and Equation 2.2...

$$\Delta E_k = W_{net} = \overrightarrow{F_{net}} \cdot \overrightarrow{\Delta x}$$

Setting these equal to each other and solving for $\Delta x$ gives...

$$\Delta x = \frac{-\frac{1}{2} m \cdot v_{0x}^2}{\overrightarrow{F_{net}}} = \frac{-5.39 \times 10^5 \text{ J}}{-2.25 \times 10^4 \text{ N}} = 23.9 \text{ m}$$

## 2.9   Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

### General

- Units in an equation cancel just like variables or numbers.

- Numbers with different dimensions can be multiplied or divided, but not added or subtracted.

- Physical quantities should be converted to SI units before being used in calculations.

- When thinking about a physical scenario, it is often helpful to make a sketch.

- Sometimes a calculator can give answers that are correct mathematically but cannot be physically correct.

- Conventionally, right is the positive "x" direction $\hat{x}$.

- Conventionally, up is the positive "y" direction $\hat{y}$.

$$+\hat{y}$$

$$+\hat{x}$$

Conventional $x$ and $y$ directions[1]
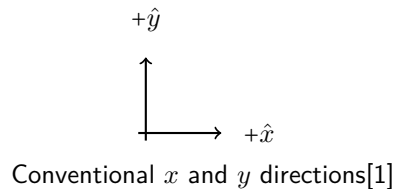
### Forces

- In U.S. Customary units, pound is the unit of force; slug is the unit of mass.

- If a force acts in the direction opposite to the velocity of an object it will slow the object down.

- A force acting in the direction of an object's displacement does positive work on the object.

- A force acting opposite the direction of an object's displacement does *negative* work on the object.

### Motion

- The change in the position of an object is called its displacement.

- Displacement can be positive or negative.

- The total distance an object moves, or the length of the path it follows, is always positive.

- Acceleration refers to a change in velocity whether the speed is increasing, decreasing, or staying the same.

- The acceleration of an object in free-fall is constant throughout the time the object is in the air, even if it is not moving at some point in time during the flight.

- The slope of the line on either a position-vs-time graph or a displacement-vs-time graph is the velocity.
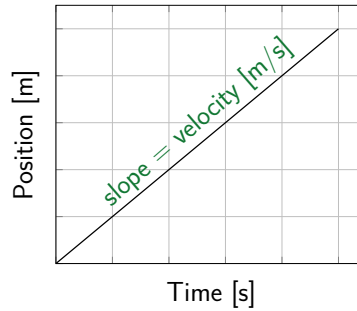
Sample position vs time graph[1]

- The slope of the line on a velocity-vs-time graph is the acceleration.

- The area under the curve of a velocity-vs-time graph is the displacement of the object.
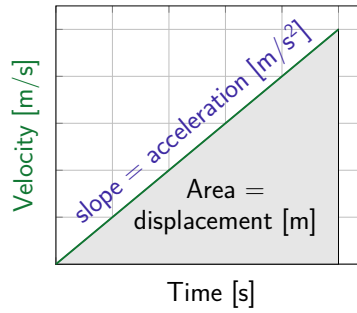


Sample velocity vs time graph[1]

## Momentum

- Momentum is negative if velocity is negative.

- The slope of the line on a momentum-vs-time graph is the net force on the object.
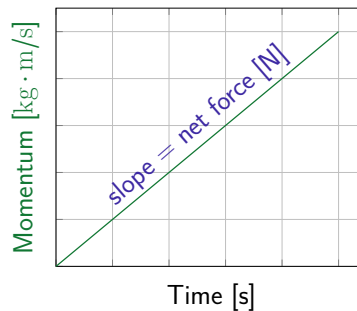


Sample momentum vs time graph[1]

## Energy

- The area under the curve of a force-vs-position graph is the amount of work done by the force.
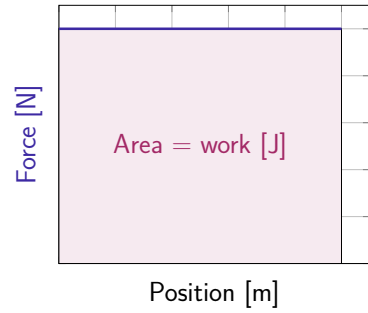
Sample force vs position graph[1]

## Mathematical Models

| equation | | restrictions on the validity of the equation |
|:---:|:---:|:---:|
| $v_{avg} = \frac{s}{\Delta t}$ | (2.1) | -none- |
| $W = E_{tot,f} - E_{tot,i}$ <br><br> "Work-Energy Theorem" | (2.2) | -none- |
| $\vec{a} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{v_f} - \vec{v_i}}{\Delta t}$ | (2.3) | only valid when the net force is constant |
| $W_{net} = \overrightarrow{F_{net}} \cdot \overrightarrow{\Delta x} = F_{net} \cdot \Delta x \cdot \cos(\theta)$ | (2.4) | only valid when the net force is constant |

## 2.10   Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

### Level 1 - Remember

2.1 **[W]** List two reasons for using SI units instead of U.S. Customary Units in physics.

2.2 **[N]** Give the appropriate SI units and U.S. Customary units for each of the following:

|        | SI (Système International) Unit | U.S. Customary Unit |
|--------|--------------------------------|---------------------|
| length |                                |                     |
| force  |                                |                     |
| mass   |                                |                     |
| speed  |                                |                     |

2.3 **[G]** How does one find work from a graph of force vs. position?

2.4 **[G]** How does one find force from a graph of momentum vs. time?

2.5 **[G]** How does one find acceleration from a graph of velocity vs. time?

2.6 **[G]** How does one find displacement from a graph of velocity vs. time?

2.7 **[G]** How does one find velocity from a graph of position vs. time?

2.8 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

### Level 2 - Understand

2.9 **[W]** The word "acceleration" is used in physics to mean a change in velocity. What is happening to an object's speed when it is accelerating?

### Level 3 - Apply

2.10 **[G]** Draw a free-body diagram for the ball in Section 2.3.

2.11 **[G & N]** What is the total displacement of the ball described in Section 2.6, from the time it is thrown to the moment just before it hits the ground? Remember that displacement is a vector.

2.12 **[G & N]** What is the total distance (path length) traveled for the ball described in Section 2.6, from the time it is thrown to the moment just before it hits the ground?

2.13 **[G]** Figure 2.22 shows a position-vs-time graph and its corresponding velocity-vs-time graph. Create a corresponding acceleration-vs-time that goes along with these other two graphs.

2.14 **[G & N]** Create position-vs-time, velocity-vs-time, and acceleration-vs-time graphs for the soccer ball from Section 1.4, horizontal direction only.

2.15 **[G & N]** Create position-vs-time, velocity-vs-time, and acceleration-vs-time graphs for the rock from Section 1.5, vertical direction only.

2.16 **[W]** Describe the momentum of a ball that is dropped from the time it leaves a person's hand until the time just before it hits the ground.

2.17 **[G]** Section 2.7 describes a physical situation of a car accelerating and then traveling at constant speed.

    (a) Draw a velocity vs. time graph for the car in this situation. Be sure to use SI units.

    (b) Use the graph to find the displacement of the car in the first 3 s.

    (c) Use the graph to find the displacement of the car in the second 3 s.

    (d) Use the graph to find the acceleration of the car in the first 3 s.

    (e) Use the graph to find the acceleration of the car in the second 3 s.

## Level 4 - Analyze

2.18 **[G]** Figure 2.4 shows a graph of position vs. time for an object that is moving at 2 m/s. Draw a similar graph for an object that is moving at 5 m/s for 10 s.

2.19 **[W, G, & N]** In Section 2.2, there was a 170-g hockey puck sliding across a sheet of ice. If the mass of the hockey puck had been 340 g and everything else in the situation stayed the same, which of the following would change?

    (a) The sketch at the beginning of Section 2.2 (and make a new sketch if any change is needed)

    (b) The free body diagram in Section 2.2 (and make a new FBD if any change is needed)

    (c) The calculation of kinetic energy in Section 2.2 (and find the new value for the kinetic energy if any change is needed)

    (d) The motion map in Section 2.2 (and make a new motion map if any change is needed)

    (e) The calculation of momentum in Section 2.2 (and find the new value for the momentum if any change is needed)

    (f) The calculation of displacement in Section 2.2 (and find the new value for the displacement if any change is needed)

2.20 **[W, G, & N]** In Section 2.2, there was a hockey puck sliding across a sheet of ice at 24 m/s. If the speed of the hockey puck had been 12 m/s and everything else in the situation stayed the same, which of the following would change?

    (a) The sketch at the beginning of Section 2.2 (and make a new sketch if any change is needed)

    (b) The free body diagram in Section 2.2 (and make a new FBD if any change is needed)

    (c) The calculation of kinetic energy in Section 2.2 (and find the new value for the kinetic energy if any change is needed)

    (d) The motion map in Section 2.2 (and make a new motion map if any change is needed)

    (e) The calculation of momentum in Section 2.2 (and find the new value for the momentum if any change is needed)

    (f) The calculation of displacement in Section 2.2 (and find the new value for the displacement if any change is needed)

2.21 **[N]** In the last line of the "Numbers" column of Section 2.7, it appears to say that $60 \cdot 3 = 80$. Should it be 180?

## Level 5 - Evaluate

2.22 **[N]** Compare the change in gravitational potential energy for an object of mass $m$ falling a distance $d$ to the work done by the force of gravity on an object of mass $m$ falling a distance $d$. Explain your reasoning.

2.23 **[N]** Compare the work done bringing an object of mass $m$ from a complete stop to a speed $v$ over a distance $d$ to the work done bringing that same object from the same speed $v$ to a stop over a distance $d/4$. Explain your reasoning.

## Level 6 - Create

2.24 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Add the main ideas from this chapter to the concept map that you began for the question at the end of Chapter 1.

2.25 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

2.26 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

# Chapter 3

# Two Objects

Up to this point we have only been looking at a single object. That object has always interacted with something else: A soccer ball rolling across the ground; a hockey puck sliding on ice; a ball falling under the influence of the earth's gravity; but we have never stopped to consider the second object and how it interacts with the first.

We will consider three new things in this chapter: reference frames (also called frames of reference or points of view), force pairs, and conversion of kinetic energy to thermal energy in collisions or when objects are sliding against each other.

Whenever we look at a physical situation, we need to be mindful of our reference frame. Up to this point, the surface of the earth has been used as a reference, and is assumed to be fixed in place as we consider the physics of each situation. But we can also use moving reference frames. For example, if you are a passenger in a moving car, it is still quite easy to pick up an object that is sitting next to you, even though that object is traveling at high speed compared to the ground outside of the car. As long as the car is moving at a constant velocity, the physics inside the car is just the same as if the car were sitting still.



Figure 3.1: Here we see two objects that have interacted with each other. To understand the collision, we need to take both objects into account, and we will have to decide whether to look at the collision from the reference frame of a person in the truck, a person in the car, or a person standing by the side of the road.[7]

The idea of force pairs comes about because forces are always interactions between objects. If an object exerts a force on another object, the second object exerts the same amount of force back on the first.

Collisions are often divided into categories, elastic or inelastic, depending on what happens during the collision. In a "completely inelastic" collision the objects stick together and a large amount of kinetic energy is converted into thermal energy. In a "perfectly elastic" collision the objects bounce off of each other and no kinetic energy is converted to thermal energy. Most collisions fall between these two extremes, with objects deforming but not sticking together. These collisions are considered elastic or inelastic depending on the amount of kinetic energy that was converted to thermal energy.

## 3.1   Reference Frames

**Words**

Imagine standing on a bridge over the highway in the picture on the right. You would look at these vehicles and say that the two cars are going North at 25 m/s and the truck is going South at 25 m/s. You are comparing the velocities of the vehicles to your own velocity, which is zero if the surface of the earth is stationary.

Of course the surface of the earth is not stationary because the earth is spinning and orbiting the sun! But in many situations we can ignore the motion of the earth itself. Since you are not moving with respect to the surface of the earth, we say that you are looking at the vehicles from the earth's reference frame.

What are the velocities of the vehicles in each others' reference frames?

To consider other reference frames, we just need to imagine ourselves in a different place. What would this situation look like if we were inside the black car? Since the white car that is ahead of us is moving at the same speed, in the same direction as us, it will actually appear not to be moving at all. If it is 20 meters ahead of us, it will stay 20 meters ahead of us as long as neither of us accelerates.

**Graphics**



Figure 3.2: Two cars traveling North and a truck traveling South, all at 25 m/s.[8]
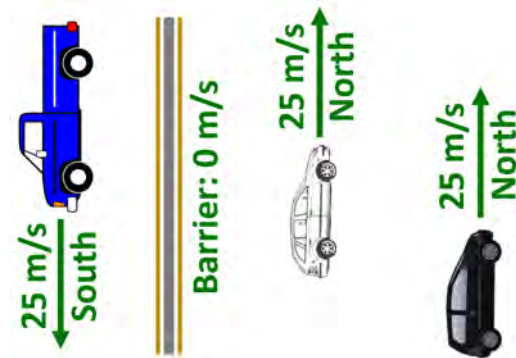


Figure 3.3: The earth's reference frame[9]

**Numbers**

We will need to introduce new notation for velocity in different reference frames: "$\overrightarrow{v_{1 \leftarrow 2}}$" will mean "the velocity of object 1 as seen by object 2." Using this notation, making North the positive direction, and referring to the cars just as "white" and "black"…

| Knowns | Unknowns |
|---|---|
| $\overrightarrow{v_{truck \leftarrow earth}} = -25\,\text{m/s}$ | $\overrightarrow{v_{white \leftarrow truck}}$ |
| $\overrightarrow{v_{white \leftarrow earth}} = +25\,\text{m/s}$ | $\overrightarrow{v_{black \leftarrow truck}}$ |
| $\overrightarrow{v_{black \leftarrow earth}} = +25\,\text{m/s}$ | $\overrightarrow{v_{truck \leftarrow white}}$ |
| | $\overrightarrow{v_{black \leftarrow white}}$ |
| | $\overrightarrow{v_{truck \leftarrow black}}$ |
| | $\overrightarrow{v_{white \leftarrow black}}$ |

To change from one reference frame to another, simply subtract the velocity of the object whose frame we are entering:

$$\boxed{\overrightarrow{v_{2 \leftarrow 3}} = \overrightarrow{v_{2 \leftarrow 1}} - \overrightarrow{v_{3 \leftarrow 1}}} \qquad (3.1)$$

To go from the earth's reference frame to the black car's reference frame, we can make "1" the earth, "2" the white car, and "3" the black car:

$$\overrightarrow{v_{white \leftarrow black}} = \overrightarrow{v_{white \leftarrow earth}} - \overrightarrow{v_{black \leftarrow earth}}$$
$$= (+25\,\text{m/s}) - (+25\,\text{m/s}) = 0$$

Staying in the black car's reference frame and looking at the truck, it appears to be moving very fast. From the bridge, the truck appeared to be moving 25 m/s, but since we are moving in the opposite direction of the truck, it appears to be moving much faster.

Interestingly, the earth itself and everything hooked to it appears to be moving in the reference frame of the black car. If you are sitting in the black car, you see houses, trees, and the cement barrier going past your window, even though they are motionless in the earth's reference frame.

Often, it will be easiest to think about situations in the earth's reference frame, but all of the laws of physics are still true in any reference frame that is not accelerating. This type of reference frame is often called an "inertial reference frame."

If you are riding in a train car and sipping from a cup of coffee, you understand the physics of taking a small careful sip, and you are able to do this as naturally on a train as you would if you were standing on solid ground. The laws of physics work just the same in the moving train as outside on solid ground. That is, unless the train is suddenly lurching out of the station, or applying the brakes, or going around a sharp curve. At those times, the inside of the train is a non-inertial reference frame, so the normal laws of physics do not apply inside the train car, and you spill your coffee.
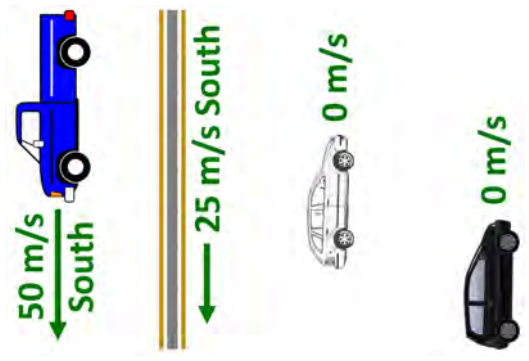


Figure 3.4: The black car's reference frame[9]
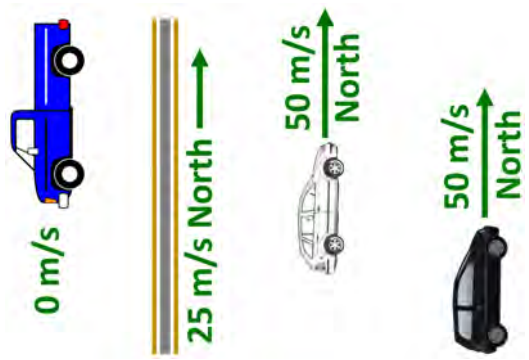


Figure 3.5: The truck's reference frame[9]

Similarly,

$$\overrightarrow{v_{truck \leftarrow black}} = \overrightarrow{v_{truck \leftarrow earth}} - \overrightarrow{v_{black \leftarrow earth}}$$
$$= -25\,\text{m/s} - (+25\,\text{m/s}) = -50\,\text{m/s}$$

What happens if we look at the *earth* from the black car's reference frame? Using Equation 3.1 where "3" is again the black car but this time the earth is both "1" and "2,"

$$\overrightarrow{v_{earth \leftarrow black}} = \overrightarrow{v_{earth \leftarrow earth}} - \overrightarrow{v_{black \leftarrow earth}}$$

...and recognizing that the velocity of the earth in the earth's reference frame is zero...

$$\overrightarrow{v_{earth \leftarrow black}} = 0 - (+25\,\text{m/s}) = -25\,\text{m/s}$$

The earth and everything solidly connected to it has a velocity of -25 m/s in the black car's reference frame!

Notice that $\overrightarrow{v_{earth \leftarrow black}}$ is the opposite of $\overrightarrow{v_{black \leftarrow earth}}$. This will be true for any pair of objects:

$$\boxed{\overrightarrow{v_{1 \leftarrow 2}} = -\overrightarrow{v_{2 \leftarrow 1}}} \tag{3.2}$$

## 3.2  An Ant Pushing a Rock

### Words

Figure 3.6 shows an ant pushing a rock. Let's imagine that at first neither the ant nor the rock is moving, but then the ant begins pushing and the rock and the ant both start moving to the right, accelerating together at a constant 0.5 m/s². The rock looks much larger than the ant, so we will say that the ant has a mass of 0.005 kg and the rock has a mass of 0.015 kg.

The ant is applying a force to the rock to make it accelerate. What are the other forces involved, and how do they compare with the force that the ant is applying to the rock? For simplicity, we will assume that the rock slides freely on the ground, and we will consider only the horizontal direction.

If a drawing is not given, it is good to make a sketch to help with understanding the situation clearly.

The possible forces in the horizontal direction are at each interface. So between the ant and the rock, between the ant and the ground, and between the rock and the ground.

At each interface there are two forces involved–for example, the ant pushes on the rock, and the rock pushes back on the ant. To some people, it may seem obvious that since the ant is the thing doing the pushing, it is applying the larger force. To others, it may seem obvious that since the rock is larger than the ant, it is applying the larger force. But in fact neither of these is correct!

### Graphics



Figure 3.6: An ant pushing a rock.[10]



Figure 3.7: A sketch of the ant, the rock, and the forces affecting them. [1]

### Numbers

**Assumptions:** $+\hat{x}$ is to the right; no friction between the rock and the ground; horizontal direction only

| Knowns | Unknowns |
|---|---|
| $m_{ant} = 0.005$ kg | $\overrightarrow{F_{ant \to rock}}$ |
| $m_{rock} = 0.015$ kg | $\overrightarrow{F_{ground \to ant}}$ |
| $\overrightarrow{a} = +0.5 \text{ m/s}^2\ \hat{x}$ | $\overrightarrow{F_{rock \to ant}}$ |
| $\overrightarrow{F_{ground \to rock}} = 0$ | |

Here, $\overrightarrow{F_{ant \to rock}}$ is the force applied by the ant on the rock, and similarly for the other subscripts.

Note that only one acceleration $\overrightarrow{a}$ is listed as an unknown, instead of listing $\overrightarrow{a_{ant}}$ and $\overrightarrow{a_{rock}}$ separately. Since the ant and the rock are in contact with each other the whole time, their motion will be identical, so they have the same acceleration. No subscript is needed.

Since we have the acceleration and masses of the objects and are looking for force, we can use Newton's Second Law:

$$\overrightarrow{F_{net}} = m \cdot \overrightarrow{a}$$

First, let's consider the rock because only one force is acting on it in the horizontal direction: the force from the ant. That force is just enough to accelerate the rock at 0.5 m/s²: 0.0075 N.

The ant has two forces acting on it, and we don't yet know either one, so it is easier instead to consider the ant and the rock to be a single system, and look at the forces affecting that system. Figure 3.7 shows that the only external horizontal force affecting the system is the frictional force of the ground pushing the ant. That force is just enough to accelerate the system at 0.5 m/s²: 0.01 N.

Now we can consider just the ant. The net force from the ground and the rock has to be just enough to accelerate the ant at 0.5 m/s². We already know the size of the force from the ground on the ant, so now we can find the force from the rock on the ant. Interestingly, it is exactly the same magnitude as the force of the ant on the rock, but in the opposite direction!

This is a general rule that will be true for all forces. Forces are interactions between two objects, and the force on one object is the same magnitude and opposite direction as the force on the second object. This is called Newton's Third Law.

When determining how forces affect an object, for example when finding an object's acceleration, its change in momentum, or its change in energy, only the external forces, those acting on the object from outside, should be considered.



Figure 3.8: FBD of rock, horizontal direction only[1]



Figure 3.9: FBD of ant & rock system, horizontal direction only[1]



Figure 3.10: FBD of ant, horizontal direction only[1]



Figure 3.11: Combined FBD[1]

It is possible to combine the FBD's of the ant and the rock. For each object, only the external forces, the arrows touching the object from the outside, affect the object's motion and momentum.

We have the mass and the acceleration of the rock, so we can find the net force on it. The only force on the rock is from the ant, so the net force is just $\overrightarrow{F_{ant \rightarrow rock}}$.

$$\overrightarrow{F_{ant \rightarrow rock}} = m_{rock} \cdot \overrightarrow{a} = +0.0075 \,\text{N}\, \hat{x}$$

We have the mass and the acceleration of the combined ant & rock system, so we can find the net force on it as well, and the only external force on the system is from the ground on the ant.

$$\overrightarrow{F_{ground \rightarrow ant}} = (m_{ant} + m_{rock}) \cdot \overrightarrow{a} = +0.01 \,\text{N}\, \hat{x}$$

We have the mass and the acceleration of the ant, so we can find the net force on it as well, and since we already know the force from the ground on the ant we can find the force of the rock on the ant.

$$\overrightarrow{F_{net}} = \overrightarrow{F_{ground \rightarrow ant}} + \overrightarrow{F_{rock \rightarrow ant}}$$

Rearranging gives…

$$\overrightarrow{F_{rock \rightarrow ant}} = \overrightarrow{F_{net}} - \overrightarrow{F_{ground \rightarrow ant}}$$
$$= \left(0.005 \,\text{kg} \cdot 0.5 \,\text{m/s}^2 - 0.01 \,\text{N}\right) \hat{x}$$
$$= -0.0075 \,\text{N}\, \hat{x}$$

We can see that the force of the ant on the rock is equal and opposite to the force of the rock on the ant. This will be true for any physical situation.

$$\boxed{\overrightarrow{F_{1 \rightarrow 2}} = -\overrightarrow{F_{2 \rightarrow 1}}} \qquad (3.3)$$

## 3.3  Kicking Horizontally

### Words

Figure 3.12 shows a soccer player about to kick a ball. The 450-gram ball was initially moving to the left at 5 m/s and after the kick it moves to the right at 15 m/s.

What can be said about what happened during the kick from just this information?

There is not really much information to work with here. The mass of the ball shouldn't change because of the kick. The only change described is the velocity of the ball.

Velocity was to the left, and after the kick velocity is to the right at a higher speed. So there has been a change in momentum, not only in magnitude but also in direction.

Imagine the kick in slow motion. The ball is moving left, then it comes into contact with a foot. The foot stops the motion to the left, removing all of the ball's initial momentum, and then gives the ball momentum to the right.

This happens quickly, so in a short time the foot created a large change in the ball's momentum. A change in momentum is often called an *impulse*, especially when the interaction occurs over a very small amount of time.

### Graphics



Figure 3.12: A soccer player kicking a ball.[11]
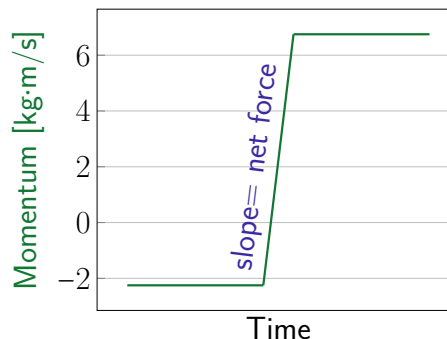


Figure 3.13: Momentum is constant before the kick, changes rapidly during the kick, and then is constant again after the kick.[1]

Since momentum is always conserved in an isolated system, something outside of the ball affected it.

### Numbers

**Assumptions:** $+\hat{x}$ is to the right; horizontal direction only

| Knowns | Unknowns |
|---|---|
| $m = 0.45$ kg | ??? |
| $v_{0x} = -5$ m/s | |
| $v_{f,x} = +15$ m/s | |

In this situation we are asked to find anything that we can, so there are no specific unknowns to look for.

Given mass and velocity, perhaps momentum would be a good place to start.

Since we have initial and final information about the ball, we can calculate its change in momentum, knowing that $\vec{p} = m \cdot \vec{v}$.

$$\begin{aligned}
\overrightarrow{\Delta p} &= \vec{p_f} - \vec{p_i} \\
&= m \cdot \vec{v_f} - m \cdot \vec{v_i} \\
&= m \cdot \overrightarrow{\Delta v} \\
&= 0.45\,\text{kg} \cdot (+15\,\text{m/s} - (-5\,\text{m/s}))\,\hat{x} \\
&= 9\,\text{kg} \cdot \text{m/s}\,\hat{x}
\end{aligned}$$

$\overrightarrow{\Delta p}$ is often called *impulse*.

Changing the momentum of the ball would have taken a force, so the foot applied a force to the ball, to the right. And since forces come in equal-and-opposite pairs, the ball must also have applied that same amount of force to the foot, but in the opposite direction.

This all happened very quickly, so the forces involved had to be very large but acting over a very small amount of time.

We also know something about the kinetic energy of the ball: After the kick it was moving more quickly than it was before the kick. So its kinetic energy increased. That means work must have been done on the ball.

The foot must have done work on the ball, and if the foot continued moving to the right the entire time, it was doing work for the entire time that it was in contact with the ball.

At the instant the ball came into contact with the foot, it was moving to the left, so it should be doing work on the foot, except that the foot was moving in the opposite direction. What actually happens during the collision is some of the energy gets stored as elastic, or spring, potential energy and some of the energy is transformed into thermal energy, or heat, warming up the ball. The ball's kinetic energy and some work from the foot is first transformed into thermal energy and elastic potential energy, and then more work from the foot and the elastic potential energy is transformed into more thermal energy and the final kinetic energy of the ball.
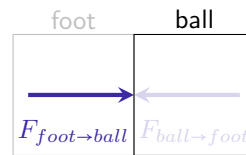


Figure 3.14: Combined FBD of the foot and ball during the kick, focusing on forces on the ball[1]
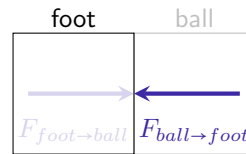


Figure 3.15: Combined FBD of the foot and ball, focusing on forces on the foot.[1]

Newton's Second Law tells us that...

$$\overrightarrow{F_{net}} = \frac{\overrightarrow{\Delta p}}{\Delta t}$$

...but we don't know $\Delta t$. We do, however, know that $\Delta t$ is very small, probably a few milliseconds, so the forces involved must be very large. Using Equation 3.3,

$$\overrightarrow{F_{foot \to ball}} = -\overrightarrow{F_{ball \to foot}}$$

...so there is also a very large force applied from the ball to the foot.

We don't know the amount of time that the force is applied, but we do know that the time that the ball was in contact with the foot is exactly equal to the time that the foot was in contact with the ball. We also know that the force on the foot from the ball was exactly equal to the force on the ball from the foot. Since we know that the times are the same and the forces are equal and opposite...

$$\overrightarrow{F_{foot \to ball}} \cdot \Delta t = -\overrightarrow{F_{ball \to foot}} \cdot \Delta t$$

...or...

$$\overrightarrow{\Delta p_{ball}} = -\overrightarrow{\Delta p_{foot}}$$
$$\overrightarrow{p_{ball,f}} - \overrightarrow{p_{ball,i}} = -\left(\overrightarrow{p_{foot,f}} - \overrightarrow{p_{foot,i}}\right)$$

...which can be rearranged to show that...

$$\boxed{\overrightarrow{p_{tot,i}} = \overrightarrow{p_{tot,f}}} \qquad (3.4)$$

This is true for any isolated system of objects, that is, a system that has no interactions with anything outside of the system.

## 3.4    Elastic Collision

### Words

One hard steel ball with a mass of 0.7 kg is sitting motionless when it is hit by an identical steel ball that is moving to the right at 3 m/s. A collision between steel balls is normally an *elastic* collision, which means that almost no kinetic energy is transformed to thermal energy in the collision. Assume that there are no external forces affecting the balls and the collision is perfectly elastic so no thermal energy is created.

What is the final velocity of each ball?

In any collision, the first thing to consider is momentum, since momentum is always conserved for any isolated system. And the forces between the objects during a collision are usually so large that all other forces can be neglected during the collision. So for practical purposes, momentum is conserved for any collision, whether there are external forces or not.

We are also assuming that the collision is perfectly elastic, so no kinetic energy is converted to thermal energy in this collision. During the collision the balls deform very slightly, storing elastic potential energy, and this elastic potential energy transforms into kinetic energy after the collision, making the final kinetic energy equal to the initial kinetic energy.

### Graphics
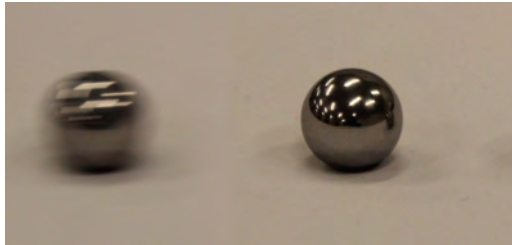


Figure 3.16: A moving ball about to hit a stationary ball[1]
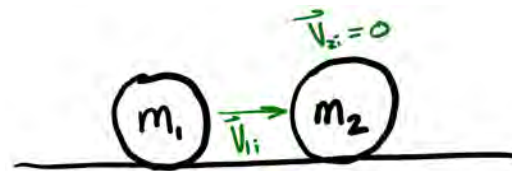


Figure 3.17: Sketch of balls before the collision[1]
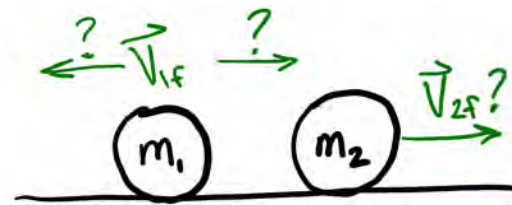


Figure 3.18: Sketch of balls after the collision[1]

### Numbers

**Assumptions:** $+\hat{x}$ is to the right; no external forces; perfectly elastic collision

| Knowns | Unknowns |
|---|---|
| $m_1 = m_2 = 0.7 \text{ kg}$ | $\overrightarrow{v_{1,f}}$ |
| $\overrightarrow{v_{1,i}} = +3 \text{ m/s} \, \hat{x}$ | $\overrightarrow{v_{2,f}}$ |
| $\overrightarrow{v_{2,i}} = 0$ | |
| $E_{k,i} = E_{k,f}$ | |

Equation 3.4 tells us that:

$$\overrightarrow{p_{1,i}} + \overrightarrow{p_{2,i}} = \overrightarrow{p_{1,f}} + \overrightarrow{p_{2,f}}$$
$$m_1 \cdot \overrightarrow{v_{1,i}} + m_2 \cdot \overrightarrow{v_{2,i}} = m_1 \cdot \overrightarrow{v_{1,f}} + m_2 \cdot \overrightarrow{v_{2,f}}$$

...but we don't know $\overrightarrow{v_{1,f}}$ or $\overrightarrow{v_{2,f}}$. We need a second equation.

Given that the collision is perfectly elastic...

$$E_{k,i} = E_{k,f}$$

...or...

$$\frac{1}{2}m_1 \cdot v_{1,i}^2 + \frac{1}{2}m_2 \cdot v_{2,i}^2 = \frac{1}{2}m_1 \cdot v_{1,f}^2 + \frac{1}{2}m_2 \cdot v_{2,f}^2$$

In this case, the ball on the right starts with no momentum, and the ball on the left has momentum. So the total momentum of the system is to the right. Whatever happens during the collision, the total momentum will still have to be to the right after the collision. That could mean both balls will end up going to the right, one ball stops and the other goes to the right, or one ball goes to the left and the other ball goes *faster* to the right.

Since there are so many options, in order to determine what actually happens in a collision like this, you have to crank through the numbers.

After going through the numbers, we find that there are three possibilities:

Possibility #1: Ball 1 stops completely and Ball 2 has a final velocity that is the same as the initial velocity of Ball 1.

Possibility #2: The velocities of both balls are the same as their initial velocities. This describes the situation if the first ball would have missed the second ball completely, so it is not the solution to what happens after a collision.

Possibility #3: Ball 1 stops completely and Ball 2 stays still. This cannot be correct, because this solution does not conserve momentum.

So the first possibility has to be the solution: Ball 1's final velocity is zero and Ball 2's final velocity is 3 m/s to the right.

Possibility #1:
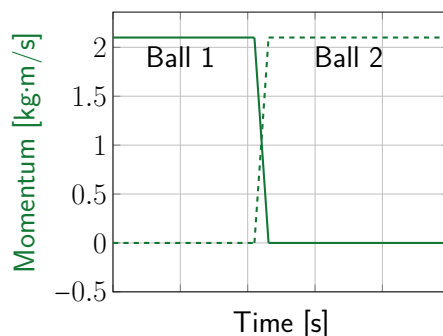


Figure 3.19: If the velocity of Ball 1 goes to zero, its momentum goes to zero. In order to conserve momentum for the system, the momentum of Ball 2 has to increase by the same amount that the momentum of Ball 1 decreased.[1]

Possibility #2:
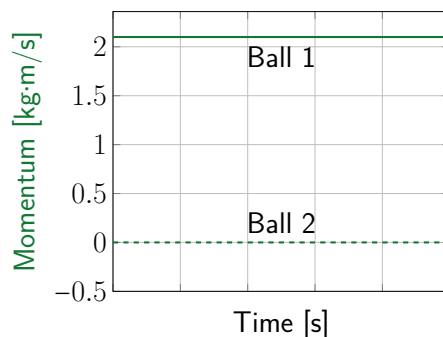


Figure 3.20: If the velocity of Ball 2 does not change, its momentum does not change. So the momentum of Ball 1 cannot change.[1]

We have two equations (one each for momentum and energy) and two unknowns, so we can solve for both velocities. We can simplify the equations, since $m_1 = m_2$ and $\overrightarrow{v_{2,i}} = 0$, giving...

$$\overrightarrow{v_{1,i}} = \overrightarrow{v_{1,f}} + \overrightarrow{v_{2,f}} \qquad (\text{from momentum})$$

and

$$v_{1,i}^2 = v_{1,f}^2 + v_{2,f}^2 \qquad (\text{from energy})$$

Since there is only motion in the $x$ direction, we can just consider the $\hat{x}$ part of the vectors:

$$v_{1,i,x} = v_{1,f,x} + v_{2,f,x}$$

$$v_{1,i,x}^2 = v_{1,f,x}^2 + v_{2,f,x}^2$$

Squaring both sides of the first equation...

$$v_{1,i,x}^2 = v_{1,f,x}^2 + 2 \cdot v_{1,f,x} \cdot v_{2,f,x} + v_{2,f,x}^2$$

Now we have two equations for $v_{1,i,x}^2$. Setting them equal to each other...

$$v_{1,f,x}^2 + v_{2,f,x}^2 = v_{1,f,x}^2 + 2 \cdot v_{1,f,x} \cdot v_{2,f,x} + v_{2,f,x}^2$$

...which can only be true if...

$$2 \cdot v_{1,f,x} \cdot v_{2,f,x} = 0$$

There are three possibilities:

| | | |
|---|---|---|
| Possibility #1 | $v_{1,f,x} = 0$ | $v_{2,f,x} \neq 0$ |
| Possibility #2 | $v_{1,f,x} \neq 0$ | $v_{2,f,x} = 0$ |
| Possibility #3 | $v_{1,f,x} = 0$ | $v_{2,f,x} = 0$ |

Which ones make physical sense?

## 3.5 Two Moving Balls

### Words

Two identical steel balls have an elastic collision with each other. Both have a mass of 0.7 kg. One is initially moving to the right at 1.5 m/s and the other is initially moving to the left at 1.5 m/s.

What is the final velocity of each ball?

In any collision, the first thing to consider is momentum. The total momentum will be conserved. In this case Ball 1 has momentum to the right and Ball 2 has the exact same amount of momentum but to the left. So the total momentum of the system is zero. That means that after the collision, the balls will also have equal and opposite momentum. And since they both have the same mass, they will also have equal and opposite velocity, so the same speed but opposite directions.

Since this is an elastic collision, we know that no thermal energy is created. That means initial and final kinetic energy must be equal in this situation. Initially, the speeds and the masses of the two balls are the same. Since the final speeds of the two balls are equal, the only way for the final kinetic energy to be the same as the initial kinetic energy is if the final speeds are the same as the initial speeds. Ball 1 ends up moving to the left at 1.5 m/s and Ball 2 ends up moving to the right at 1.5 m/s.

### Graphics



Figure 3.21: Two steel spheres that are about to collide.[1]



Figure 3.22: Sketch of balls before the collision[1]
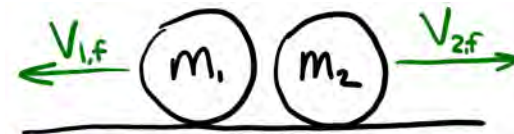


Figure 3.23: Sketch of balls after the collision[1]

### Numbers

**Assumptions:** $+\hat{x}$ is to the right; no external forces

| Knowns | Unknowns |
|---|---|
| $m_1 = m_2 = 0.7\,\text{kg}$ | $\overrightarrow{v_{1,f}}$ |
| $\overrightarrow{v_{1,i}} = +1.5\,\text{m/s}\,\hat{x}$ | $\overrightarrow{v_{2,f}}$ |
| $\overrightarrow{v_{2,i}} = -1.5\,\text{m/s}\,\hat{x}$ | |
| $E_{k,i} = E_{k,f}$ | |

As in Section 3.4, conservation of momentum tells us that…

$$\overrightarrow{p_{1,i}} + \overrightarrow{p_{2,i}} = \overrightarrow{p_{1,f}} + \overrightarrow{p_{2,f}}$$

and conservation of energy tells us that…

$$\frac{1}{2}m_1 \cdot v_{1,i}^2 + \frac{1}{2}m_2 \cdot v_{2,i}^2 = \frac{1}{2}m_1 \cdot v_{1,f}^2 + \frac{1}{2}m_2 \cdot v_{2,f}^2$$

But in this case $\overrightarrow{p_{1,i}} + \overrightarrow{p_{2,i}} = 0$, so…

$$\overrightarrow{p_{2,f}} = -\overrightarrow{p_{1,f}}$$

Since the masses are equal, $\overrightarrow{v_{2,f}} = -\overrightarrow{v_{1,f}}$. Since the speeds $v_{1,i} = v_{2,i}$ and $v_{1,f} = v_{2,f}$, the energy equation simplifies to:

$$v_{1,i} = v_{2,i} = v_{1,f} = v_{2,f}$$

So we are left only with determining the final directions of motion. This can be done by considering conservation of momentum.

Let's try looking at this question from a different reference frame, one that is moving 1.5 m/s to the left. In that reference frame, Ball 2 is not moving, and Ball 1 is moving at 3 m/s to the right. After the collision, Ball 1 is not moving in this reference frame, but Ball 2 is now moving to the right at 3 m/s.

This is exactly the same as the physical scenario in Section 3.4! So these two sections have presented exactly the same physical scenario, as seen from two different reference frames. Sometimes a clever choice of reference frame can simplify the analysis of a collision.

The graphs in Figures 3.24, 3.25, & 3.26 show the motion of the two balls over a very brief period of time just before, during, and just after the collision. Since the only force is when the two balls are colliding with each other, there is no acceleration except for sharp spikes during the collision itself.
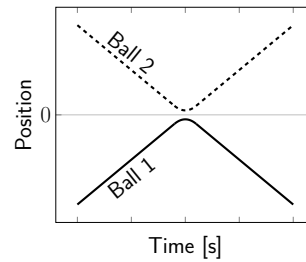


Figure 3.24: Positions in elastic collision[1]
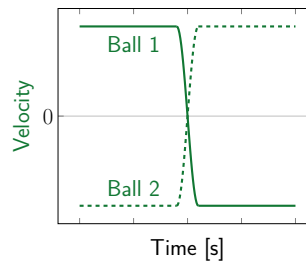


Figure 3.25: Velocities in elastic collision[1]
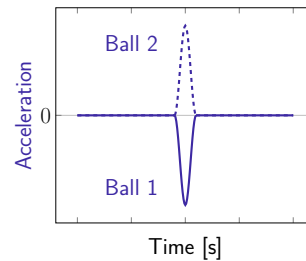


Figure 3.26: Accelerations in elastic collision [1]

65

## 3.6  Railway Couplers

### Words

When railway cars are assembled into a train, they are connected using railway couplers. This is an example of an inelastic collision, because the two train cars stick together after the collision that couples them.

Consider a 90,000 kg boxcar sitting motionless on a track in a railyard. A 500,000 kg engine backs into it at 0.2 m/s. The collision between the cars takes 0.08 seconds, after which they are locked together. We should be able to describe their velocity after the collision, the change in kinetic energy during the collision, the amount of force they applied to each other during the collision, and the acceleration of the engine and the boxcar during the collision.

In any collision, you can't go wrong by starting to think about it in terms of momentum. Initially the boxcar was not moving, so it had no momentum. The engine did have momentum. After the collision, the two train cars were stuck together, so they have the same velocity. Since momentum is conserved, there must be momentum after the collision. So after the collision both of the train cars have momentum in the same direction as the initial momentum. The engine transfers just enough of its momentum to the boxcar to bring them both to the same final velocity.

### Graphics



Figure 3.27: Two train cars connected by railway couplers.[13]
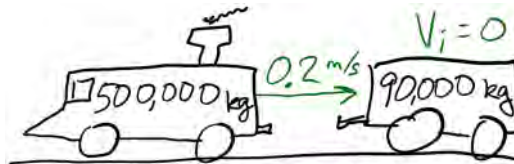


Figure 3.28: Sketch of the train cars before coupling.[1]

### Numbers

| Knowns | Unknowns |
|---|---|
| $m_{boxcar} = 90,000 \, \text{kg}$ | $\overrightarrow{F_{boxcar \rightarrow engine}}$ |
| $m_{engine} = 500,000 \, \text{kg}$ | $\overrightarrow{F_{engine \rightarrow boxcar}}$ |
| $\overrightarrow{v_{i,boxcar}} = 0$ | $\overrightarrow{v_f}$ |
| $\overrightarrow{v_{i,engine}} = 0.2 \, \text{m/s} \, \hat{x}$ | $\Delta E_k$ |
| $t_{collision} = 0.08 \, \text{s}$ | $\overrightarrow{a_{boxcar}}$ |
| | $\overrightarrow{a_{engine}}$ |

Since a direction is not given, it is easiest to set the problem up so that the initial velocity is in the positive direction. There is only one $\overrightarrow{v_f}$ because the train cars are stuck together after the collision.

We can start by finding the initial momentum of the system:

$$\overrightarrow{p_{i,tot}} = \overrightarrow{p_{i,engine}} + \overrightarrow{p_{i,boxcar}}^{0}$$
$$= m_{engine} \cdot \overrightarrow{v_{i,engine}}$$
$$= (500,000 \, \text{kg}) \cdot (0.2 \, \text{m/s} \, \hat{x})$$
$$= 100,000 \, \text{kg} \cdot \text{m/s} \, \hat{x}$$

Since momentum is conserved in a collision, we can set the final momentum equal to the initial momentum, and use that to find the final velocity of the stuck-together engine-boxcar system:

$$\overrightarrow{p_{i,tot}} = \overrightarrow{p_{f,tot}}$$
$$100,000 \, \text{kg} \cdot \text{m/s} \, \hat{x} = m_{tot} \cdot \overrightarrow{v_f}$$

No work is done on the engine-boxcar system by anything outside of that system, so the total energy of the system doesn't change.

We know that this is an inelastic collision, since the two railway cars stick together following the collision. That tells us that some of the kinetic energy was transformed into thermal energy during the collision as the couplings rubbed together and latched into place. Any inelastic collision results in the conversion of some kinetic energy into thermal energy.

Because of conservation of momentum, the less massive boxcar has a larger change in velocity than the more massive engine. And since this change in velocity occurs in the same amount of time, we also know that the acceleration of the boxcar is larger than the acceleration of the engine during the collision.

The force that the boxcar exerts on the engine would have the same magnitude as the force that the engine exerts on the boxcar, but these two forces would be in opposite directions.
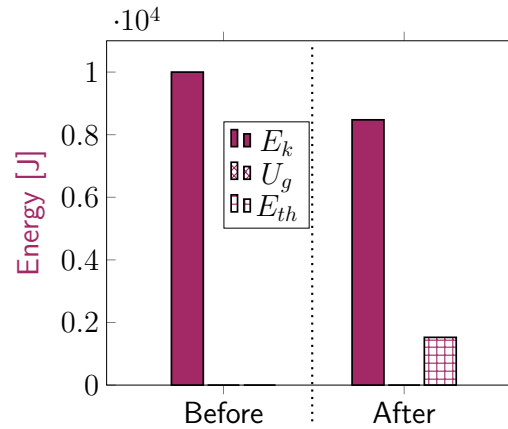
Figure 3.29: Energy bar graph for the collision. Initial thermal energy is taken to be zero. Gravitational energy is zero because both train cars are on the ground before and after the collision.[1]

Note that the total heights of the columns on the left is equal to the total heights of the columns on the right.
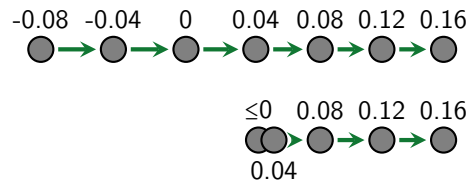
Figure 3.30: Motion map of the engine (top) and the boxcar (bottom). The labels are in seconds, and the collision is from 0 to 0.08 s.[1]

Note that the change in the length of the arrows is much larger for the boxcar than for the engine.

$$\overrightarrow{v_f} = \frac{100,000\,\text{kg}\cdot\text{m/s}\,\hat{x}}{590,000\,\text{kg}}$$
$$= 0.169\,\text{m/s}\,\hat{x}$$

Now we can find the change in kinetic energy.

$$\Delta E_k = E_{k,f,tot} - E_{k,i,tot}$$
$$= \frac{1}{2}m_{tot}\cdot v_f^2 - \frac{1}{2}m_{engine}\cdot v_{i,engine}^2$$
$$= (8,425\,\text{J}) - (10,000\,\text{J})$$
$$= -1\,575\,\text{J}$$

Conservation of energy tells us that this energy doesn't just disappear. It has to go somewhere. In an inelastic collision, it changes to thermal energy.

$$\Delta E_{th} = -\Delta E_k = 1\,575\,\text{J}$$

If we assume that the force is constant during the collision, then the acceleration would be…

$$\overrightarrow{a} = \frac{\overrightarrow{\Delta v}}{\Delta t}$$

This gives a larger acceleration for the boxcar than for for the engine: $\overrightarrow{a_{boxcar}} = +2.11\,\text{m/s}^2\,\hat{x}$ and $\overrightarrow{a_{engine}} = -0.388\,\text{m/s}^2\,\hat{x}$

## 3.7   Curling

### Words

The conversion of kinetic energy to thermal energy is not limited to collisions. It happens any time that two surfaces rub together. Take, for example, the Olympic sport of curling. One person pushes a heavy "stone" on a sheet of ice and then releases it, and the rest of the curling team sweeps the area in front of the stone as it slides, trying to control the friction to get the stone to stop in a target area.

When it leaves the curler's hand, the stone has momentum and kinetic energy, but the force of friction opposes the momentum, doing negative work on the stone and slowing it until it eventually comes to rest. After the stone has stopped moving, all of the kinetic energy is gone, changed into thermal energy.

Let's consider an 18 kg curling stone that was initially released with a momentum of $60 \, \text{kg} \cdot \text{m/s}$, and stops after traveling 45 m across the ice. We should be able to describe the velocity of the stone for the whole time it is sliding, the force of friction on the stone, the energy conversions that take place as it slides across the ice, and the amount of time that the stone is in motion after it has been released.

### Graphics



Figure 3.31: Top: A curler preparing to push a stone. Bottom: Stones in the target area.[14]



Figure 3.32: A sketch of the curling stone before and after sliding across the ice.[1]

### Numbers

Many of our mathematical models are only valid if the force is constant. Sometimes, like for this situation, the assumption is valid; other times, like in a collision, it is not. Even when the force is not constant, versions of these mathematical models are useful for finding average values. For example:

$$\overrightarrow{F_{net,avg}} = \frac{\overrightarrow{\Delta p}}{\Delta t} \qquad (3.5)$$

and

$$\overrightarrow{a_{avg}} = \frac{\overrightarrow{\Delta v}}{\Delta t} \qquad (3.6)$$

…where $\overrightarrow{F_{net,avg}}$ and $\overrightarrow{a_{avg}}$ are the average values over time. One of our other mathematical models is true for the average force over a distance:

$$\boxed{W_{net} = F_{net,avg} \cdot \Delta x \cdot \cos(\theta)} \qquad (3.7)$$

**Assumptions:** $+\hat{x}$ is to the right

| Knowns | Unknowns |
|---|---|
| $m = 18 \, \text{kg}$ | $\overrightarrow{v}$ |
| $\overrightarrow{p_i} = 60 \, \text{kg} \cdot \text{m/s} \, \hat{x}$ | $\overrightarrow{F_f}$ |
| $\overrightarrow{p_f} = 0$ | Energy transformations |
| $\overrightarrow{\Delta x} = 45 \, \text{m} \, \hat{x}$ | $\Delta t$ |
| $\overrightarrow{v_f} = 0$ | |

Since specific directions aren't given, we can choose a direction that is easy to draw or easy to think about. Since "to the right" is conventionally taken to be the positive direction, for simplicity Figure 3.32 is set up so that the initial momentum is to the right. Given that, we know that the initial velocity is also to the right, or positive. And since the stone stops at the end, its final velocity is zero.

The force of friction always opposes the relative motion of two objects, so the force on the stone is to the left, or negative.

The amount of time that the stone spends sliding across the ice depends on the force of friction and the initial momentum. The larger the force, the less time it will take to stop; and the larger the initial momentum, the more time it will take to stop.

Note that the question doesn't ask about just the initial or final velocity. It says we should be able to describe the velocity for the whole time. So what about the time during which it is sliding? The velocity will always be to the right, but dropping continuously while the stone slides across the ice, as is shown in Figure 3.34. We could speak of an average velocity of the stone during this time, which would be to the right, and if the force is constant, the average velocity would be half of the initial velocity.
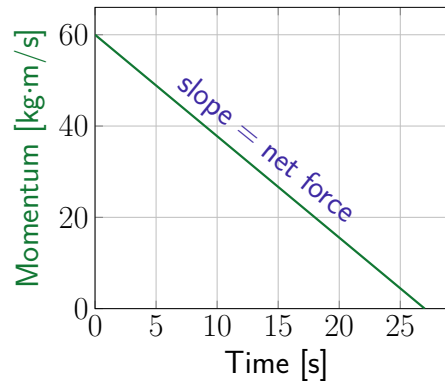


Figure 3.33: Momentum is decreasing over time, indicating a negative net force.[1]
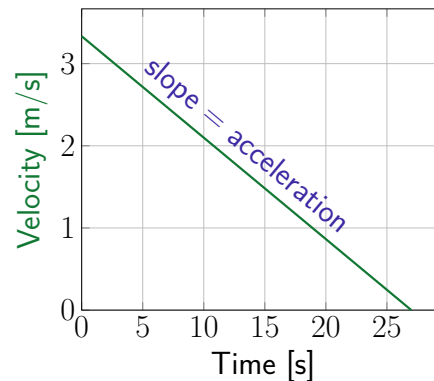


Figure 3.34: Velocity is decreasing over time, indicating a negative acceleration.[1]

It is relatively easy to find the initial velocity, and from it the initial kinetic energy:

$$\overrightarrow{p_i} = m \cdot \overrightarrow{v_i}$$

$$\overrightarrow{v_i} = \frac{\overrightarrow{p_i}}{m} = \frac{60 \, \text{kg} \cdot \text{m/s}}{18 \, \text{kg}} = 3.33 \, \text{m/s} \, \hat{x}$$

$$E_{k,i} = \frac{1}{2} m \cdot v_i^2 = \frac{1}{2} (18 \, \text{kg}) (3.33 \, \text{m/s})^2 = 100 \, \text{J}$$

All of the initial kinetic energy is converted to thermal energy, so $\Delta E_k = -100$ J and $\Delta E_{th} = 100$ J. Now we can use the Work-Energy Theorem and 3.7 to find the average force of friction.

$$\Delta E_k = W_{net} = F_f \cdot \Delta x \cdot \cos(180°)$$

$$F_f = \frac{\Delta E_k}{-\Delta x} = \frac{-100 \, \text{J}}{-45 \, \text{m}} = 2.22 \, \text{N}$$

180° was used as the angle because we assume that the force is opposite the direction of the displacement. Since we got a positive value for the force, we know that our assumption was correct. If the number turned out to be negative then we would know that our assumption was not correct. The velocity varies continuously over the time the stone is moving. We have already found the initial and final velocity. Another useful piece of information is the average velocity, which is given by...

$$\boxed{\overrightarrow{v_{avg}} = \frac{\overrightarrow{\Delta x}}{\Delta t}} \tag{3.8}$$

In situations where the net force (and therefore the acceleration) is constant, average velocity is also given by...

$$\boxed{\overrightarrow{v_{avg}} = \frac{\overrightarrow{v_i} + \overrightarrow{v_f}}{2}} \tag{3.9}$$

## 3.8  Car crash, initial impact

### Words

Let's consider the car crash in Figure 3.1. We can imagine what happened. Perhaps the 2500 kg truck was stopped, and the 1500 kg car hit it from behind at 25 m/s. The truck would have been shoved forward by the car, and the two stuck together. The time in which the car and truck were smashing into each other would have been very short, let's say 30 milliseconds. The drivers of both vehicles probably were applying the brakes, and the vehicles skidded together to a stop after traveling together for 4 m. For now let's focus on what happened during the early part of the crash when they were smashing into each other, before the vehicles skidded together.

We should be able to find the average force that each vehicle applied to the other during the early part of the crash, the velocity of the car-truck wreckage just after they had finished smashing into each other but before they started skidding together, and the kinetic energy of the vehicles just before and just after they smashed into each other.

Before the car hit the truck, the truck was not moving but the car was; a short time after contact the car and the truck were connected together and moved as if they were a single object with the combined mass of both vehicles. Because of this, there is only one velocity after the collision, and it is the velocity of both the car and the truck.

### Graphics
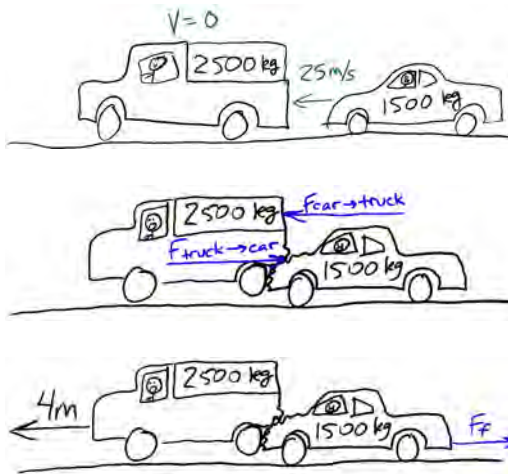


Figure 3.35: Sketches of what could have happened in the crash. In this section we will consider only the top two sketches.[1]

### Numbers

**Assumptions:** $+\hat{x}$ is to the right; external forces are negligible

| Knowns | Unknowns |
|---|---|
| $m_{car} = 1500 \text{ kg}$ | $\overrightarrow{F_{car \to truck}}$ |
| $m_{truck} = 2500 \text{ kg}$ | $\overrightarrow{F_{truck \to car}}$ |
| $\overrightarrow{v_{i,car}} = -25 \text{ m/s } \hat{x}$ | $\overrightarrow{v_f}$ |
| $\overrightarrow{v_{i,truck}} = 0$ | $E_{k,i}$ |
| $t_{collision} = 0.03 \text{ s}$ | $E_{k,f}$ |

The place to start in any collision is with conservation of momentum, which we can use to find the final velocity of the car and truck:

$$\overrightarrow{p_{i,tot}} = \overrightarrow{p_{f,tot}}$$
$$\overrightarrow{p_{i,car}} = \overrightarrow{p_{f,tot}}$$
$$m_{car} \cdot \overrightarrow{v_{i,car}} = m_{tot} \cdot \overrightarrow{v_f}$$

$$\overrightarrow{v_f} = \left( \frac{m_{car}}{m_{tot}} \right) \overrightarrow{v_{i,car}} = -9.375 \text{ m/s } \hat{x}$$

This is the velocity of the two vehicles after they have smashed together and before they start sliding together down the road.

In the crash, the car transferred some of its momentum to the truck, applying a force to the truck. So the truck applied an equal-but-opposite force to the car. Since momentum is conserved in an isolated system, the change in the truck's momentum is the same as the change in the car's momentum, but in the opposite direction.

This tells us something about the acceleration. They have the same change in momentum, but different masses. The more massive truck has a smaller change in velocity than the car, so the acceleration of the truck is less than the acceleration of the car.

Initially the car had kinetic energy, but the truck did not; after the collision they both have some kinetic energy. But in the collision the two vehicles stuck together, a completely inelastic collision, which means that some of the kinetic energy was transformed into thermal energy as the car and truck were deforming.



Figure 3.36: Free body diagram (FBD) of the truck during the collision, horizontal direction only. The force of friction is small compared to the force caused by the collision, so it is being neglected.[1]
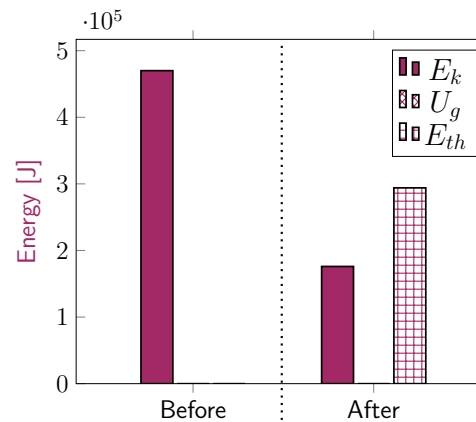


Figure 3.37: Energy bar graph for the collision. Initial thermal energy is taken to be zero. Gravitational energy is zero because both vehicles are on the ground before and after the collision.[1]

Note that the total heights of the columns on the left is equal to the total heights of the columns on the right.

We can use Newton's Second Law to find the force on the truck during the collision:

$$\overrightarrow{F_{car \to truck}} = \frac{\overrightarrow{\Delta p_{truck}}}{\Delta t} = \frac{m_{truck} \cdot (\overrightarrow{v_f} - \overrightarrow{v_{i,truck}})}{t_{collision}}$$
$$= \frac{2500 \text{ kg} \cdot (-9.375 - 0) \text{ m/s } \hat{x}}{0.03 \text{ s}}$$
$$= -7.8 \times 10^5 \text{ N } \hat{x}$$

From Equation 3.3, we know that

$$\overrightarrow{F_{truck \to car}} = -\overrightarrow{F_{car \to truck}} = +7.8 \times 10^5 \text{ N } \hat{x}$$

We already know all of the velocities and masses, so we can find the change in kinetic energy.

$$E_{k,i} = \frac{1}{2} m_{car} \cdot v_{i,car}^2 + \frac{1}{2} m_{truck} \cdot v_{i,truck}^2$$
$$= \frac{1}{2}(1500 \text{ kg}) \cdot (-25 \text{ m/s})^2 + 0 = 4.7 \times 10^5 \text{ J}$$

$$E_{k,f} = \frac{1}{2} m_{total} \cdot v_f^2$$
$$= \frac{1}{2}(4000 \text{ kg}) \cdot (-9.375 \text{ m/s})^2 = 1.76 \times 10^5 \text{ J}$$

Energy is conserved whenever no work is being done by an external force. Gravity and springs do not play a role in this situation, so lost kinetic energy changes into thermal energy.

$$E_{k,i} + E_{th,i} = E_{k,f} + E_{th,f}$$
$$E_{k,i} - E_{k,f} = E_{th,f} - E_{th,i}$$

$$\Delta E_{th} = (4.7 - 1.76) \times 10^5 \text{ J} = 2.94 \times 10^5 \text{ J}$$

## 3.9 Car crash, sliding

### Words

Now we consider the same collision as in Section 3.8, but focus on what happened when the vehicles skidded together across the ground. During that time, the vehicles were braking and they skidded to a stop after traveling for 4 m.

We should be able to describe the total braking force that was required, which is a frictional force that we will assume to be constant, acting between the tires and the road; the amount of time that they were skidding; and the energy transformations.

It is usually best to start any analysis with whatever part seems easiest. In this case, it is easiest to think about the energy involved. We know the vehicles are moving when they start to skid down the road, so they have kinetic energy. The road is flat, so we don't have to worry about any changes in gravitational potential energy. And there are not any springs that are trying to stop them or make them go faster, so we also don't need to worry about spring potential energy. So we started with only kinetic energy. And after the vehicles stop moving, they no longer have kinetic energy. So where did all of the energy go? The only option left is thermal since we have ruled out potential energies. All of the kinetic energy is transformed into thermal energy.

### Graphics



Figure 3.38: Sketch of the car and truck skidding together after the initial impact.[1]
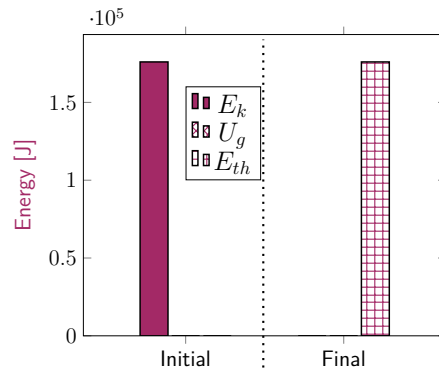


Figure 3.39: Energy bar graph for the vehicles skidding on the road. Initial thermal energy is taken to be zero. Gravitational energy is zero because both vehicles are on the ground before and after the collision.[1]

### Numbers

**Assumptions:** $+\hat{x}$ is to the right; frictional force is constant Note that the initial velocity for the

| Knowns | Unknowns |
|---|---|
| $m_{wreckage} = 4000\ \text{kg}$ | $\overrightarrow{F_f}$ |
| $\overrightarrow{v_i} = -9.375\ \text{m/s}\ \hat{x}$ | $t_{skidding}$ |
| $\overrightarrow{v_f} = 0$ | $\Delta E_{th}$ |
| $\overrightarrow{\Delta x} = -4\ \text{m}\ \hat{x}$ | $\Delta E_k$ |

skidding part of the collision is the final velocity of impact part of the collision.

We already know from the last section that $E_{k,i}$ for the skidding portion of the collision is $1.76 \times 10^5\,\text{J}$. And since $\overrightarrow{v_f} = 0$, $E_{k,f} = 0$. Using the Work-Energy Theorem, we can find the work done by friction on the vehicles:

$$W = E_{tot,f} - E_{tot,i} = -1.76 \times 10^5\,\text{J}$$

Work done by friction is not like work done by gravity. If gravity does negative work on an object, gravitational potential energy is stored. If friction does negative work on something that is sliding, it creates thermal energy. So in this case, $\Delta E_{th} = 1.76 \times 10^5\,\text{J}$

As the tires skid across the ground, the frictional force does negative work on the vehicles, slowing them down. Whenever two surfaces slide against each other with friction, the friction fights the relative motion of the two surfaces, converting kinetic energy into thermal energy.

This only happens when the surfaces move relative to each other. If a car is sitting on the side of a hill with the brakes locked, there is still friction between the tires and the road, but there is no relative motion, so no work is being done and kinetic energy is not being transformed by friction into thermal energy.

We can find the time needed to stop the vehicles by considering momentum. The car-truck wreckage has momentum after the initial impact, and then the force of friction reduces the momentum to zero over a certain amount of time.

The larger the force of friction, the less time needed to reduce the momentum to zero.

We could also consider the question of the force between the car and the truck during the time they were skidding. It would depend on how much of the frictional force is coming from the truck's tires and how much from the car's tires. If the car were not braking at all then there would be a large force applied on the car from the truck during the time they were skidding. If they were both braking then the force between them would be smaller.

$\cdot 10^4$

Area = Work
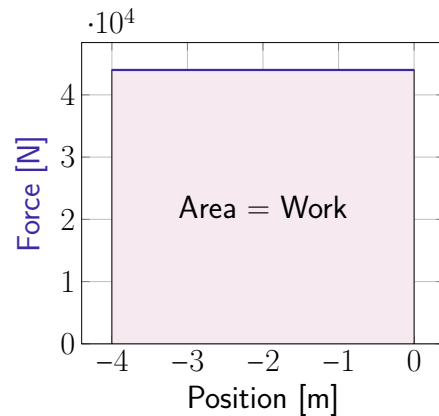
Figure 3.40: A force opposite the direction of motion does negative work equal to the area under the curve in a Force-vs-Position graph.[1]

$\cdot 10^4$

slope = net force
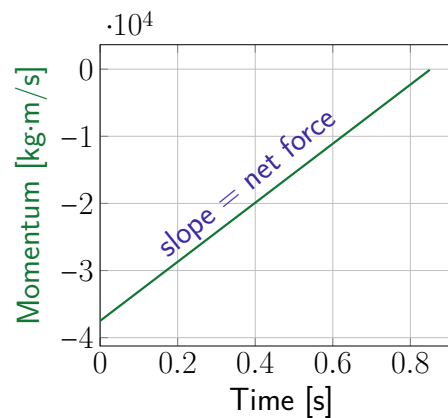
Figure 3.41: Momentum starts with a large negative value, and goes to zero over time because of the force of friction.[1]

We can find the force of friction from the work done by friction:

$$W_{net} = F_{net} \cdot \Delta x \cdot \cos\theta$$

$$F_{net} = \frac{W_{net}}{\Delta x \cdot \cos\theta} = \frac{-1.76 \times 10^5 \text{ J}}{(4 \text{ m}) \cdot (-1)} = 4.4 \times 10^4 \text{ N}$$

The force is positive, so it is in the positive direction, opposite the direction of motion. Since friction is the only force in the horizontal direction, $F_{net,x} = F_f$.

We can use Newton's Second Law to find the time needed to stop.

$$\overrightarrow{F_{net}} = \frac{\overrightarrow{\Delta p}}{\Delta t} = \frac{\overrightarrow{p_f} - \overrightarrow{p_i}}{t_{skidding}}$$

$$t_{skidding} = \frac{m_{wreckage} \cdot v_{f,x} - m_{wreckage} \cdot v_{i,x}}{F_f}$$

$$= \frac{4000 \text{ kg} \cdot 0 - 4000 \text{ kg} \cdot (-9.375 \text{ m/s})}{4.4 \times 10^4 \text{ N}}$$

$$= 0.85 \text{ s}$$

73

## 3.10  Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

### General

- Physical situations can be analyzed from different reference frames.

- The laws of physics are true in any inertial (not accelerating) reference frame.

- It is often useful to consider the physics of a system of objects, not just the physics related to a single object.

- When you make an assumption about the direction of a vector quantity and then you calculate a positive magnitude, your assumption about direction was correct.

- When you make an assumption about the direction of a vector quantity and then you calculate a negative magnitude, your assumption about direction was not correct.

### Forces

- Forces are interactions between two objects, and the force on one object is the same magnitude and opposite direction as the force on the second object.

- A free body diagram of an object includes only external forces acting *on* the object, never forces *caused by* the object.

- During a collision, the forces caused by the collision are usually so much larger than any other forces that all other forces can be neglected.

- During a collision, forces change rapidly, so usually average forces during a collision are calculated.

- The force of friction opposes the relative motion of two objects.

- The force of friction converts kinetic energy into thermal energy when two surfaces slide against each other.

### Motion

- Objects, even stationary objects, change velocity when we change to a different reference frame.

- In a collision between two objects with different masses, the more massive object experiences less acceleration than the less massive object.

### Momentum

- Change in momentum is often called "impulse."

- Considering momentum is a good way to begin any investigation of a collision.

### Energy

- An elastic collision is one in which little kinetic energy is transformed to thermal energy.

- A perfectly elastic collision is one in which no kinetic energy is transformed to thermal energy.

- An inelastic collision is one in which a large amount of kinetic energy is transformed to thermal energy.

- Completely inelastic collisions are collisions in which the two objects stick together after the collision.

- If the force is opposite the direction of motion in a force-vs-displacement graph, the area "under" the curve represents negative work. In this case either the displacement or the force is shown as a negative number on the graph.
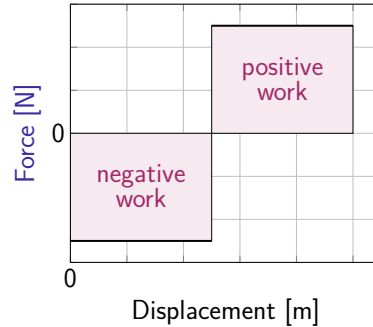


Negative and positive work on a force vs displacement graph[1]

## Mathematical Models

| equation | | restrictions on the validity of the equation |
|---|---|---|
| $\overrightarrow{v_{2\leftarrow3}} = \overrightarrow{v_{2\leftarrow1}} - \overrightarrow{v_{3\leftarrow1}}$ | (3.1) | -none- |
| $\overrightarrow{v_{1\leftarrow2}} = -\overrightarrow{v_{2\leftarrow1}}$ | (3.2) | -none- |
| $\overrightarrow{F_{1\rightarrow2}} = -\overrightarrow{F_{2\rightarrow1}}$ <br><br> "Newton's Third Law" | (3.3) | -none- |
| $\overrightarrow{p_{tot,i}} = \overrightarrow{p_{tot,f}}$ | (3.4) | only valid when there is no external net force |
| $\overrightarrow{F_{net,avg}} = \frac{\overrightarrow{\Delta p}}{\Delta t}$ | (3.5) | -none- |
| $\overrightarrow{a_{avg}} = \frac{\overrightarrow{\Delta v}}{\Delta t}$ | (3.6) | -none- |
| $W_{net} = F_{net,avg} \cdot \Delta x \cdot \cos(\theta)$ | (3.7) | -none- |
| $\overrightarrow{v_{avg}} = \frac{\overrightarrow{\Delta x}}{\Delta t}$ | (3.8) | -none- |
| $\overrightarrow{v_{avg}} = \frac{\overrightarrow{v_i} + \overrightarrow{v_f}}{2}$ | (3.9) | only valid when the net force is constant |

## 3.11 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

### Level 1 - Remember

3.1 **[W]** What does "impulse" mean in physics?

3.2 **[W]** What type of energy conversion always happens during an inelastic collision?

3.3 **[W]** Differentiate between what is meant by elastic, inelastic, and completely inelastic collisions.

3.4 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

### Level 2 - Understand

3.5 **[W]** Which of the following describe inertial reference frames where the normal laws of physics are valid?

   (a) A typical physics classroom

   (b) A physics classroom during a violent earthquake

   (c) The back of a truck that is traveling at constant speed on a long, straight road

   (d) The back of a truck that is traveling at constant speed on a long, straight, uphill road

   (e) The back of a truck that is accelerating away from a stoplight

   (f) A spaceship floating far off in space, away from any sources of gravity

   (g) A spaceship firing its rockets far off in space, away from any sources of gravity

3.6 **[N]** Confirm that Equation 3.8 and Equation 3.9 give the same result for the physical scenario described in Section 3.7. Show your work.

3.7 **[W & N]** In Section 3.8 we found the horizontal force on the car and the truck during the impact.

   (a) Is the magnitude of the force the same for both the car and the truck? Explain why or why not.

   (b) Find the accelerations of the car and the truck during the impact.

   (c) Is the magnitude of the acceleration the same for both the car and the truck? Explain why or why not.

3.8 **[W]** Explain why the initial velocity for the physical scenario that is considered in Section 3.9 is the same as the final velocity for the physical scenario that is considered in Section 3.8.

3.9 **[W & N]** Find the acceleration of the car and the truck from Section 3.9 during the skidding. Is the acceleration of the car the same as the acceleration of the truck? Why or why not?

## Level 3 - Apply

3.10 **[N]** There are six "Unknowns" listed at the beginning of Section 3.1, but numerical values are only found for two of them. Find the rest of the numerical values.

3.11 **[G]** Section 3.1 includes illustrations in the reference frames of the earth, the black car, and the truck. Draw a similar illustration in the white car's reference frame.

3.12 **[W & G]** Draw an energy bar graph for the soccer ball in Section 3.3 for the time just before and just after the kick. Is energy conserved in this situation? If not, where did the extra energy come from, or where did it go?

3.13 **[N]** In Section 3.6 two forces were listed in the unknowns, but they were never found. What are those forces, including magnitude and direction?

3.14 **[N]** In Section 3.7 one of the unknowns was time, but it was never explicitly found, though it can be seen in some of the graphs. Exactly how much time does the stone spend sliding across the ice?

3.15 **[G]** Section 3.8 includes a free body diagram for the truck during the collision, but ignoring the force of friction.

   (a) Re-draw that free body diagram for the forces on the truck in the horizontal direction only, including the force of friction. Assume that the driver of the truck was applying the brakes during the time of the impact.

   (b) Does including the force of friction result in a larger or smaller magnitude of the net force on the truck in the horizontal direction?

   (c) Draw another free body diagram for the forces on the *car* in the horizontal direction only, including the force of friction. Assume that the driver of the car was applying the brakes during the time of the impact.

   (d) Does including the force of friction result in a larger or smaller magnitude of the net force on the car in the horizontal direction?

## Level 4 - Analyze

3.16 **[W & N]** In Section 3.7 it is mentioned that some of the team members use brooms to change the force of friction as the stone slides across the ice. If they were able to reduce the force of friction by 10%, what effect would that have on the displacement and the time? Would they increase or decrease? Would they also change by 10%, or more, or less? Explain your answers.

## Level 5 - Evaluate

3.17 **[W]** In the situation given in Section 3.2 it is stated that the ant and rock start motionless. Now imagine what would happen if the forces applied stayed the same but the ant and rock started with an initial velocity in one direction or the other.

   (a) Would the initial velocity affect the acceleration? Explain why or why not.

   (b) Would the initial velocity affect the force applied by the ant to the rock or the rock to the ant? Explain why or why not.

3.18 **[W]** Two people are moving toward each other across an open field. One is running and the other is walking. Rank the following in terms of the amount of time needed for the two people to come together:

   (a) Considering this situation in the earth's reference frame

(b) Considering this situation in the reference frame of the person who is running

(c) Considering this situation in the reference frame of the person who is walking

*Ignore the effects of "special relativity," if you know what that is!*

3.19 **[N]** An assertion is made in this chapter that during a collision the forces caused by the impact are usually so much larger than any other forces that all other forces can be neglected. Compare the force of the impact in Section 3.8 to the force of friction in Section 3.9 for the same collision. How many times larger is the force of impact than the force of friction with the road? Probably both cars were braking even during the impact. Does ignoring the force of friction cause a significant (which we will define for this question as larger than 10%) error in the calculation of the force in the impact?

3.20 **[W, G, & N]** At the end of Section 3.9 it is noted that the force between the car and the truck during the time that they are skidding depends on the relative amount of frictional force applied by the tires of each vehicle.

(a) Find the amount of force applied by the truck on the car while they are skidding if the car is not braking at all and all $4.4 \times 10^4\,\mathrm{N}$ of frictional force comes from the truck's tires.

(b) Find the amount of frictional force that would need to be applied by the car's tires and by the truck's tires so that their total frictional force would still be $4.4 \times 10^4\,\mathrm{N}$ and there would be no net force applied by the truck on the car while they are skidding.

## Level 6 - Create

3.21 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Add the main ideas from this chapter to the concept map that you began for the question at the end of Chapter 1.

3.22 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

3.23 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

# Chapter 4

# Working in Two Dimensions

So far, we have only considered objects and forces in one dimension, either horizontally or vertically. But our universe is not one-dimensional. Usually we think of it as being three-dimensional, and that is correct. We live in three spatial dimensions. It's also possible to consider time as another "dimension," in which case we can think of living in a four-dimensional "space-time." And in some theoretical models of physics called "string theories" there have been attempts made to describe the universe as being 10-, 11-, or even 26-dimensional! For now, let's just work on expanding our understanding from one dimension to two.

At first we will consider a flat plane that is lying flat on the ground, so North and South, East and West; or left and right, forward and backward. Then we will consider vertical planes, so up and down, left and right. Finally, we will consider planes that are tilted. In all cases, it is important to remember that for our analyses, the two dimensions have to be at right angles to each other. That allows us to consider each dimension separately from the other.

Figure 4.1: A compass rose, showing the traditional direction of North as "up." [15]

For example, if your aunt's home is West of yours, you have to get there by traveling West[a]and not North or South. In fact, if you do go North, you will then have to undo that motion by going South. North and South have to be considered as completely separate from East and West.

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[a]You could also get there by going really far East, since the earth is spherical–but we will imagine everything as being flat for now. Curved space gets complicated.

## 4.1 Floating on the Water

### Words

Imagine sitting on an inner tube, just floating in the water with a group of friends. If the river isn't moving and all of you are just lazily floating in the water, you would say that none of you is moving. And if a person were standing on a nearby shore, that person would also say that you aren't moving.

These ideas should sound very familiar, because it is very similar to our discussion of reference frames in an earlier chapter. The only difference now is that some of the objects we are considering (the inner tubes, you, and your friends) are actually riding on one of the objects (the water). But all of the ideas are the same.

But what if the water were moving, and carrying you and your friends along? If a river is flowing North at 2 m/s, and you and your friends are just floating along with the river, you will also go North at 2 m/s. The person standing on the shore would be able to watch you going North along with the river.

### Graphics

Figure 4.2: A group of people floating in inner tubes on a river. If they don't try to push themselves through the water by paddling, they will all move together, along with the river.[16]

### Numbers

**Knowns**

$\overrightarrow{v_{river \leftarrow shore}} = 0$ or $2\,\text{m/s}\,\hat{\text{N}}$

$\overrightarrow{v_{you \leftarrow river}} = 0$

$\overrightarrow{v_{friend \leftarrow river}} = 0$

**Unknowns**

$\overrightarrow{v_{you \leftarrow shore}}$

$\overrightarrow{v_{friend \leftarrow you}}$

$\overrightarrow{v_{shore \leftarrow you}}$

Our equations for motion of one object relative to another work here, though we have never tried to use them where one object is riding on another. Taking objects 1, 2, and 3 to be the river, you, and the shore, respectively…

$$\overrightarrow{v_{you \leftarrow shore}} = \overrightarrow{v_{you \leftarrow river}} - \overrightarrow{v_{shore \leftarrow river}}$$

We aren't given $\overrightarrow{v_{shore \leftarrow river}}$, but we are given $\overrightarrow{v_{river \leftarrow shore}}$, and we have already learned that…

$$\overrightarrow{v_{shore \leftarrow river}} = -\overrightarrow{v_{river \leftarrow shore}}$$

…So…

$$\overrightarrow{v_{you \leftarrow shore}} = \overrightarrow{v_{you \leftarrow river}} + \overrightarrow{v_{river \leftarrow shore}}$$

If $\overrightarrow{v_{river \leftarrow shore}} = 0$ then $\overrightarrow{v_{you \leftarrow shore}} = 0$, and if $\overrightarrow{v_{river \leftarrow shore}} = 2\,\text{m/s}\,\hat{\text{N}}$ then $\overrightarrow{v_{you \leftarrow shore}} = 2\,\text{m/s}\,\hat{\text{N}}$.

Now what if you and your friend were deep in conversation, not paying any attention to anything except each other? If you didn't happen to notice that the shore and the shoreline was moving past you, it would feel to you exactly like you and your friend were sitting still, and meanwhile the shore is slipping by at 2 m/s to the South.
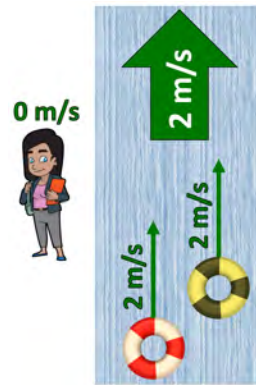


Figure 4.3: From the frame of reference of a person standing on the shore, you and your friend are moving along with the river.[17]
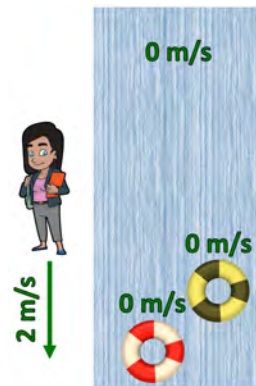


Figure 4.4: From your frame of reference, you and your friend are not moving, and a person standing on the shore is moving South.[17]

Interestingly, $\overrightarrow{v_{friend \leftarrow you}}$ does not depend at all on the velocity of the river. We can again use our equations for the motion of one object as seen from the frame of another, this time taking objects 1, 2, and 3 to be the river, your friend, and you, respectively.

$$\overrightarrow{v_{friend \leftarrow you}} = \overrightarrow{v_{friend \leftarrow river}} - \overrightarrow{v_{you \leftarrow river}}$$

$\overrightarrow{v_{river \leftarrow shore}}$ does not appear in this expression at all. Regardless of the velocity of the river relative to the shore, the velocity of your friend relative to you will always be zero in this scenario.

The same technique can also tell us the velocity of the shore in your frame of reference as you float down the river…

$$\overrightarrow{v_{shore \leftarrow you}} = \overrightarrow{v_{shore \leftarrow river}} - \overrightarrow{v_{you \leftarrow river}}$$
$$= -2\,\mathrm{m/s}\,\hat{N}$$

In other words, $2\,\mathrm{m/s}$ South.

## 4.2  Boating in a River

### Words

If you are in a boat on a still body of water, and you start rowing South, you will go South. If you row East, you will go East. And whatever speed you achieve in the water, that is the speed that somebody standing on a dock would say that you are moving. But, things change if you are in a moving river.

If you are in a river that is slowly flowing East, and you point your boat south and start rowing, you will indeed go South. Compared to the water surrounding you, you will be going directly South. But to somebody who is standing on a dock, they would say that you are going both South and East, because as you row South the current carries you East.

The actual direction that you move, as seen by someone on the dock, depends on how fast you are rowing relative to the speed of the water in the river.

It is tempting to say that the first boat in Figure 4.5 is moving South, which is correct, and the others are moving SouthEast, which is not correct. As shown in figure 4.1, SouthEast is a specific direction, halfway between South and East. So unless we are sure that the boat is traveling in exactly that direction, we should say that it is traveling South *and* East.

### Graphics
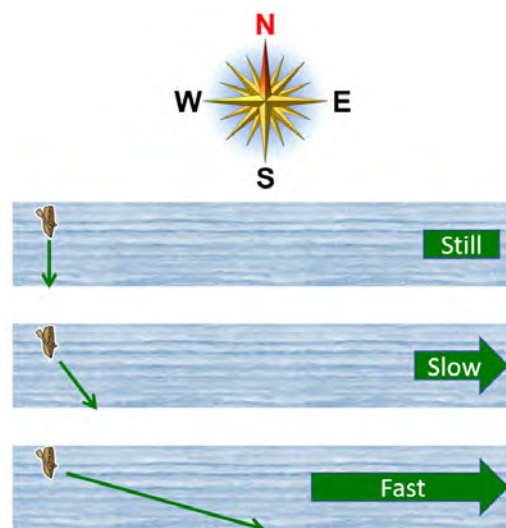


Figure 4.5: The path taken by a boat in a river, as seen by someone on a dock, depends on both the velocity of the boat in the river and the velocity of the river.[18]

The path taken by the boat can be found graphically by adding the velocity vectors "tip to tail," that is, draw the first vector, then draw the second so that its tail starts at the tip of the previous one. Continue with any other vectors that need to be added together. The resultant vector starts at the tail of the first vector and ends at the tip of the last one.

### Numbers

**Assumptions:** all parts of the river flow with the same velocity.

| Knowns | Unknowns |
|---|---|
| $\overrightarrow{v_{boat \leftarrow river}}$ | $\overrightarrow{v_{boat \leftarrow dock}}$ |
| $\overrightarrow{v_{river \leftarrow dock}}$ | |

Again, our equations for relative motion work here. Taking objects 1, 2, and 3 to be the river, the boat, and the dock, respectively...

$$\overrightarrow{v_{boat \leftarrow dock}} = \overrightarrow{v_{boat \leftarrow river}} - \overrightarrow{v_{dock \leftarrow river}}$$

We aren't given $\overrightarrow{v_{dock \leftarrow river}}$, but we are given $\overrightarrow{v_{river \leftarrow dock}}$, and we know that...

$$\overrightarrow{v_{dock \leftarrow river}} = -\overrightarrow{v_{river \leftarrow dock}}$$

...So...

$$\overrightarrow{v_{boat \leftarrow dock}} = \overrightarrow{v_{boat \leftarrow river}} + \overrightarrow{v_{river \leftarrow dock}}$$

In order to add the velocity vectors numerically, you have to break up each vector into its component parts, in this case the East/West component (E subscript below) and the North/South component (N subscript).

Let's consider a situation where you can row the boat at a speed of 5 m/s, and the river is flowing at a uniform speed of 4 m/s to the East.

If you were to row directly East, along with the river, you would move along very quickly. Since you are rowing the same direction as the current, the speed of the river would add to your rowing speed, and someone on the dock would see you going past at 9 m/s. This is the highest possible total speed when combining 5 m/s and 4 m/s.

If you were to row directly West, you would be fighting the river and moving very slowly. At 5 m/s, you can row slightly faster than the flow of the river. So you would be able to go West but only at 1 m/s. This is the lowest possible speed when combining 5 m/s and 4 m/s. If the river were flowing faster than you could row, you would not be able to go West at all, but would slowly go down the river even when paddling upstream as quickly as you could.

If you were to row directly South, you would go both South and East, and your speed would be somewhere between 1 m/s and 9 m/s. For this example, since you are rowing faster than the flow of the river, your direction would be somewhere between South and SouthEast.

Remember, vectors have magnitude and direction, so as long as you keep the magnitude and direction of a vector the same you are free to shift it up, down, left, or right on the page to get the tips & tails to line up correctly. This same technique will work with any type of vectors, for example forces, momentum, displacement, and acceleration.
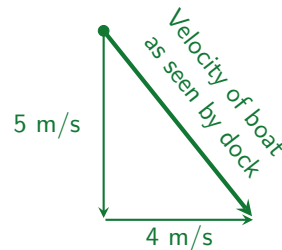


Figure 4.6: Adding a vector of 5 m/s South to a vector 4 m/s East gives a resultant vector at an angle between South and East.[1]

If the velocity vectors are drawn to scale, the resultant vector can be measured with a ruler to get a good estimate of the speed of the boat as seen by the dock. A good estimate of the direction can be found by using a protractor.

From the diagram in Figure 4.6, measurement with a ruler and protractor gives a speed of approximately 6.5 m/s, at an angle approximately 40° East of South.

$$v_{boat \leftarrow dock, E} = v_{boat \leftarrow river, E} + v_{river \leftarrow dock, E}$$

$$v_{boat \leftarrow dock, N} = v_{boat \leftarrow river, N} + v_{river \leftarrow dock, N}$$

In this example,

$$\overrightarrow{v_{boat \leftarrow dock}} = -5 \,\text{m/s}\, \hat{N} + 4 \,\text{m/s}\, \hat{E}$$

The North/South direction has been assigned a negative value in the "North" direction. In other words, South. It is usually easier to define North as positive and East as positive when doing calculations, and then if the final answer is negative you can describe it as a positive value in the South (or West) direction.

A vector can also be described in terms of magnitude and direction. Since the North/South and East/West directions are at right angles to each other, the magnitude of the resultant vector has to be found using the Pythagorean theorem:

$$\boxed{A^2 = A_x^2 + A_y^2} \qquad (4.1)$$

…where $A$ is the magnitude of any vector $\overrightarrow{A}$, $A_x$ is the $\hat{x}$ component of the vector, and $A_y$ is the $\hat{y}$ component of the vector. In our case, we are using East and North in place of $x$ and $y$.

The magnitude of the velocity, in other words the speed, of the boat as seen from the dock would be…

$$v_{boat \leftarrow dock} = \sqrt{5^2 + 4^2} \,\text{m/s} = 6.4 \,\text{m/s}$$

## 4.3 Straight Across a River

**Words**

Let's consider attempting to cross a river directly in a boat. If there is any current flowing in the river, aiming your boat directly across the river will mean that you will reach the opposite bank at a point downstream. Is there a way to aim a boat in such a way that it goes directly across the river?

Suppose we have a river that is 50 meters wide from North to South, and it is uniformly flowing East at 4 m/s. If a boat is rowed at a speed of 5 m/s relative to the river, in what direction should it be pointed so that it crosses directly from the North bank of the river to the South bank, as seen from a dock? How much time would be needed to cross the river in this way?

If we want to end up directly South from our starting point, then we need to make sure that our velocity as seen from a dock is directly South. We will be in the river, which is flowing East, so that means that the way we row our boat will have to exactly cancel out the Eastward flow of the river. That means our boat will have to be going 4 m/s West compared to the river, since the river is going 4 m/s East.

**Graphics**
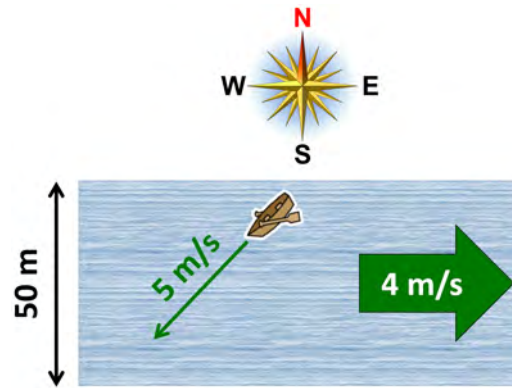


Figure 4.7: In order to go directly South across the river as seen from a dock, the boat will need to be pointed slightly upstream.[18]

If we start by drawing the vector we know, 4 m/s East, we can rotate the 5 m/s until the resultant vector is vertical.
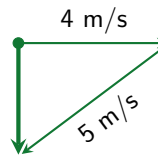


Figure 4.8: Finding the angle to make the resultant vector vertical.[1]

**Numbers**

**Assumptions:** all parts of the river flow with the same velocity.

| Knowns | Unknowns |
|---|---|
| $\overrightarrow{v_{river \leftarrow dock}} = 4\,\text{m/s}\,\hat{E}$ | $\theta_{boat}$ |
| $v_{boat \leftarrow river} = 5\,\text{m/s}$ | $t$ |
| $\overrightarrow{\Delta x} = -50\,\text{m}\,\hat{N}$ | |

The angle of any vector is related to its $\hat{x}$ and $\hat{y}$ components by sine, cosine, and tangent.

$$\sin\theta = \frac{A_y}{A} \qquad (4.2)$$

$$\cos\theta = \frac{A_x}{A} \qquad (4.3)$$

$$\tan\theta = \frac{A_y}{A_x} \qquad (4.4)$$

...where $\theta$ is the angle opposite the $\hat{y}$ component and adjacent to the $\hat{x}$ component of $\overrightarrow{A}$.



84
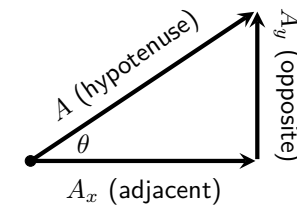
The time needed to cross the river will be longer than if there were no current in the river, because some of the boat's velocity has to be used to fight the flow of the river. The faster the river is flowing, the longer it will take to cross.

Using a protractor to measure between the 4 m/s vector and the 5 m/s vector, we find an angle of approximately $35°$, so the direction is approximately $35°$ South of West.

It is important to clearly specify the reference direction and which way the angle is measured from that direction, because compass bearings are measured differently from angles in mathematics and physics.



Figure 4.7: Compass bearings start with zero at North and proceed clockwise. Angles in physics usually start at East and proceed counterclockwise.[19]

The speed of the boat as seen from the dock can be found by measuring the vertical vector in Figure 4.8. It is approximately 3 m/s. This can be used to determine the amount of time that is needed to cross the river.

"SOHCAHTOA" is a mnemonic that many people use to remember Equations 4.2, 4.3, and 4.4:

**S**ine=**O**pposite/**H**ypotenuse,
**C**osine=**A**djacent/**H**ypotenuse,
**T**angent=**O**pposite/**A**djacent.

In this case, using $\hat{x}$ and $\hat{y}$ notation, $\overrightarrow{v_{boat \leftarrow river}}$ has a magnitude $v = 5\,\text{m/s}$ and a horizontal component $v_x = -4\,\text{m/s}$. So we can use Equation 4.3, solving for $\theta$:

$$\theta = \arccos\left(\frac{-4\,\text{m/s}}{5\,\text{m/s}}\right) = 143°$$

The normal way to define angles is clockwise from the positive x axis, so $143°$ is in a direction North and West. Unfortunately, in many cases the inverse trigonometric functions like arccos do not give the correct angles on a calculator, because there are multiple possible correct answers mathematically. For that reason, it is usually best to start with a diagram like that in Figure 4.8 and use only positive values for each part. Finding the $\theta$ in the upper right of that figure,

$$\theta = \arccos\left(\frac{4\,\text{m/s}}{5\,\text{m/s}}\right) = 37°$$

...so, $37°$ South of West. Using Equation 4.4 we can find that the North/South component of the velocity is -3 m/s. Since velocity is constant in this problem, we can use the average velocity to find the time needed to cross the river.

$$\Delta t = \frac{\overrightarrow{\Delta x}}{\overrightarrow{v_{avg}}} = \frac{-50\,\text{m}\,\hat{N}}{-3\,\text{m/s}\,\hat{N}} = 17\,\text{s}$$

85

## 4.4   Kicking in a New Direction

### Words

Since we have started considering two dimensions, we have only looked at motion, not forces, momentum, or energy. Now it's time to branch out.

A 0.45-kg soccer ball is initially moving at 20 m/s in a direction $30°$ West of North. A player kicks the ball so that it begins moving at 20 m/s in a direction $30°$ West of South. The time that the player's foot is in contact with the ball is 0.02 seconds. How can we describe the ball's acceleration, energy, momentum, and the force applied to it during the kick?

In the description, the velocity appears to be constant before and after the kick, so the momentum also must be constant before and after the kick, and that also means there is no acceleration before and after the kick.

What happens during the kick?

The initial and final speeds are the same, but the direction changes, so the velocity changes during the kick. That means that there was an acceleration, since acceleration is a change in velocity over time. And the time involved, the time of the kick, is very short. That means the acceleration had to be very large to get a large change in velocity in a small amount of time.

### Graphics



Figure 4.8: A soccer player kicking a ball.[11]
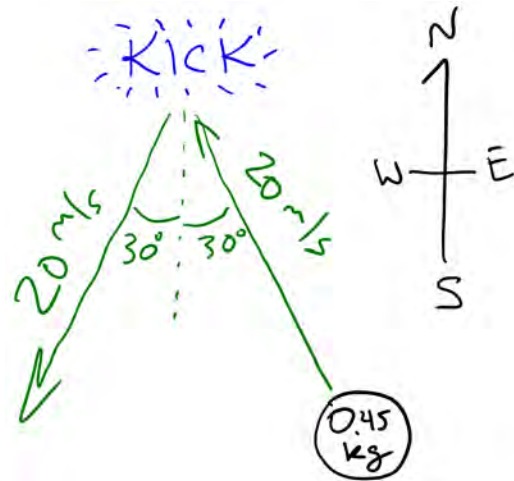


Figure 4.9: A sketch of the physical scenario.[1]

### Numbers

**Assumptions:** constant velocity before & after the kick

| Knowns | Unknowns |
|---|---|
| $m = 0.45\,\text{kg}$ | $\vec{a}$ |
| $\vec{v_0} = 20\,\text{m/s} @ 30° \text{ W of N}$ | $E$ |
| $\vec{v_f} = 20\,\text{m/s} @ 30° \text{ W of S}$ | $\vec{p}$ |
| $\Delta t = 0.02\,\text{s}$ | $\overrightarrow{F_{foot \to ball}}$ |

We should start by breaking the velocities up into their component parts using Equations 4.2, 4.3, and 4.4. Since the directions are given in terms of North, South, East, and West, we can use that notation.

$$v_{0,N} = v_0 \cdot \cos\theta_0 = 17.3\,\text{m/s}$$
$$v_{0,E} = -v_0 \cdot \sin\theta_0 = -10\,\text{m/s}$$
$$v_{f,N} = -v_0 \cdot \cos\theta_f = -17.3\,\text{m/s}$$
$$v_{f,E} = -v_0 \cdot \sin\theta_0 = -10\,\text{m/s}$$

Where did the minus signs come from, and why weren't the angles converted to normal physics notation, with $0°$ pointing East? The sketch in Figure 4.9 was used to determine positive and negative signs, and to find appropriate triangles using the angles given in the question. This is a simpler approach than converting to normal physics notation, and avoids the ambiguity in angles that comes from trusting the output of a calculator.

If the velocity changes, that means that the momentum also changes during the kick, which means that a force must have been applied to the ball.

We know that a force applied in the direction of an object's motion speeds it up, and a force applied opposite the direction of motion slows it down. In this case, a force acted that didn't actually change the speed of the ball! So the force must have been in a direction other than the direction of motion.

The ball was initially moving North and West, and after being kicked it was moving South and West. So we know that there was definitely a change in the North/South direction. There must have been an acceleration to the South, which means that there must have been a force applied to the South, and a change in momentum to the South.

In order to determine whether a force was also applied to the East or West, details about the velocity or momentum in that direction need to be known.

If we assume that the ball stayed on the ground the whole time, we don't need to worry about gravitational energy. Kinetic energy depends only on the speed of the ball, and since the initial and final speeds are the same, the kinetic energy is the same before and after the collision.

To find a change in momentum, the momentum vectors can be subtracted graphically. So far, we have only learned about adding vectors tip-to-tail. Subtracting a vector is the same as adding its opposite, and the opposite of a vector is another vector with the same magnitude and opposite direction.
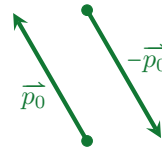
Figure 4.10: Finding the opposite of the initial momentum.[1]

Change in momentum is final momentum minus initial momentum, so...
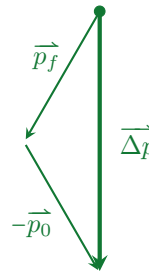
Figure 4.11: Finding the change in the momentum of the soccer ball.[1]

The change in momentum of the ball is due South. This means that the force applied to the ball and the acceleration are also due South.

If we assume a constant net force during the kick, we can find the acceleration:

$$a_E = \frac{v_{f,E} - v_{0_E}}{\Delta t} = 0 \, \text{m/s}^2$$

$$a_N = \frac{v_{f,N} - v_{0_N}}{\Delta t} = -1730 \, \text{m/s}^2$$

Combining these, we find $\vec{a} = -1730 \, \text{m/s}^2 \, \hat{N}$

We can use the masses and velocities to find initial and final momentum:

$$p_{0,N} = m \cdot v_{0,N} = 7.79 \, \text{kg} \cdot \text{m/s}$$

$$p_{0,E} = m \cdot v_{0,E} = -4.5 \, \text{kg} \cdot \text{m/s}$$

$$p_{f,N} = m \cdot v_{f,N} = -7.79 \, \text{kg} \cdot \text{m/s}$$

$$p_{f,E} = m \cdot v_{f,E} = -4.5 \, \text{kg} \cdot \text{m/s}$$

Subtracting initial from final momentum gives us the impulse caused during the kick: $\overrightarrow{\Delta p} = -15.6 \, \text{kg} \cdot \text{m/s} \, \hat{N}$.

Then Newton's Second Law can be used to find the force on the ball during the kick:

$$\overrightarrow{F_{net,foot \to ball}} = \frac{\overrightarrow{\Delta p}}{\Delta t} = -779 \, \text{N} \, \hat{N}$$

Since the speed is the same before and after the kick, we know that the kinetic energy is also the same before and after the kick:

$$E_{k,f} = E_{k,0} = \frac{1}{2} m \cdot v_0^2 = \frac{1}{2}(0.45 \text{kg})(20 \, \text{m/s})^2 = 90 \, \text{J}$$

Using the velocity vector components instead of the speed gives the same result for energy.

## 4.5  Checking In Ice Hockey

### Words

Ice hockey players commonly run into each other on the ice, often intentionally. During any collision, external forces can generally be neglected because the force of the colliding bodies against each other is so much larger than any other forces that are acting on them.

If a 90-kg hockey player is moving South across the ice at 5 m/s and is struck by an 80-kg hockey player who is moving NorthEast at 6 m/s, what can we say about the collision and the two players after the collision? Assume that the collision is perfectly inelastic, so the two become entangled together, not bouncing off of each other.

Since this is a collision, we should begin by thinking about momentum. We know that momentum is conserved in any collision, whether it is elastic or inelastic.

Since the 90-kg hockey player was moving South, that player's initial momentum is to the South. The 80-kg hockey player was moving NorthEast, so that player's momentum was partially North and partially East.

After the two collide and become entangled, their total momentum has to be the same as their initial momentum, just as in the one-dimensional problems we have considered previously. But this time we have to think about both directions.

### Graphics



Figure 4.12: Ice hockey players.[20]

### Numbers

**Assumptions:** external forces are negligible; perfectly inelastic collision

| Knowns | Unknowns |
| --- | --- |
| $m_1 = 90\,\text{kg}$ | ??? |
| $\overrightarrow{v_{1,i}} = -5\,\text{m/s}\,\hat{N}$ | |
| $m_2 = 80\,\text{kg}$ | |
| $\overrightarrow{v_{2,i}} = 6\,\text{m/s} \text{ @ } 45° \text{ N of E}$ | |

The $45°$ angle is given, since the direction is listed as NorthEast, which is half-way between North and East. $\overrightarrow{v_{2,i}}$ can be broken into its component parts using Equations 4.2 & 4.3.

$$\overrightarrow{v_{2,i}} = 4.24\,\text{m/s}\,\hat{N} + 4.24\,\text{m/s}\,\hat{E}$$

As in one dimension, the place to start in any two-dimensional collision is conservation of momentum. But we need to separate the North/South components from the East/West components:

$$p_{N,f,tot} = p_{N,i,tot}$$
$$p_{E,f,tot} = p_{E,i,tot}$$

The North/South component is...

$$\begin{aligned} p_{N,f,tot} &= m_1 \cdot v_{N,i,1} + m_2 \cdot v_{N,i,2} \\ &= (90\,\text{kg})\,(-5\,\text{m/s}) + (80\,\text{kg})\,(4.24\,\text{m/s}) \\ &= -111\,\text{kg}\cdot\text{m/s} \end{aligned}$$

And the East/West component is...

Let's start with the East/West direction. Initially, the first hockey player has no momentum in that direction, and the second hockey player has momentum to the East. After they collide, they will still have momentum to the East. So they will end up moving to the East, at a slower speed than the second hockey player had initially, because the momentum that hockey player had before the collision is shared by the two hockey players after the collision.

The North/South direction is more complicated. One hockey player initially has momentum to the North and the other has momentum to the South. With their different masses, different speeds, and the angle, it is difficult to determine whether after the collision they will be moving slightly North or slightly South, so the final direction should be mostly to the East.

Since the first hockey player was initially moving South and ended up moving mostly East, that player must have experienced a force during the collision that was North and East. That makes sense, because that was the direction the second player was moving before the collision. This means that the second hockey player would have experienced an equal and opposite force, South and West. South is easy to see, because the first hockey player was initially moving South, and there is a force to the West because some of the second player's Eastward momentum was transferred to the first player.

There was a loss of kinetic energy during the collision; the lost kinetic energy was converted into thermal energy.
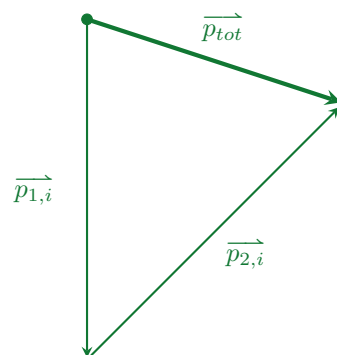


Figure 4.13: Finding the total momentum of the two hockey players.[1]
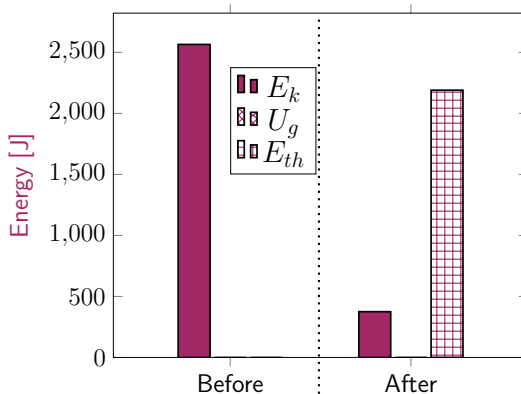


Figure 4.14: Energy bar graph for the collision. Initial thermal energy is taken to be zero. Gravitational energy is zero because both players are on the ground before and after the collision.[1]

$$p_{E,f,tot} = m_1 \cdot v_{E,i,1} + m_2 \cdot v_{E,i,2}$$
$$= 0 + (80\,\text{kg})\,(4.24\,\text{m/s})$$
$$= 339\,\text{kg} \cdot \text{m/s}$$

We can convert this into magnitude and direction using Equations 4.1 & 4.4.

$$p_{f,tot} = \sqrt{(111\,\text{kg} \cdot \text{m/s})^2 + (339\,\text{kg} \cdot \text{m/s})^2}$$
$$= 357\,\text{kg} \cdot \text{m/s}$$

$$\theta_f = \arctan \frac{p_{N,f,tot}}{p_{E,f,tot}} = -18°$$

The final momentum is correct, but we need to check the angle against a sketch of the situation to be sure. $-18°$ is South and East, and we can use Figure 4.13 to see that this is the correct general direction, so our angle is correct.

Now we can find the final velocity of the hockey players:

$$\overrightarrow{v_f} = \frac{\overrightarrow{p_{f,tot}}}{m_{tot}} = 2.10\,\text{m/s} \ @ \ 18°\,\text{S of E}$$

Next we can find the initial and final kinetic energy:

$$E_{k,i} = \frac{1}{2}m_1 \cdot v_{1,i}^2 + \frac{1}{2}m_2 \cdot v_{2,i}^2 = 2565\,\text{J}$$

$$E_{k,f} = \frac{1}{2}m_{tot} \cdot v_f^2 = 375\,\text{J}$$

Total energy is conserved, and there is no spring energy or gravitational potential energy stored after the collision, so the lost kinetic energy is converted to thermal energy.

## 4.6 Standing on a Rope

**Words**

We have looked in detail at motion and momentum in two dimensions. Let's turn now to look at forces and energy. Consider the fiddler who is standing on a rope in Figure 4.15. We will assume that the fiddler is not moving. What are the forces that are acting on him? And how much work is he doing?

Take the mass of the fiddler to be 70 kg. The rope in front of him and behind him is angled up 25° above the horizontal.

First, it is important to establish the meaning of the question. One way to look at the situation is to say that the only two forces that are affecting the fiddler are the force of gravity, directed downward, and the normal force of the rope pushing up on his feet. That makes this problem essentially the same as the rock sitting motionless on the ground from Section 1.3, which is a one-dimensional question that by now is something that we should be able to answer without much difficulty.

The intent of this question is to investigate the impact of having a rope going off at an angle, so we will use as our system the fiddler and also the section of the rope on which he is standing. We aren't given the mass of the rope, and we will assume that it is negligible compared to the mass of the fiddler.

**Graphics**



Figure 4.15: A fiddler balancing on a rope.[21]



Figure 4.16: Sketch of the fiddler balancing on a rope. The blue dotted line indicates the "system" to be considered when making a free body diagram.[1]

**Numbers**

**Assumptions:** The fiddler is not moving; the mass of the rope is negligible; the part of the rope under the fiddler's feet is part of the system

| Knowns | Unknowns |
|---|---|
| $m = 70\,\text{kg}$ | Forces on fiddler |
| $\theta_1 = 25°$ above horizontal | $W$ |
| $\theta_2 = 25°$ above horizontal | |
| $g = 9.8\,\text{m/s}^2$ | |

Note that the angles are not necessarily the standard angles in math–they are as described in the text and in Figures 4.16 & 4.17.

To analyze this system, we need to separate the forces shown in Figure 4.17 into $\hat{x}$ & $\hat{y}$ components, using Equations 4.2 & 4.3. There are three forces to consider, as can be seen in Figure 4.17.

Using $+\hat{x}$ to the right and $+\hat{y}$ up…

$$F_{net,x} = F_{t,2} \cdot \cos 25° - F_{t,1} \cdot \cos 25°$$

$$F_{net,y} = F_{t,2} \cdot \sin 25° + F_{t,1} \cdot \sin 25° - F_g$$

Since the fiddler is not moving in this scenario, his acceleration is zero. That means, according to Newton's Second Law, that $\overrightarrow{F_{net}} = 0$. The $\hat{x}$ direction tells us that

$$F_{t,2} = F_{t,1}$$

Once we have decided on our system, the forces that we need to consider are only the forces that act on our system, not the forces acting inside the system or the forces of the system acting on something else. One force that acts on the system is the force of gravity, which depends on the mass of the system (in this case, the mass of the fiddler, since we are neglecting the mass of the rope).

There are also two ropes that connect our system to things that are outside of the system, and each of those ropes can have a tension force pulling on the system. Tension is always a pulling force, just as the normal force is always a pushing force. Tension is in the direction of the rope, string, chain, *etc*, and if the rope or other object that is being used to pull is light (often referred to as massless), the tension is the same along the entire length of the rope.

Looking at the sketch in Figure 4.16, we can see that there is symmetry. The angles are the same in front of the fiddler and behind the fiddler, so if the fiddler turned around the picture would be exactly the same. That symmetry tells us that whatever the tension is in the rope in front of the fiddler is the same as the tension behind the fiddler.

When considering the amount of work the fiddler is doing, it was already stated that the fiddler is not moving, and since work is caused by a force acting over a distance, if there is no distance moved then there is no work that is being done.
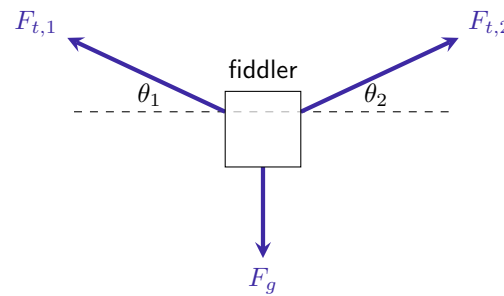


Figure 4.17: FBD for the fiddler[1]
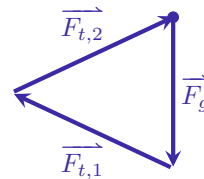


Figure 4.18: Adding the force vectors graphically. The net force is zero, so we know that the arrows have to end in the same place where they started.[1]

Graphically adding the vectors shows that the horizontal components of the two tension forces have to cancel out, because the gravitational force does not have a horizontal component. Graphical addition also shows that the tension force has a slightly larger magnitude than the gravitational force, since all the tension arrows are slightly longer.

...so the $\hat{y}$ direction tells us...

$$2 \cdot F_{t,1} \cdot \sin 25° = F_g$$

The magnitude of the force of gravity in this situation is given by $F_g = m \cdot g$, so...

$$F_g = m \cdot g = 686 \, \text{N}$$

Solving for $F_{t,1}$, we get...

$$F_{t,1} = \frac{F_g}{2 \cdot \sin 25°} = 812 \, \text{N}$$

Because of the angles involved, the tension in the rope is actually larger than the force of gravity!

Now we can find the work that the person is doing:

$$W = \overrightarrow{F} \cdot \overrightarrow{\Delta x}$$

And since there is no motion $\overrightarrow{\Delta x} = 0$, so $W = 0$. No work is being done.

## 4.7 Pulling a Sled

### Words

Figure 4.19 shows a moose that is connected to a sled. Imagine that the moose is pulling the sled across level ground at a constant velocity of 3 m/s to the left. The strap that it pulls with is at an angle of 30° above the horizontal, the mass of the loaded sled is 200 kg, and the magnitude of the frictional force between the sled and the ground is 800 N.

Analyze this physical scenario as thoroughly as you can.

We are given very little information about the moose, so we should instead focus on the sled for our analysis. Since the velocity is constant, there are several things that we know immediately:

- The momentum has to be constant.

- The kinetic energy has to be constant.

- The acceleration, which is the change in velocity over time, has to be zero.

- Since the momentum is constant (or since the acceleration is zero), the net force on the sled has to be zero. In other words, the forces are balanced.

### Graphics



Figure 4.19: A moose strapped to a sled.[22]

Motion Map

4 s    3 s    2 s    1 s    0 s
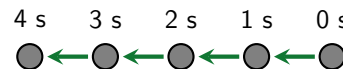
Figure 4.20: Motion map of the sled moving to the left at constant velocity.[1]

We could draw a momentum vs time graph, or an energy bar graph with only one bar, but usually those are only useful when we want to show a change in something. Since momentum and energy are both constant, drawing them is not very useful.

### Numbers

**Assumptions:** $+\hat{x}$ direction is to the right; $+\hat{y}$ direction is upward

| Knowns | Unknowns |
|---|---|
| $m = 200\,\text{kg}$ | ??? |
| $\theta = 30°$ above horizontal | |
| $\overrightarrow{v} = -3\,\text{m/s}\,\hat{x}$ | |
| $F_f = 800\,\text{N}$ | |

We can find the momentum of the sled is given by:

$$\overrightarrow{p_{sled}} = m \cdot \overrightarrow{v} = -600\,\text{kg} \cdot \text{m/s}\,\hat{x}$$

We can also find the kinetic energy:

$$E_{k,sled} = \frac{1}{2}m \cdot v^2 = 900\,\text{J}$$

Acceleration is zero since velocity is not changing:

$$\overrightarrow{a} = \frac{\overrightarrow{\Delta v}}{\Delta t} = 0$$

Since $\overrightarrow{a} = 0$, Newton's Second Law tells us that the net force is also zero. So the sum of the forces in both the $\hat{x}$ and the $\hat{y}$ directions must be zero.

$$F_{net,x} = F_f - F_t \cdot \cos\theta = 0$$

We are given $F_f$, so we can solve for $F_t$:

$$F_t = \frac{F_f}{\cos\theta} = 923\,\text{N}$$

The force of gravity is balanced partly by the normal force but also partly by the upward part of the tension in the strap, so the normal force should be less than the gravitational force in this case. The forward part of the tension force needs to be just large enough to balance the force of friction.

The net work that is being done is zero, since the energy is not changing, but in fact the moose is doing work by pulling the sled, and the frictional force is doing negative work to try to stop the sled.

Work is caused by a force being applied through a displacement, and the displacement is constantly increasing in time since the sled is moving at a constant velocity. That means that the moose is doing a constant amount of work per time.

In physics we have a special name for work per time: Power. Power is measured in watts, where 1 watt is 1 joule per second.
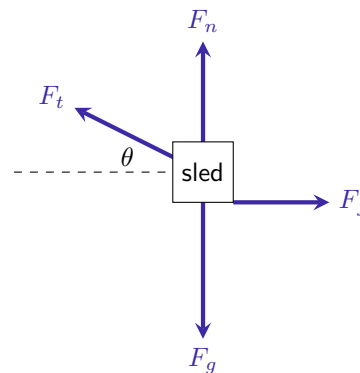


Figure 4.21: FBD for the sled, with guesses about the magnitudes of each arrow.[1]
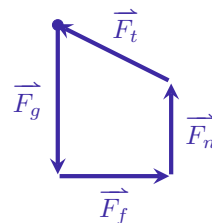


Figure 4.22: Adding the force vectors graphically. The net force is zero, so we know that the arrows have to end in the same place where they started.[1]

Just from adding the force vectors graphically with correct directions but without doing calculations for magnitudes, we can see that the only way the net force can be zero is for the normal force to be smaller than the gravitational force and for the horizontal component of the tension force to exactly cancel the frictional force.

Figure 4.21 shows us that

$$F_{net,y} = F_n + F_t \cdot \sin\theta - F_g = 0$$

$F_g$ is the mass times the acceleration of gravity, or 1960 N. We can solve for the only thing we don't know in that mathematical model, $F_n$.

$$F_n = F_g - F_t \cdot \sin\theta = 1500\,\text{N}$$

The force of friction is opposing the direction of motion, so as the sled moves, friction is doing negative work on the sled. The moose is doing positive work on the sled, so that the net work is zero. The amount of work done by the moose is...

$$W = F \cdot \Delta x \cdot \cos\theta$$

But we don't have a displacement in this scenario. The displacement increases over time, so the work also increases over time. The moose is doing a constant amount of work per unit time, or supplying a constant power:

$$\boxed{P = \frac{W}{\Delta t} = \frac{F \cdot \Delta x \cdot \cos\theta}{\Delta t}} \quad (4.5)$$

...where $P$ is power, and the equation is valid while the force is constant. Power can also be expressed as...

$$\boxed{P = F \cdot v \cdot \cos\theta} \quad (4.6)$$

So the power supplied by the moose is:

$$P_{moose} = F_t \cdot v \cdot \cos\theta = 2400\,\text{W}$$

## 4.8 Pulling a Frictionless Sled

### Words

Let's consider the same physical scenario as in Section 4.7, but this time consider what would happen if there were no friction between the sled and the ground.

If the moose started at rest and pulled the 200 kg sled with the same force as before, 923 N, at an angle of 30° above the horizontal, across a horizontal distance of 50 m, what would happen?

Again, we are given very little information about the moose, so we should instead focus on the sled for our analysis.

The moose is generating a force that is acting over a certain distance, so a good starting point would be to consider the work that the moose is doing. The moose does work on the sled, which increases the kinetic energy of the sled. At first the sled is not moving, so it begins with zero kinetic energy. The sled stays on level ground the entire time, and there are no springs to worry about, so all of the work done by the moose is converted into kinetic energy.

### Graphics

Figure 4.23: A moose strapped to a sled.[22]
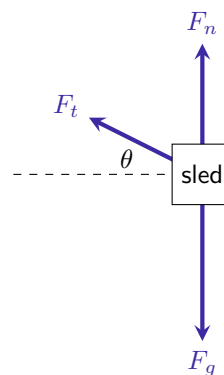
Figure 4.24: FBD for the sled, with arrow lengths based on the forces found in Section 4.7.[1]

### Numbers

**Assumptions:** $+\hat{x}$ direction is to the right; $+\hat{y}$ direction is upward; no air resistance

| Knowns | Unknowns |
|---|---|
| $m = 200\,\text{kg}$ | ??? |
| $F_t = 923\,\text{N}$ | |
| $\theta = 30°$ above horizontal | |
| $\Delta x = -50\,\text{m}\,\hat{x}$ | |
| $F_f = 0\,\text{N}$ | |

Since there is no acceleration in the vertical direction, our analysis of the $y$ components of force in Section 4.7 where $F_{net,y} = 0$ remains valid. But we need to reconsider $F_{net,x}$

$$F_{net,x} = -F_t \cdot \cos\theta = -800\,\text{N}$$

Since we know the mass of the sled, we can also find its acceleration:

$$\vec{a} = \frac{\vec{F_{net}}}{m} = -4\,\text{m/s}^2$$

Given net force and displacement, we can use work to find final kinetic energy:

$$W_{net} = F_{net} \cdot \Delta x \cdot \cos\theta = \Delta E_k = E_{k,f} - E_k, i$$

94

The net force on the sled gives it an acceleration in the same direction as the net force, to the left. Since the sled started at rest, its speed will increase to the left over the entire time that the moose is pulling. That also means that the momentum will be increasing to the left the entire time that the moose is pulling.

In the physical situation examined in Section 4.7, the velocity was constant, so the power supplied by the moose was constant. In this situation, the speed is increasing, so every second the moose is covering more distance than it did in the previous second. That means every second the moose is doing more work than it did in the previous second.

So the amount of power that the moose is producing increases as its speed increases. For any real creature or machine, the power it can produce limits the amount of force that it is able to apply to an object. Typically a larger force can be applied when the object is moving at a low speed, and as speed increases power becomes the limiting factor.
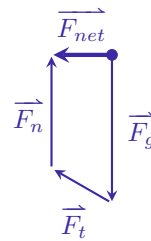


Figure 4.25: Adding the force vectors graphically shows that there is a net force to the left.[1]
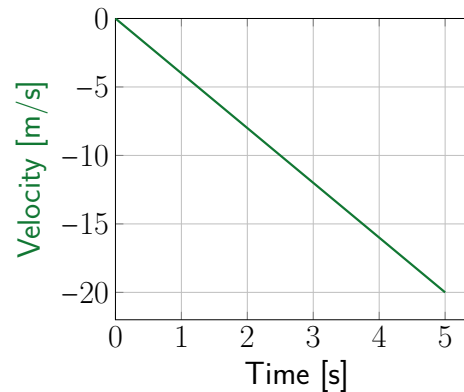


Figure 4.26: Velocity of the sled.[1]

…where $\theta$ is the angle between $\overrightarrow{F_{net}}$ and $\overrightarrow{\Delta x}$, not the angle of the strap that the moose is using to pull the sled.

$$E_{k,f} = (800\,\text{N}) \cdot (50\,\text{m}) \cdot \cos 0° = 40\,000\,\text{J}$$

From here, we can find the final speed of the sled:

$$E_{k,f} = \frac{1}{2} m \cdot v_f^2$$

Interestingly, since we found acceleration earlier we could have found the velocity in a more direct route. Multiplying the above equation for $W_{net}$ by 2, dividing by $m$ , and rearranging it slightly gives:

$$\boxed{2a \cdot \Delta x \cdot \cos \theta = v_f^2 - v_i^2} \qquad (4.7)$$

This is commonly considered to be a standard equation of motion.

Once we have the final velocity and the acceleration, we can also find the time needed for the moose to pull the sled 50 m.

The power that the moose uses to pull the sled increases over time, even though the force stays the same. This can be seen from Equation 4.6, knowing that the velocity is constantly increasing in time since the force is constant. The maximum power will be needed at the maximum speed:

$$P_{max} = F_{net} \cdot v_f \cdot \cos 0°$$

## 4.9 Waterslide

### Words

The waterslide shown in Figure 4.27 has an extremely steep slope at the top. Water is used to lubricate the slide during operation, making it essentially frictionless.

The slide has a height of 9 m and the slope goes down at an angle 70° from the horizontal. If a 50 kg rider starts at the top at rest and slides down the steep slope, what is her acceleration down the steep slope, what is her speed when she reaches the bottom of the steep slope, and how much time does it take her to reach the bottom of the steep slope?

We need to consider the acceleration down the slide. Acceleration is a vector, so it includes direction. We could specify the direction of the acceleration using $x$ and $y$ components, or a magnitude in some direction, but it would be easier if we just create a way to refer to the direction down the slope. We can call that the "parallel" direction, and then the "perpedicular" direction would be at a right angle to the direction of the slope.

If the slope were completely flat, then the rider would just sit there, not accelerating. If the slope were completely vertical then the rider would be in free-fall, accelerating down the slope with the acceleration of gravity $g$. For slopes in between these extremes, the acceleration should be between zero and $g$. The steeper the slope, the closer the acceleration would be to $g$.

### Graphics



Figure 4.27: A waterslide.[23]
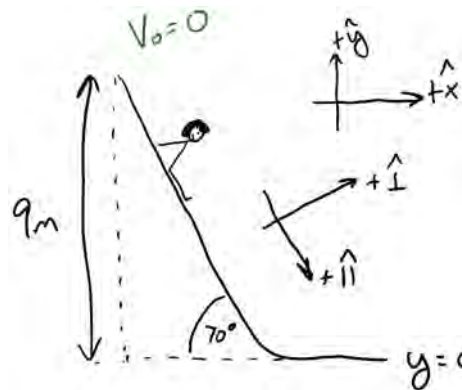


Figure 4.28: A sketch of the rider on the waterslide. $\hat{x}$ & $\hat{y}$ axes are shown, as are $\hat{\parallel}$ & $\hat{\perp}$ axes.[1]

### Numbers

| Knowns | Unknowns |
|---|---|
| $m = 50\,\text{kg}$ | $\vec{a}$ |
| $\theta = 70°$ above horizontal | $v_f$ |
| $y_0 = 9\,\text{m}$ | $\Delta t$ |
| $y_f = 0\,\text{m}$ | |
| $v_0 = 0\,\text{m/s}$ | |
| $F_f = 0\,\text{N}$ | |

We can use work and energy to find the speed at the bottom of the slope. There are no external forces or springs, and no friction to generate thermal energy, so we only need to consider kinetic and gravitational potential energy.

$$E_{k,i} + U_{g,i} = E_{k,f} + U_{g,f}$$

Taking ground level to be $y = 0$...

$$0 + m \cdot g \cdot y_0 = \frac{1}{2} m \cdot v_f^2 + 0$$

$$v_f = \sqrt{2g \cdot y_0} = 13.3\,\text{m/s}$$

If there is really no friction on the waterslide, then the only forces affecting the rider are gravity and the normal force. And since the motion is perpendicular to the normal force, the normal force doesn't do any work on the rider. So it is only gravity that affects the rider's energy. Since energy is conserved, all of the rider's gravitational potential when she is at the top of the waterslide will be converted to kinetic energy when she reaches the bottom. Surprisingly, that means it doesn't actually matter how steep the slope is–the speed at the bottom will be the same if the rider has dropped the same vertical distance.

The amount of time that the rider takes to reach the bottom will depend on how steep the slope is. If the slope is steep then the distance traveled will be short and the acceleration will be large. If the slope is shallow, then the distance traveled will be longer and the acceleration will be smaller, and both of these changes will mean a longer time to the bottom with a more shallow slope.

In our everyday experience, we would say that the rider would be going faster at the bottom if the slope is steep. That is correct, because in everyday life the friction is never really zero, and in fact the frictional force would be smaller when the slope is steeper.
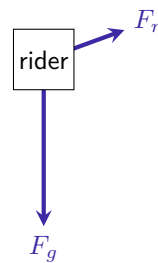


Figure 4.29: FBD of the rider on the waterslide.[1]

Since the acceleration of the rider is in the parallel direction, the net force also has to be in the parallel direction, giving us the relative magnitudes of $F_g$, $F_n$, and $F_{net}$.
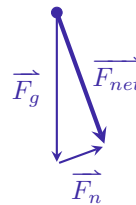


Figure 4.30: Adding the force vectors graphically.[1]

If we knew the rider's displacement from the top to the bottom of the steep slope, we could use Equation 4.7 to find her acceleration down the slope. The magnitude of the displacement is the length of the steep slope, which we can find from Figure 4.28 using Equation 4.2:

$$\Delta x = \frac{y_0}{\sin 70°}$$

Since the displacement and the acceleration are both in the positive parallel direction, Equation 4.7 gives…

$$2a_\parallel \cdot \Delta x_\parallel \cdot \cancel{\cos 0}^{\,1} = v_f^2 - 0^2$$

$$a_\parallel = \frac{v_f^2}{2\Delta x_\parallel} = \frac{2g \cdot y_0}{2\frac{y_0}{\sin 70°}} = g \cdot \sin 70°$$

Which means that the force due to gravity down a slope is given by

$$F_{g,\parallel} = m \cdot g \cdot \sin \theta$$

…where the positive parallel direction is down the slope and $\theta$ is the angle of the slope above the horizontal.

In the direction perpendicular to the slope, the gravitational force pointing into the slope exactly cancels the normal force out of the slope. Using Figure 4.30 and Equation 4.3…

$$F_{g,\perp} = -m \cdot g \cdot \cos \theta$$

…where the positive perpendicular direction is up out of the slope and $\theta$ is the angle of the slope above the horizontal.

## 4.10  Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

### General

- For our analyses, the two dimensions being considered must be at right angles to each other.

- The two dimensions can be considered completely independent from each other in terms of forces, momentum, and motion. For example, a force in the vertical direction does not affect motion in the horizontal direction.

- Vectors can be added numerically by breaking them up into components and adding the components separately.

- In a problem involving compass directions, it is often convenient to call South "negative North" and West "negative East" while doing calculations, but the final answer should be given in positive values North, South, East, and/or West.

- SouthEast, NorthEast, *etc.* are specific directions exactly halfway between, for example, South and East. "South and East" can refer to any direction between South and East.

- The bearings on a compass are not the same as the standard angles used in mathematics and physics. In physics, angles are normally measured counterclockwise from the positive x axis.

- It is important to sketch a physical situation when dealing with angles, so you have a general idea of the directions. Trusting in a calculator alone to give a correct answer for an angle is risky, because for every trigonometric function there are multiple angles that give the same answer.

- Vectors can be added graphically by combining them "tip to tail," with the resultant vector starting at the tail of the first vector and ending at the tip of the last vector. This provides a good estimate of the resultant vector if the drawing is made and measured carefully.

- Subtracting a vector is the same as adding the opposite of the vector. The opposite of a vector is another vector with the same magnitude and opposite direction.



| Vector 1 | Vector 2 | -Vector 2 | Vector 1 + Vector 2 | Vector 1 - Vector 2 |

Sample addition and subtraction of vectors [1]
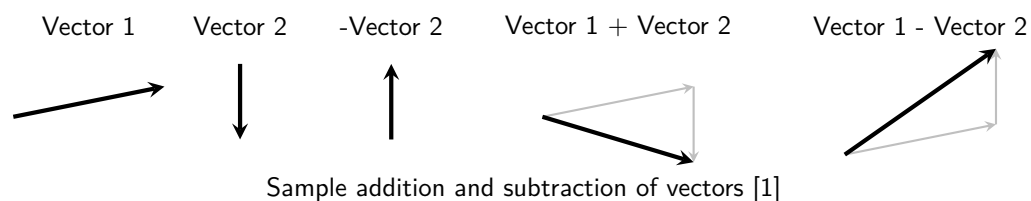
### Forces

- If a force is applied in a direction that is not in the direction of an object's motion and not opposite the direction the object's motion, the speed of the object may not change, but the direction of its motion will change.

- Tension is a pulling force that is in the direction of the string, rope, chain, *etc*.

- If a rope or other object that is being used to pull is light (often referred to as massless), the tension is the same along the entire length of the rope.

**Motion**

- The mathematical models that are used to compare relative motion between different objects also work when one object is riding on another.

## Momentum

- For isolated systems in 2-D, the momentum in one dimension (*e.g.* the $x$ direction) is conserved *and* the momentum in the other dimension (*e.g.* the $y$ direction) is conserved.

## Energy

- Since kinetic energy does not depend on the direction of an object's motion but on its speed, it doesn't matter whether you calculate the kinetic energy using vector components separately or simply using the speed.

- Power is work per time. It is measured in watts.

- 1 watt is 1 joule per second.

- For any real creature or machine, power limits the force that it can produce. Typically a larger force can be applied at low speeds and power becomes the limiting factor at high speeds.

## Mathematical Models

| equation | restrictions on the validity of the equation |
|---|---|
| $A^2 = A_x^2 + A_y^2$     (4.1) <br><br> "Pythagorean theorem" | right triangle |
| $\sin\theta = \frac{A_y}{A}$     (4.2) | calculator may give wrong theta–confirm with a sketch |
| $\cos\theta = \frac{A_x}{A}$     (4.3) | calculator may give wrong theta–confirm with a sketch |
| $\tan\theta = \frac{A_y}{A_x}$     (4.4) | calculator may give wrong theta–confirm with a sketch |
| $P = \frac{W}{\Delta t} = \frac{F \cdot \Delta x \cdot \cos\theta}{\Delta t}$     (4.5) | only valid when the force is constant |
| $P = F \cdot v \cdot \cos\theta$     (4.6) | only valid when the force is constant |
| $2a \cdot \Delta x \cdot \cos\theta = v_f^2 - v_i^2$     (4.7) | only valid when the net force is constant |

## 4.11 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

### Level 1 - Remember

4.1 **[G & N]** Which gives the most accurate result, adding vectors graphically or numerically?

4.2 **[W & G]** Describe the direction that the boat in Figure 4.5 is moving, in terms of compass directions.

4.3 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

### Level 2 - Understand

4.4 **[W & N]** Why is it especially important to sketch physical scenarios instead of just trusting a calculator when it comes to angles?

4.5 **[N]** How much thermal energy was generated in the collision described in Section 4.5?

4.6 **[W & G]** Motion maps can be drawn in two dimensions. Create a motion map showing the soccer ball from Section 4.4 from three seconds before the kick to three seconds after the kick.

4.7 **[N]** What are the initial momenta for each of the ice hockey players in Section 4.5, in the earth's reference frame?

4.8 **[G & N]** Equation 4.6 includes an angle $\theta$, but the text doesn't say what the angle is. Based on what you know about Equation 2.4, what would this $\theta$ be measured between? Make a sketch that shows the angle $\theta$ in relation to the other variables in this mathematical model.

4.9 **[N]** In the calculation of $E_{k,f}$ in Section 4.8, $0°$ is used for $\theta$, but in the "knowns" it says $\theta = 30°$. Is this an error in the text? Explain your answer.

### Level 3 - Apply

4.10 **[G & N]** For the physical scenario considered in Section 4.2, the velocity of the boat as seen by the dock is calculated in component form, and the magnitude of the velocity is also found, but the direction is not found. Find the direction.

4.11 **[N]** At the end of Section 4.4, a statement is made that using velocity vector components instead of speed gives the same result when calculating kinetic energy. Verify this statement by calculating the initial and final kinetic energy of the soccer ball in Section 4.4 using the $\hat{N}$ and $\hat{E}$ components of the velocity and comparing to the given result that was calculated using the speed.

4.12 **[G & N]** If everything else stayed the same, what initial speed of the first hockey player in Section 4.5 would have resulted in a final momentum that was due East?

4.13 **[G & N]** If everything else stayed the same, what initial speed of the second hockey player in Section 4.5 would have resulted in a final momentum that was due East?

4.14 **[G]** Create a motion map representing the position of the sled in Section 4.8 as the moose pulls it 50 m.

4.15 **[G & N]** The final velocity and maximum power are not calculated in Section 4.8. Calculate values for them.

## Level 4 - Analyze

4.16 **[W, G, & N]** If the river in Section 4.2 were 50 m wide like the river in Section 4.3, how much time would the boat need to cross the river if it were pointed directly South? What is the best direction in which to angle a boat if you want to cross the river as quickly as possible with no regard for the distance the boat travels downstream?

4.17 **[W & G]** First, draw the motion map for the soccer ball from the question in the "Level 2 - Understand" section. Then, explain how the motion map shows the acceleration during the kick.

4.18 **[G & N]** Find the direction of the force that would have to be applied to the soccer ball in Section 4.4 in order for its final velocity to be...

 (a) ...due East at 20 m/s.
 (b) ...due West at 10 m/s.
 (c) ...20 m/s @ $30°$ East of South.

4.19 **[N]** Find the magnitudes and directions of the forces on each hockey player in the collision described in Section 4.5, if the length of time that the two players are crashing into each other is 0.05 seconds.

4.20 **[W, G]** Redraw Figures 4.16, 4.17, and 4.18 for the rope at the following angles, and describe how the magnitude of the tension force in the ropes would be different from that with the rope at the original angle.

 (a) $\theta = 75°$ above the horizontal
 (b) $\theta = 5°$ above the horizontal

## Level 5 - Evaluate

4.21 **[W, G, & N]** According to Section 4.2, for a boat moving at a speed of 5 m/s relative to a river whose speed is 4 m/s, the highest possible speed for the boat as seen from the dock is 9 m/s, and the slowest possible speed is 1 m/s. Is there any direction that the boat could travel such that its speed relative to the water is still 5 m/s, and the speed of the boat relative to the dock is also 5 m/s? If not, explain why not. If so, find the direction graphically or numerically.

4.22 **[W, G, & N]** Describe using words and either graphics or numbers how the acceleration, final energy, and final momentum of the soccer ball in Section 4.4 would have been different if the force applied to the ball during the kick were doubled and the time remained the same.

4.23 **[W, G, & N]** Consider the physical scenario described in Section 4.5. Are the speeds and masses reasonable? Is the assumption about the collision being perfectly inelastic reasonable? Give reasons for your answers.

4.24 **[W, G, & N]** Analyze the physical scenario of the moose pulling the sled in Section 4.8 if the mass of the sled were to be cut in half and all other given quantities remained the same. What effect would this have on the motion, momentum, forces, energy, and power?

4.25 **[W, G, & N]** Analyze the physical scenario of the moose pulling the sled in Section 4.8 if the mass of the sled were to be cut in half and the pulling force of the moose were also cut in half. What effect would this have on the motion, momentum, forces, energy, and power?

4.26 **[W, G, & N]** Is the final speed attained by the moose and sled in Section 4.8 reasonable? Explain your answer. If it is not reasonable, what faulty assumptions or unrealistic starting parameters result in an unreasonable answer?

## Level 6 - Create

4.27 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Add the main ideas from this chapter to the concept map that you began for the question at the end of Chapter 1.

4.28 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

4.29 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

# Chapter 5

# Variable Forces

So far, we have considered only situations where force is constant. That makes life easier, and it allows us to use reasonably simple mathematical models. But the world isn't simple, and now it is time to begin addressing situations where forces change.

Many machines are built in such a way that they can use a small force to generate a larger force. This is true, for example, with pulleys, screwdrivers, pliers, and systems of gears. We will look at a system of pulleys to demonstrate how a small force can be used to create a larger force.

In the spring scale shown in Figure 5.1, the distance between the ends of the scale is related to the amount of force applied by the scale. That's what makes it function as a scale–you can read the position to know the amount of force. So with a spring, force is dependent upon displacement.

Another example of a force that changes depending on the physical scenario is frictional force. Friction opposes the relative motion between two objects, so if the direction of motion changes, the direction of the frictional force also changes. If the two objects are not moving relative to each other, the frictional force changes depending on the net force created by all other forces, adjusting to keep the total net force at zero so the objects remain stationary relative to each other.



Figure 5.1: A spring scale, which changes in length as the tension force changes.[24]

Finally, we will look again at gravitational force. We have been using $g$ as the acceleration due to gravity, but that is only true at the surface of the earth. In fact, the force of gravity changes with distance from the center of the earth...and with distance from the center of everything else in the universe.

## 5.1 Pulleys

### Words

A pulley is a wheel that can spin around an axle, with a rope, chain, *etc.*, that goes around the outer edge of the wheel. For now we will assume that both friction and the mass of the pulleys is negligible. Such pulleys can freely rotate, so a rope that is wrapped around a pulley has the same tension on each side of the pulley. Wrapping a rope around a pulley effectively changes the direction of the motion of the rope and the direction of the tension.

We will consider a 4-kg mass hanging from a single pulley that is held up using a rope that is wrapped around two pulleys as shown in Figure 5.2. Note that there are three ropes in the figure: one wrapped around the pulleys, and one connected to the center of each pulley. One end of the longest rope is connected to the ceiling and the other end is pulled down by an applied force. The mass is moving upward at a constant speed of 3 m/s. We should be able to find the tension in all three ropes and the magnitude of the applied force that is needed to lift the mass at a constant speed. We should also be able to find the amount of power that is being used to lift the mass and the amount of power supplied by the applied force.

Looking at Figure 5.2, there are actually two sections of rope 1 that together hold up the mass. Since the mass is not accelerating, the forces are balanced; the tension in rope 1 must be half of the weight of the mass.

### Graphics
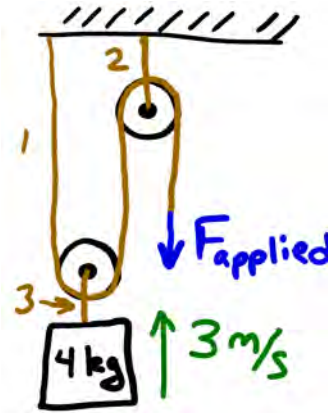


Figure 5.2: A system of two pulleys being used to lift a 4 kg mass. Note that there are three different ropes, labeled 1-3.[1]
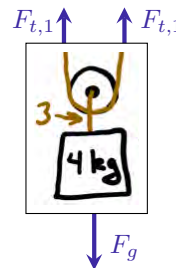


Figure 5.3: FBD of the 4 kg mass and the pulley closest to it.[1]

### Numbers

**Assumptions:** friction is negligible; mass of pulleys is negligible; near the surface of the earth; upward is $+\hat{y}$.

| Knowns | Unknowns |
|---|---|
| $m = 4\,\mathrm{kg}$ | $F_{t,1}$ |
| $v_{mass} = 3\,\mathrm{m/s}$ | $F_{t,2}$ |
| $g = 9.8\,\mathrm{m/s^2}$ | $F_{t,3}$ |
| | $F_{applied}$ |
| | $P_{lift}$ |
| | $P_{applied}$ |

Since the mass is moving at constant speed, its acceleration is zero. That means the net force on the mass is zero. Using Newton's Second Law with the free-body diagram in Figure 5.3…

$$\overrightarrow{F_{net}} = m \cdot \overrightarrow{a}^{\,0}$$
$$2F_{t,1} - m \cdot g = 0$$
$$2F_{t,1} = m \cdot g$$
$$F_{t,1} = \frac{m \cdot g}{2}$$
$$F_{t,1} = 19.6\,\mathrm{N}$$

Since the tension force in rope 1 is the same everywhere, we can see from Figure 5.2 that $F_{applied}$ is the same as $F_{t,1}$. A similar analysis on the pulley closest to the mass shows that $F_{t,3}$ is 39.2 N.

Rope 3 only has one section, so its tension has to be equal to the weight of the mass since the mass is not accelerating.

The applied force supplies the tension on one side of rope 1, so the applied force is also half of the weight of the mass. This system of pulleys creates a type of machine that creates an output force (used to lift the mass) that is twice as large as the input (applied) force, so we say that this machine has a "mechanical advantage" of two. Mechanical advantage is simply the ratio of the output force to the input force.

At first, it may seem that doubling the force should also double the power since power is proportional to force and velocity. That would violate conservation of energy, so that shouldn't be possible. As can be seen in Figure 5.5, the rope at the position of the applied force is moving faster than the mass. The increased speed is just enough to make the output power the same as the input power. So energy is conserved even in a machine with a mechanical advantage that is greater than 1.
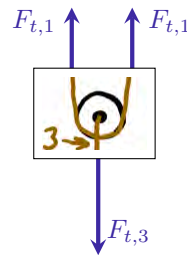


Figure 5.4: FBD of the pulley closest to the 4 kg mass.[1]
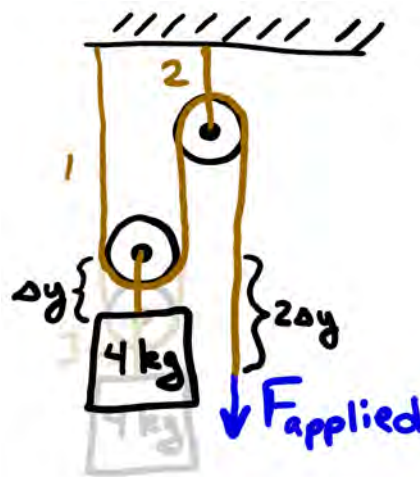


Figure 5.5: The applied force has to move twice as quickly as the mass in this system. Notice that when the mass rises by $\Delta y$ both the left section and the middle section of rope 1 get shorter by $\Delta y$, so for the rope to stay the same length the right section has to get longer by $2\Delta y$.[1]

The mechanical advantage $MA$ of a machine is given by…

$$MA = \frac{F_{output}}{F_{input}} \qquad (5.1)$$

In this situation…

$$MA = \frac{F_{output}}{F_{input}}$$
$$= \frac{39.2\,\text{N}}{19.6\,\text{N}}$$
$$= 2$$

We can also find the power needed to lift the mass:

$$P = \frac{W}{\Delta t} = \frac{\Delta E}{\Delta t}$$
$$= \frac{m \cdot g \cdot \Delta y}{\Delta t}$$
$$= m \cdot g \frac{\Delta y}{\Delta t}$$
$$= (4\,\text{kg})\left(9.8\,\text{m/s}^2\right)(3\,\text{m/s})$$
$$= 118\,\text{W}$$

The power supplied by the applied force can be found using the velocity. But we need to be careful, because as illustrated in Figure 5.5, the applied force is moving at twice the speed of the mass.

$$P = F \cdot v \cdot \cancel{\cos\theta}^{\,1}$$
$$= (19.6\,\text{N})(6\,\text{m/s})$$
$$= 118\,\text{W}$$

## 5.2   Spring Scale

**Words**

Springs are physical objects that are usually made from metal because of many metals' ability to return to their original shape after being flexed. Springs can produce force, and they can also store potential energy. For an ideal spring, the force required to stretch or compress it is proportional to the amount of extension or compression. This is referred to as "Hooke's Law." We will consider only ideal springs. Real springs are more complicated. They do not obey Hooke's Law when the compression or extension is large, and can become permanently deformed if stretched too far.

Let's consider the spring inside the scale shown in Figure 5.6. If the markings represent the mass of an object that is hanging from the hook, measured in kg, and the spacing between each number is 0.01 m, what is the stiffness of the spring, and how much potential energy is stored in the spring if a 5 kg mass is hanging from the hook?

It is important to notice that the numbers on the scale do not refer to the length of the spring. They refer to how far the spring is compressed from its natural length.

**Graphics**



Figure 5.6: A spring scale and a schematic of the way the spring is connected inside.[25]

The ring at the top of the scale is connected to the orange body of the scale, which is connected to the bottom of the (purple) spring inside. The black hook at the bottom of the scale is connected to a rod that passes up through the spring and is connected to the top of the spring. Pulling down on the hook presses down on the top of the spring.

**Numbers**

**Assumptions:** ideal spring

| Knowns | Unknowns |
|---|---|
| 0.01 m/kg | $k_s$ |
| | $U_s$ for 5 kg |

$k_s$, the spring constant, is a measure of the stiffness of a spring. It is defined using Hooke's Law:

$$\boxed{\overrightarrow{F_s} = -k_s \cdot \overrightarrow{\Delta x}} \qquad (5.2)$$

...where $\overrightarrow{F_s}$ is the force of the spring acting on whatever is extending or compressing it, $k_s$ is the spring constant, and $\overrightarrow{\Delta x}$ is the displacement of the end of the spring from its unstretched and uncompressed "equilibrium" position.

$k_s$ always has a positive value. The minus sign in Equation 5.2 tells us that the force applied by the spring is opposite the direction of the displacement. For example, if a spring is stretched to the right, it will be pulling to the left.

Hanging one kg from this scale compresses the spring by 0.01 m. We can use this information to find the spring constant with Equation 5.2. As can be seen in the free body diagram in Figure 5.7 when the system is in equilibrium, all of the forces that are acting on the 1 kg mass are in the vertical direction. So we only need to consider the $\hat{y}$ components of the forces.

The stiffness of the spring is measured in terms of its spring constant, in newtons per meter. A stiffer spring (higher spring constant)will compress less when a force is applied.

If a spring is not stretched or compressed, it cannot do any work, so it has no spring potential energy. But if work is done on the spring to stretch or compress it the work gets stored as spring potential energy. The farther a spring is stretched or compressed, the more energy it stores. A stiffer spring also stores more energy than a less stiff spring if both are stretched or compressed by the same amount.

An ideal spring stores the same amount of energy whether it is stretched or compressed, if the amount of extension or compression is the same. Spring potential energy is another type of mechanical energy.

Spring force, like other forces we have considered so far, is referred to as a "conservative force." This means that energy is conserved by the force– it is essentially converting energy between potential (spring) energy and kinetic energy. It doesn't matter how many times the spring compresses and expands, the energy will be conserved. A nonconservative force would be one that transforms energy into a form other than kinetic or potential energy. With a non-conservative force, the amount of kinetic and potential energy that is "lost" to another form of energy depends on the path that was traveled by the object. If it went back and forth multiple times, for example, the final total kinetic and potential energy would be lower.

mass, hook, & bar

$F_s$ $F_g$

Figure 5.7: FBD of the mass, hook, and metal bar when the system is in equilibrium.[1]
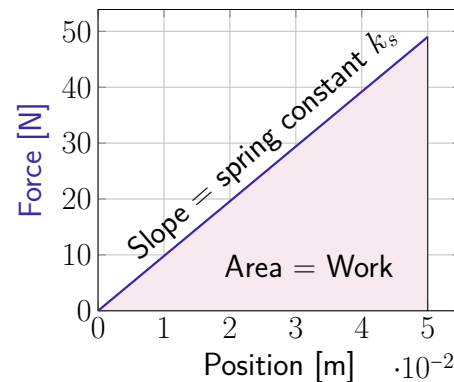


Figure 5.8: As the distance the spring is stretched (or compressed) increases, the force required to stretch (or compress) it also increases in linear fashion.[1]

$$F_{net,y} = 0 = (-k_s \cdot (-0.01\,\mathrm{m})) - (m \cdot g)$$

$$k_s = \frac{m \cdot g}{0.01\,\mathrm{m}} = \frac{(1\,\mathrm{kg}) \cdot \left(9.8\,\mathrm{m/s^2}\right)}{0.01\,\mathrm{m}} = 980\,\mathrm{N/m}$$

We have not only found the value for the spring constant; we have also found that the unit for the spring constant is $[\mathrm{N/m}]$.

To find energy stored in the spring, we need to consider how much work is used to stretch or compress the spring. Work is done by a force acting over a distance, so we need the distance that the spring is compressed when a 5 kg mass is hung from it. We could use Hooke's law, but it is simpler to note that we found $k_s$ by knowing that the spring compresses by 0.01 m/kg. So a simple ratio tells us the distance:

$$(5\,\mathrm{kg}) \cdot (0.01\,\mathrm{m/kg}) = 0.05\,\mathrm{m}$$

We can use Figure 5.8 to find the work that was done in compressing the spring by finding the area under the curve. It is triangular, so the area is $\frac{1}{2} b \cdot h$. The base is $0.05\mathrm{m}$, and the height is the magnitude of the force associated with the maximum distance:

$$F_s = k_s \cdot \Delta x$$

The area under the curve is therefore:

$$W = \frac{1}{2} \cdot \Delta x \cdot k_s \cdot \Delta x = \frac{1}{2} \cdot k_s \cdot \Delta x^2$$

So the potential energy stored in a spring is given by…

$$\boxed{U_s = \frac{1}{2} \cdot k_s \cdot \Delta x^2} \qquad (5.3)$$

## 5.3  Bouncing Ball

### Words

The collision between a tennis ball and a concrete driveway was recorded using high-speed video at a frame rate of 240 frames per second. Five consecutive frames are shown from left to right in Figure 5.9. The tennis ball has a mass of 57 g and a diameter of 6.6 cm. Analyze the motion, forces, energy, and momentum in this situation, considering the vertical direction only.

By using the tennis ball as a length scale, we should be able to determine the initial velocity from the first two images and the final velocity from the last two images.

We can also use the distance above the ground to determine the gravitational potential energy in each frame.

We do not know the exact length of the time the collision lasted, the time that the ball was touching the ground, but we can see that the collision had not yet started in the second image and was finished by the fourth image, so the time of the collision was less than 1/120 s.

Looking at the series of images, it is clear that the most significant changes revolve around the time of the collision. The ball is badly misshapen when it is touching the ground, and the direction of motion (and the direction of the momentum) reverses during that time as well. So we will focus our attention on the center frame.

### Graphics
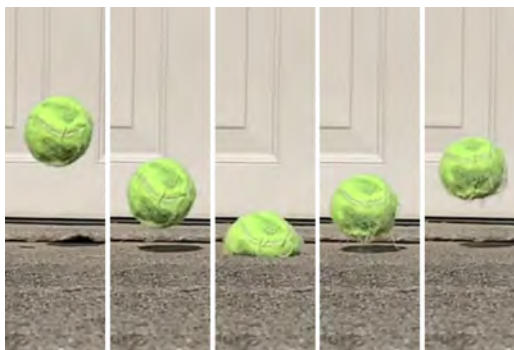


Figure 5.9: Still frames of a tennis ball bouncing off of concrete. The images, left to right, were taken at a rate of 240 frames per second. Note the flattening of the ball in the center frame.[1]

The still frames in Figure 5.9 are shifted compared to each other. The ball is not actually moving to the right. Careful comparison with the background shows that in fact the ball has a small horizontal velocity to the left. For this analysis, the slight horizontal motion is neglected and the ball is assumed to be moving only vertically.

### Numbers

**Assumptions:** No horizontal motion; $+\hat{y}$ direction is upward; no initial thermal energy

| Knowns | Unknowns |
|---|---|
| $m = 0.057\,\text{kg}$ | $\vec{v_i}$ |
| $d = 0.066\,\text{m}$ | $\vec{v_f}$ |
| $t_{frame} = \frac{1}{240}\,\text{s}$ | $\vec{a}$ |
| $t < \frac{1}{120}\,\text{s}$ | $\vec{F}$'s |
| $y_0 = 0.11\,\text{m}$ | $E$'s & $U$'s |
| $y_1 = 0.06\,\text{m}$ | $\vec{\Delta p}$ |
| $y_2 = 0.02\,\text{m}$ | |
| $y_3 = 0.05\,\text{m}$ | |
| $y_4 = 0.08\,\text{m}$ | |

$t$ is used for the duration of the collision. The different values for $y$ of the center of the ball have been estimated from the images and the diameter of the ball. There is not significant motion in the $\hat{x}$ direction, so we will restrict our analysis to the $\hat{y}$ direction.

$$\vec{v_i} = \frac{y_1 - y_0}{t_{frame}} = \frac{(0.06 - 0.11)\,\text{m}\,\hat{y}}{\frac{1}{240}\,\text{s}} = -12\,\text{m/s}\,\hat{y}$$

$$\vec{v_f} = \frac{y_4 - y_3}{t_{frame}} = \frac{(0.08 - 0.05)\,\text{m}\,\hat{y}}{\frac{1}{240}\,\text{s}} = +7.2\,\text{m/s}\,\hat{y}$$

$$\overline{a_{avg}} = \frac{\overline{\Delta v}}{\Delta t} = \frac{(+7.2 - (-12))\,\text{m/s}\,\hat{y}}{3 \cdot \frac{1}{240}\,\text{s}} = 1\,540\,\text{m/s}^2\,\hat{y}$$

The ball must be pressing down very hard on the concrete at that time, so there must also be a correspondingly large normal force pushing up on the ball from the concrete. This is what is able to change the direction of motion (and therefore also the momentum) of the ball.

What about the energy during that center frame? The force from the ground is external to the system we are considering, which is the ball. But the normal force does no work on the ball because work is done by a force acting over a distance moved, and the position of the concrete doesn't change at all.

Before and after the center frame, the ball clearly has kinetic energy. But in the center frame the ball does not. Also before and after the center frame the ball has more gravitational potential energy than in the center frame.

In all of our previous examples, when kinetic and gravitational potential energy were reduced it meant that the energy was converted to thermal energy. And that is partly true here as well. In the last frame the ball apparently has less kinetic energy and less gravitational potential energy than in the first frame. So some was lost to thermal energy. But not all!

The rest of the energy in the center frame was stored as spring (sometimes called elastic) potential energy in the ball. Like a compressed spring, a compressed ball also stores spring potential energy.
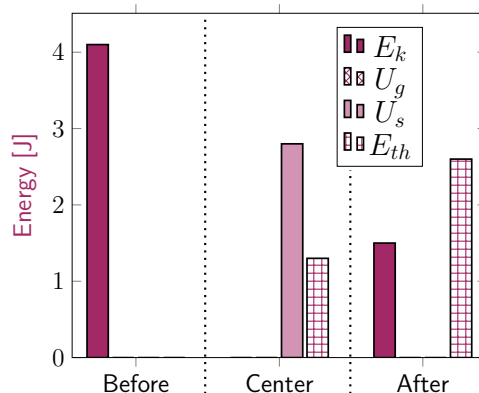


Figure 5.10: Energy bar graph for the collision. Initial thermal energy is taken to be zero. Gravitational potential energy is not zero, but is so small that it looks like zero compared to the heights of the other bars. [1]

The series of images provides a good estimate of all types of energy before and after the collision. During the collision, in the center frame, it is clear that kinetic and potential energy should both be at or near zero. But it is not clear for the center frame how the energy should be split between spring and thermal. For this bar graph it has been assumed that the thermal energy for the center frame is roughly half of the thermal energy that is present after the collision.

The time used above for the calculation of acceleration was three frames because $v_i$ was calculated between frames 0 & 1 and $v_f$ was calculated between frames 3 & 4, so three frames later. The acceleration found is so much larger than the acceleration due to gravity that we can safely ignore the effects of gravity, so in fact nearly all of the acceleration occurs between frames 2 & 4. The maximum acceleration is given by

$$\overrightarrow{a_{max}} \geq \frac{\overrightarrow{\Delta v}}{\Delta t} = \frac{(+7.2 - (-12)) \text{ m/s}\,\hat{y}}{2 \cdot \frac{1}{240} \text{ s}} = 2\,300 \text{ m/s}^2\,\hat{y}$$

Now we can find the maximum force during the collision:

$$\overrightarrow{F_{net,max}} = m \cdot \overrightarrow{a_{max}} \geq 131 \text{ N}\,\hat{y}$$

The initial and final kinetic energy is given by:

$$E_{k,i} = \frac{1}{2}m \cdot v_i^2 = 4.1 \text{ J}$$

$$E_{k,f} = \frac{1}{2}m \cdot v_f^2 = 1.5 \text{ J}$$

The gravitational potential energy when the ball is at the highest point in these frames can be found using $U_g = m \cdot g \cdot y$:

$$U_{g,max} = m \cdot g \cdot y_0 = 0.06 \text{ J}$$

Again with gravity, its contribution to energy is so small compared to the other types of energy that it can safely be ignored in this scenario.

## 5.4  Pushing a Barrel

### Words

A group of sailors in Figure 5.11 is attempting to slide a barrel across the deck of an aircraft carrier using water from a hose.

If the force that the water applies to the initially motionless 180 kg barrel starts at zero and slowly increases, the barrel remains in place until the applied force reaches $1\,200\,\text{N}$, at which point it begins to slide with an acceleration of $3\,\text{m/s}^2$.

Find the force of friction between the barrel and the deck of the aircraft carrier over the time that the sailors are spraying water at it.

Before the water is sprayed at the barrel, the barrel is motionless. Since the barrel is motionless, we know that the net force on it has to be zero, so the normal force exactly balances the force of gravity in the vertical direction. In the horizontal direction the forces are also balanced, and there is no applied force so the force of friction at that time is zero.

That sounds wrong, because if you were to push on the barrel the friction would probably be strong enough to prevent you from moving it. But the key here is that phrase, *if you were to push*. If you push to the left, friction will push to the right. If you push to the right, friction will push to the left. Friction will do whatever it has to do to keep the barrel in place. If nothing else tries to push the barrel, friction does nothing!

### Graphics



Figure 5.11: Sailors trying to move a barrel using water from a fire hose.[26]
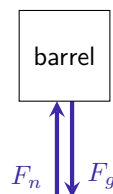


Figure 5.12: FBD of the stationary barrel with no applied force from the water.[1]

### Numbers

**Knowns**

$m = 180\,\text{kg}$

$g = 9.8\,\text{m/s}^2$

$0 \le F_{x,applied} \le 1\,200\,\text{N}$

$\vec{a}_f = 3\,\text{m/s}^2\,\hat{x}$

$\vec{a} = 0$ when $F_{x,applied} < 1\,200\,\text{N}$

**Unknowns**

$\overrightarrow{F_f}$

There are two types of frictional force, "static" $\left(\overrightarrow{F_{f,s}}\right)$ when the objects are not moving with respect to each other and "kinetic" $\left(\overrightarrow{F_{f,k}}\right)$ when they are sliding against each other.

The magnitude of the maximum static frictional force between two objects is...

$$\boxed{F_{f,s,max} = \mu_s \cdot F_n} \tag{5.4}$$

...where $\mu_s$ is the coefficient of static friction and $F_n$ is the magnitude of the normal force. "$\mu$" is the lowercase Greek letter "mu."

The magnitude of the kinetic frictional force between two objects is...

$$\boxed{F_{f,k} = \mu_k \cdot F_n} \tag{5.5}$$

...where $\mu_k$ is the coefficient of kinetic friction and $F_n$ is the magnitude of the normal force.

When the water is sprayed at the barrel, it applies a force to the right. As this applied force increases, the frictional force increases along with it, preventing the barrel from moving…until the force from the water reaches $1\,200\,\mathrm{N}$. Then something changes. The friction is no longer strong enough to hold the barrel in place–we have found the maximum "static" frictional force between the barrel and the deck.

Static frictional force is present when two objects are not moving with respect to each other. Once they start moving, it is "kinetic" frictional force that takes over. Unlike static frictional force, the magnitude of the kinetic frictional force does not depend upon any other applied forces. Kinetic frictional force is always less than or equal to the maximum static frictional force–typically it is much lower.

Since this situation is on an aircraft carrier, we can also consider the situation where the deck is rising and falling. If the deck is accelerating downward, the normal force will be smaller than the force of gravity; in other words, the "apparent weight" (the normal force required to keep the barrel on the deck) will be smaller. This reduces the force of friction between the surfaces, making the barrel easier to move. On the other hand, if the deck is accelerating upward the apparent weight of the barrel will be larger, increasing the frictional force and making it more difficult to move.
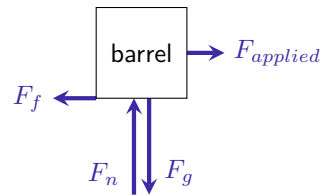


Figure 5.13: FBD of the stationary barrel with 600 N of applied force.[1]
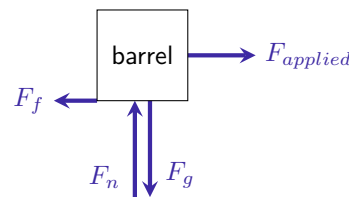


Figure 5.14: FBD of the moving barrel with 1200 N of applied force.[1]
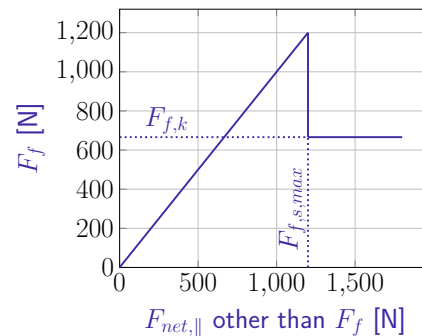


Figure 5.15: Force of friction as a function of net force from other sources parallel to a surface.[1]

In this physical scenario, the magnitude of the normal force is the same as the magnitude of the force of gravity and the maximum static frictional force is $1\,200\,\mathrm{N}$, because when the applied force reaches this value the barrel begins to slide. Using Equation 5.4 we can find $\mu_s$:

$$\mu_s = \frac{F_{f,s,max}}{F_n} = \frac{1\,200\,\mathrm{N}}{(180\,\mathrm{kg}) \cdot (9.8\,\mathrm{m/s^2})} = 0.68$$

To find $\mu_k$, we need to know $F_{f,k}$, which we can find by applying Newton's Second Law to Figure 5.14.

$$F_{net,x} = F_{applied} - F_{f,k} = m \cdot a_x$$

Solving for $F_{f,k}$ gives 660 N. Then, using Equation 5.5…

$$\mu_k = \frac{F_{f,k}}{F_n} = \frac{660\,\mathrm{N}}{(180\,\mathrm{kg}) \cdot (9.8\,\mathrm{m/s^2})} = 0.37$$

Notice that there are no units on these coefficients, and that $\mu_k$ is considerably smaller than $\mu_s$, which is typical. Tables of $\mu_k$ & $\mu_s$ for various combinations of surfaces can be found by searching the internet for a "table of coefficients of friction."

## 5.5   Sledding at White Sands

### Words

At White Sands National Monument in New Mexico, USA, the sand is slippery enough that children can slide down the sand dunes on plastic sleds. If the angle of the slope is less than $35°$ from the horizontal, a 25 kg child cannot slide down. But at $35°$ from the horizontal the child can slide down with an acceleration of $2\,\mathrm{m/s^2}$.

What are the static coefficient of friction and the kinetic coefficient of friction for the surface between the sled and the sand? What is the final speed of the child upon reaching the bottom of a 2.5-m-long sand dune if they started at the top with zero speed? What else can we find for this physical scenario?

Coefficients of friction are ratios of the frictional force to the normal force between two surfaces. Frictional force is not dependent upon the surface area. That is important in this example, because the curved bottoms of the sleds make it difficult to determine the surface area of the sled that is in contact with the sand.

If surface area doesn't affect friction, why are car tires and most shoes patterned with treads? The main function of the treads is to allow the shoe or the tire to reach the ground firmly if there is water on the surface. A completely smooth car tire would have the same amount of friction as a treaded tire but would easily go out of control, "hydroplaning" if there were any water on the ground.

### Graphics



Figure 5.16: Children sledding at White Sands National Monument.[27]
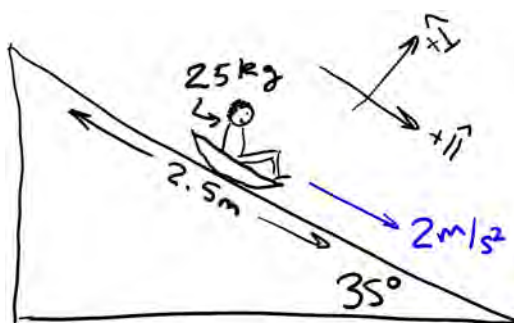


Figure 5.17: Sketch of a child sledding down the sand dune.[1]

### Numbers

**Assumptions:** $+\hat{\parallel}$ direction is down the slope; $+\hat{\perp}$ direction is normal to the slope

| Knowns | Unknowns |
|---|---|
| $\theta = 35°$ | $\mu_s$ |
| $g = 9.8\,\mathrm{m/s^2}$ | $\mu_k$ |
| $\overrightarrow{a} = 2\,\mathrm{m/s^2}\;\hat{\parallel}$ | $v_f$ |
| $\Delta x_\parallel = 2.5\,\mathrm{m}$ | ??? |
| $v_0 = 0$ | |
| $m = 25\,\mathrm{kg}$ | |

We can use acceleration and displacement to find $v_f$:

$$2a_\parallel \cdot \Delta x_\parallel = v_f^2 - v_i^2$$

$$v_f = \sqrt{2a_\parallel \cdot \Delta x_\parallel} = 3.16\,\mathrm{m/s}$$

By comparing the figures in this section, it can be seen that the net force down the slope is given by:

$$F_{net,\parallel} = F_{g,\parallel} - F_f = m \cdot g \cdot \sin\theta - F_f$$

Which $F_f$ we use depends on whether the sled is moving or not. Since the angle given is just at the point where the child starts to slide, we can use it to find $F_{s,max}$ and if we include the acceleration given at this angle we can also use it to find $F_{f,k}$. Since $a_\perp = 0$, we can see from the figures in this section that $F_n = F_{g,\perp}$. This is needed to help us find $F_{f,s,max}$ and $F_{f,k}$:

Some shoes, for example golf shoes or crampons that are used for ice climbing, are built with very sharp spikes on the bottom. These do not actually increase the friction with the surface, but increase traction by breaking into the surface, making vertical surfaces where the golfer or climber can apply horizontal normal forces instead of relying on friction.

If we consider the work and energy involved in this scenario, the child starts out not moving at the top of a sand dune, so they have gravitational potential energy but no kinetic energy. As they slide down they are accelerating in the direction of their motion, so they are speeding up, increasing their kinetic energy. They are also losing gravitational potential energy as they go down.

The frictional force opposes the child's motion as they go down. A force opposite the direction of motion does negative work on the system, so in this case the friction is doing negative work on the child, removing kinetic energy. That energy is transformed into thermal energy.

So at the bottom of the sand dune the child has kinetic energy but no gravitational potential energy. And thermal energy has also been released in the process, warming the sand and the sled.

Frictional force is the first nonconservative force that we have seen. Frictional force transforms kinetic energy into thermal energy, not potential energy.
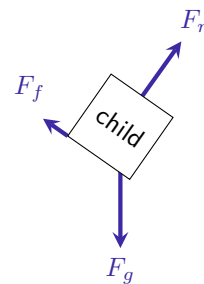


Figure 5.18: FBD of child on sand dune.[1]
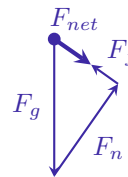


Figure 5.19: Sum of forces for the child.[1]
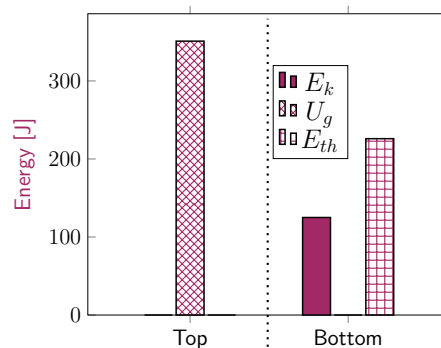


Figure 5.20: Energy bar graph for the child.[1]

$$F_{f,s,max} = \mu_s \cdot F_n = \mu_s \cdot F_{g,\perp} = \mu_s \cdot m \cdot g \cdot \cos\theta$$

Similarly, $F_{f,k} = \mu_k \cdot m \cdot g \cdot \cos\theta$. $F_{net,\parallel} = 0$ when considering $\mu_s$, so combining the equations above…

$$0 = m \cdot g \cdot \sin\theta - \mu_s \cdot m \cdot g\cos\theta$$

Solving for $\mu_s$ and using the trigonometric identity that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ …

$$\mu_s = \tan\theta = 0.70$$

Following the same steps for $\mu_k$ and using Newton's Second Law since the sled is accelerating gives…

$$m \cdot a_\parallel = m \cdot g \cdot \sin\theta - \mu_k \cdot m \cdot g\cos\theta$$

$$\mu_k = \frac{g \cdot \sin\theta - a_\parallel}{g \cdot \cos\theta} = 0.45$$

What about energy? $E_{k,i} = 0$; no springs, so no $U_s$; if we set $y = 0$ at the bottom of the slope then $U_{g,f} = 0$; and we can think of the external force of friction not as doing work but as a source of thermal energy. Setting $E_{th,i} = 0$, conservation of energy gives…

$$\cancelto{0}{E_{k,i}} + \cancelto{0}{U_{s,i}} + U_{g,i} + \cancelto{0}{E_{th,i}}$$
$$= E_{k,f} + \cancelto{0}{U_{s,f}} + \cancelto{0}{U_{g,f}} + E_{th,f}$$

We can find $E_{k,f}$ from $v_f$ and $U_{g,i}$ from Figure 5.17 and some trigonometry:

$$E_{th,f} = U_{g,i} - E_{k,f}$$
$$= \left(m \cdot g \cdot \left(\Delta x_\parallel \cdot \sin\theta\right)\right) - \left(\frac{1}{2}m \cdot v_f^2\right) = 226 \text{ J}$$

## 5.6   The Truth About Gravity

**Words**

So far, we have considered gravity to be an acceleration with a magnitude of $g = 9.8\,\mathrm{m/s^2}$, caused by a force that is always pointing downward.

That is a good model of the force of gravity near the surface of the earth, which is where most of us will probably spend most of our lives. So it is an approximation that works well in many situations that we will face. But when you leave the surface of the earth, going up or down, that model doesn't work any more.

We will now consider the force of gravity acting on a 1 kg mass that is placed either at the exact center of the earth or one earth radius above the surface of the earth.

At the earth's surface, the force of gravity from the earth acting on a 1-kg mass is 9.8 N, pointing downward. But what if you were at the center of the earth? There is no longer a "downward" direction! Imagine the earth being two pieces, the Southern and Northern hemispheres. At the center of the earth, the gravitational force caused by the Southern hemisphere would pull the mass toward the South pole, but the Northern hemisphere would pull with an equal but opposite force toward the North pole. These forces cancel, so we can use the symmetry of the situation to show that the earth's gravitational force acting on a mass at the center of the earth is zero. Symmetry arguments like this often provide helpful insights into physical situations.

**Graphics**
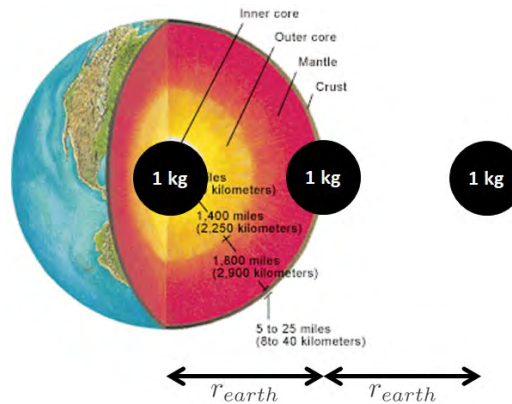


Figure 5.21: Cutaway of the earth, with a 1 kg mass shown at three different positions.[28]
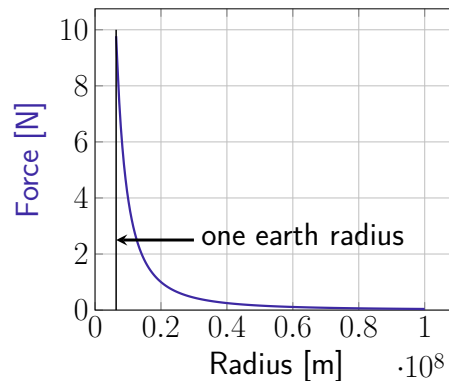


Figure 5.22: Magnitude of gravitational force between the earth and a 1 kg mass, as a function of distance.[1]

**Numbers**

| Knowns | Unknowns |
|---|---|
| $m = 1\,\mathrm{kg}$ | $\overrightarrow{F_g}$ at position $r_0$ |
| $r_0 = 0$ | $\overrightarrow{F_g}$ at position $\overrightarrow{r_1}$ |
| $r_1 = 2\,r_{earth}$ | |

In the table above, radii "$r$" are measured from the center of the earth to the center of the 1 kg mass. The $+\hat{r}$ direction is directly away from the center of the earth.

For any location other than the surface of the earth, the mathematical model $\overrightarrow{F_g} = -m \cdot g\,\hat{y}$ does not work well. Instead, we need to use Newton's Law of Universal Gravitation:

$$F_g = \frac{G \cdot m_1 \cdot m_2}{r^2} \qquad (5.6)$$

…where $G$ is the Universal gravitation constant $6.67 \times 10^{-11}\,\frac{\mathrm{N \cdot m^2}}{\mathrm{kg^2}}$, $m_1$ & $m_2$ are the masses of two spherical objects between which the force is acting, and $r$ is the distance between the centers of the two objects. Gravitational force is always attractive, so the force on each object is directed toward the other object. We can use Equation 5.6 to find the magnitude of the force when the 1 kg mass is one radius above the surface of the earth, so at a distance of two earth radii from the center of the earth. An internet search can be used to find that the radius of the earth is $6.37 \times 10^6\,\mathrm{m}$ and the mass of the earth is $5.97 \times 10^{24}\,\mathrm{kg}$.

As we move from the center of the earth to the surface of the earth, the force of gravity on our 1 kg mass increases smoothly, reaching a force of 9.8 N at the surface of the earth.

Above the surface, gravity follows an "inverse square law," meaning that the strength of the force is inversely proportional to the square of the distance from the earth's center. At *one* earth radius above the *surface* of the earth, we are *two* earth radii from the *center* of the earth. Our distance has doubled, so the force of gravity is reduced by a factor of four $\left(\frac{1}{2^2}\right)$.

What about gravitational potential energy? Work is required to lift a 1 kg mass up away from the center of the earth, regardless of the distance from the center of the earth. So gravitational potential energy is at a minimum at the center of the earth and increases as we move away from the center.

These descriptions are valid for anything that has mass; the gravitational force caused by anything is proportional to its mass.

The more refined model of gravity in this section is consistent with 19th century understanding. But in the 20th century, gravity began to be understood as a curvature of space-time. And in the 21st century perhaps quantum gravity or a better understanding of "dark matter" and "dark energy" will give us an entirely new and better model for gravity. The models in physics and all sciences are constantly growing and evolving as we grow in our understanding of the universe.
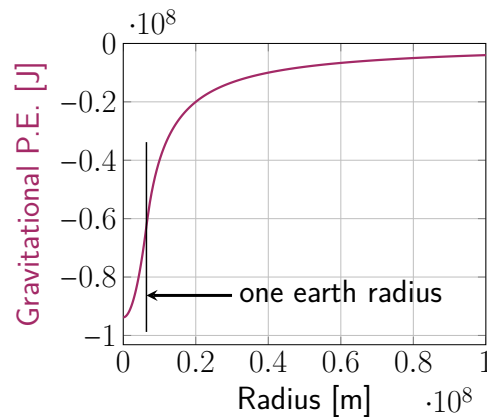


Figure 5.23: Gravitational potential energy of the earth and a 1 kg mass, as a function of distance.[1]
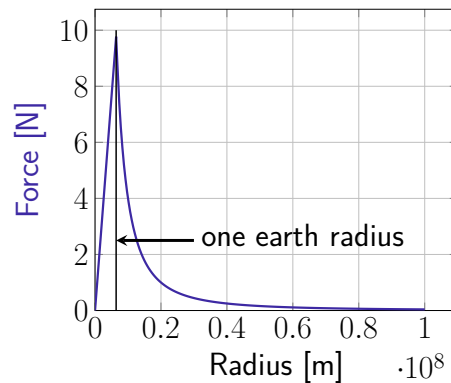


Figure 5.24: Magnitude of gravitational force between the earth and a 1 kg mass, as a function of distance, including when the mass is inside the earth.[1]

$$F_g = \frac{\left(6.67 \times 10^{-11} \, \frac{\text{N·m}^2}{\text{kg}^2}\right)\left(5.97 \times 10^{24} \, \text{kg}\right)\left(1 \, \text{kg}\right)}{\left(2 \cdot 6.37 \times 10^6 \, \text{m}\right)^2}$$

$$\overrightarrow{F_g} = -2.45 \, \text{N} \, \hat{r}$$

Let's try Equation 5.6 again to find the magnitude of the force at the center of the earth.

$$F_g = \frac{\left(6.67 \times 10^{-11} \, \frac{\text{N·m}^2}{\text{kg}^2}\right)\left(5.97 \times 10^{24} \, \text{kg}\right)\left(1 \, \text{kg}\right)}{\left(0 \, \text{m}\right)^2}$$

$$F_g = \infty \, \text{N}$$

An infinitely large force! That cannot be correct. This model only works outside of the mass distribution of the objects themselves, so not, for example, inside the earth.

Along with this new model of gravitational force, we have a new expression for gravitational potential energy, which is also only valid outside of the mass distributions of the objects themselves:

$$\boxed{U_g = \frac{-G \cdot m_1 \cdot m_2}{r}} \tag{5.7}$$

Notice that with this expression we no longer have the option of choosing our height where $U_g = 0$. Instead, $U_g = 0$ at $r = \infty$, and $U_g$ is negative everywhere else.

Inside a uniform mass distribution, the magnitude of the force is smaller than the value calculated from Equation 5.6, increasing linearly with radial distance. The gravitational potential energy inside a mass distribution is higher than the value calculated from Equation 5.7.

## 5.7   Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

### General

- The stiffness of a spring is defined by its spring constant $k$, which always has a positive value and is measured in units of newtons per meter.

### Forces

- A pulley can be used to change the direction of the tension in a rope.

- "Mechanical advantage" is the ratio of the output force to the input force for any machine.

- "Conservative" forces are those that involve energy transformations with kinetic and potential energy.

- "Nonconservative forces" are those that involve energy transformations that include forms of energy other than kinetic and potential energy.

- Ideal springs follow Hooke's Law, where force is proportional to the amount of extension or compression.

- The force of friction between two surfaces depends upon the surfaces themselves and the normal force that the surfaces exert on each other.

- If two surfaces are motionless with respect to each other, the static force of friction between them will be exactly enough to cancel out all other forces in the direction parallel to the surfaces, unless the sum of all of the other forces is larger than the maximum static force of friction, in which case the surfaces begin to move with respect to each other.

- If two surfaces are moving with respect to each other, the kinetic force of friction between them will be in the direction opposing the motion.

- If the floor on which an object sits is accelerating upward, the apparent weight of the object as measured by the normal force is larger than if the floor were stationary or moving at constant velocity.

- If the floor on which an object sits is accelerating downward, the apparent weight of the object as measured by the normal force is smaller than if the floor were stationary or moving at constant velocity.

- Frictional force does not depend upon the contact area of the surfaces.

- The gravitational force between two objects is proportional to the masses of the objects and inversely proportional to the square of the distance between their centers.

- Gravitational force is always attractive.

- Gravitational force drops to zero as the radius gets smaller if one object is inside the mass distribution of the other object.

### Motion

- A pulley can be used to change the direction of the motion of a rope.

- A position vector can be defined in a radial direction, along a radius from a center point. The positive direction is away from the center point.

## Energy

- Stretched or compressed springs store spring (also called elastic) potential energy.

- Spring potential energy is a form of mechanical energy.

- Deformed objects can store spring potential energy.

- Gravitational potential energy of two objects increases as the distance between the objects increases.

## Mathematical Models

| equation | | restrictions on the validity of the equation |
|---|---|---|
| $MA = \frac{F_{output}}{F_{input}}$ | (5.1) | -none- |
| $\overrightarrow{F_s} = -k_s \cdot \overrightarrow{\Delta x}$ <br><br> "Hooke's Law" | (5.2) | For objects that obey Hooke's Law |
| $U_s = \frac{1}{2} \cdot k_s \cdot \Delta x^2$ | (5.3) | For objects that obey Hooke's Law |
| $F_{f,s,max} = \mu_s \cdot F_n$ | (5.4) | -none- |
| $F_{f,k} = \mu_k \cdot F_n$ | (5.5) | -none- |
| $F_g = \frac{G \cdot m_1 \cdot m_2}{r^2}$ <br><br> "Newton's Law of Universal Gravitation" | (5.6) | Outside of the mass distribution of spherical objects. |
| $U_g = \frac{-G \cdot m_1 \cdot m_2}{r}$ | (5.7) | Outside of the mass distribution of spherical objects. |

## 5.8 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

### Level 1 - Remember

5.1 **[W]** Mechanical advantage is the ratio of output _____ to input _____.

5.2 **[W]** The kinetic force of friction is only relevant if the two surfaces are _____ relative to one another.

5.3 **[W]** In what direction does the kinetic force of friction act?

5.4 **[W]** The static force of friction is only relevant if the two surfaces are _____ relative to one another.

5.5 **[W]** In what direction does the static force of friction act?

5.6 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

### Level 2 - Understand

5.7 **[W & N]** A certain ideal (perfect) machine has a mechanical advantage of 10. A force of 1 000 N is supplied at the input of the machine. How large is the output force?

5.8 **[W & N]** A certain ideal (perfect) machine has a mechanical advantage of 10. 1 000 J of energy is supplied at the input of the machine. How much is the output energy?

5.9 **[G]** Draw free body diagrams for the tennis ball in the left, center, and right frames of Figure 5.9.

5.10 **[W, G, & N]** Find the momentum of the child at the top and bottom of the sand dune in Section 5.5. Momentum is conserved for an isolated system, so if the child's momentum changed, what caused that change?

### Level 3 - Apply

5.11 **[W, G, & N]** The tension force in rope 2 was never found in Section 5.1. Find the amount of tension. Include a free-body diagram along with your reasoning.

5.12 **[G & N]** The amount of potential energy stored in the spring in Section 5.2 is never actually calculated. How much is it?

5.13 **[W & N]** Find the change in the momentum of the tennis ball during the collision in Section 5.3. Use the change in momentum to find the force applied to the tennis ball during the collision. Does it agree with the $\overrightarrow{F_{net,max}}$ that is found in the text? Explain why or why not.

5.14 **[W & N]** If the mass of the barrel in Section 5.4 were doubled, what effect would that have on the maximum static frictional force and the kinetic frictional force?

5.15 **[W, G, & N]** Now that we have Newton's Universal Law of Gravitation, does that mean that we can no longer use $9.8\,\mathrm{m/s^2}$ as the acceleration due to gravity at the earth's surface? Explain why or why not. You will need to look up information about the Earth's mass and radius to answer this question.

## Level 4 - Analyze

5.16 **[W, G, & N]** How much force does the pulley system in Section 5.1 apply to the ceiling?

5.17 **[W, G, & N]** If the deck of the ship in Section 5.4 remained level but started accelerating downward at $1 \mathrm{\ m/s^2}$ because of stormy seas...

    (a) ...would the horizontal acceleration of the barrel increase, decrease, or remain the same if it had already started moving? Explain your reasoning.

    (b) ...would the static force of friction increase, decrease, or remain the same if the barrel was not yet moving and there was no applied force in the horizontal direction? Explain your reasoning.

    (c) ...would the static force of friction increase, decrease, or remain the same if the barrel was not yet moving and the applied force in the horizontal direction was 200 N? Explain your reasoning.

    (d) ...would the horizontal motion of the barrel change if it was not yet moving and the applied force in the horizontal direction was $1\,100 \mathrm{\ N}$? Explain your reasoning.

5.18 **[W & N]** Given the physical scenario described in Section 5.5, is there an angle at which a child would go down the slope at constant speed if they were given an initial push to start them moving? Explain why or why not. If it is possible, find the angle.

## Level 5 - Evaluate

5.19 **[W, G, & N]** How would the values of each of the following change (increase, decrease, or stay the same) in Section 5.1 if the friction and the mass of the pulleys were not negligible? Assume that all of the "knowns" keep their same values. Explain your reasoning.

    (a) $F_{t,3}$

    (b) $F_{applied}$

    (c) $P_{lift}$

    (d) $P_{applied}$

    (e) $MA$

5.20 **[G & N]** If the spring constant were doubled and the same mass was hung from the scale in Section 5.2, how would that affect the amount of energy stored in the spring? Explain your reasoning.

5.21 **[G & N]** If the mass were doubled and the spring constant was kept the same in Section 5.2, how would that affect the amount of energy stored in the spring? Explain your reasoning.

5.22 **[W, G, & N]** The height of the bars in the "Center" position of Figure 5.10 are not well defined just from examining Figure 5.9. What are the maximum and minimum possible heights for the bars representing spring potential energy and thermal energy?

5.23 **[W & N]** In Section 5.5 the coefficients of friction are found. Given those coefficients of friction, if the child's mass were doubled, how would that affect the angle at which the maximum static frictional force can no longer keep the child from sliding down the sand dune? Explain your reasoning.

5.24 **[W & N]** In Section 5.5 the coefficients of friction are found. Given those coefficients of friction, if the child's mass were doubled, how would that affect the acceleration of the child as they slid down the $35°$ slope? Explain your reasoning.

5.25 **[N]** Up until now we have been approximating the force of gravity on an object at the surface of the earth using $m \cdot g$. Now that we know a better way to describe gravity, what percentage difference is there in the actual force of gravity between the highest point on the earth's surface (the peak of Mount Everest) and the lowest known point on the earth's surface (Challenger Deep, in the Pacific Ocean)? Be sure to list any assumptions that you are making.

## Level 6 - Create

5.26 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Add the main ideas from this chapter to the concept map that you began for the question at the end of Chapter 1.

5.27 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

5.28 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

# Chapter 6

# Curving Paths

We have already learned that if an object experiences an unbalanced force in the direction in which it is already moving, the force does positive work on the object, increasing its kinetic energy, its momentum, and also its speed. Conversely, if an object experiences an unbalanced force opposite the direction in which it is already moving, the force does negative work on the object, decreasing its kinetic energy, its momentum, and also its speed.

Now it is time to start looking at unbalanced forces that are pointing in directions that are not parallel to the motion of an object. When that happens, the object follows a curved path.

Looking at the image of the fire dancer, try to imagine the forces that are involved. The dancer is holding two flaming balls that are hanging from chains. She spins them around, making intricate patterns in the air. The flaming balls experience forces due to both gravity and the tension in the chains. How do those combined forces make the balls move in such complicated patterns?

What is happening to the momentum of the balls as they follow these curved paths?



Figure 6.1: A fire poi dance. The flaming ball on the near side of the dancer traces out a circular path, while the ball on the far side of the dancer follows a more complicated path.[29]

## 6.1  Cliff Diving

### Words

Figure 6.2 shows a diver jumping horizontally off of the edge of a cliff. After leaving the cliff, the diver's body is in free fall, affected only by the force of gravity. Because of the initial horizontal velocity and the vertical acceleration, the diver's body follows a curved path.

If the camera took four images per second and the diver jumped horizontally at 3 m/s, for how much time was the diver in the air before hitting the water, what is the height of the cliff, at what speed did the diver enter the water, and what was the horizontal displacement of the diver over that time?

The key to understanding this situation is realizing that the horizontal direction is independent of the vertical direction. This idea was explored in Chapter 4, where the focus was on velocity, momentum, and forces. The same principle applies to position as well.

The force of gravity pulls the diver down into the water in exactly the same amount of time as if she had fallen straight down into the water from the same height. This is perhaps most easily understood by thinking of it in terms of frames of reference. If you are standing on a motionless train and you hold a ball straight out and drop it to the floor, the ball will be in the air for a specific amount of time before hitting the floor.

### Graphics

Figure 6.2: A cliff diver jumps horizontally off of a cliff. The individual images of the diver are at equal time intervals.[30]

### Numbers

**Assumptions:** $+\hat{x}$ is to the right; $+\hat{y}$ is upward; air resistance is negligible

| Knowns | Unknowns |
|---|---|
| $\vec{v_0} = -3\,\mathrm{m/s}\,\hat{x}$ | $t_{tot}$ |
| $t_{image} = 0.25\,\mathrm{s}$ | $h_{cliff}$ |
| $g = 9.8\,\mathrm{m/s^2}$ | $\Delta x$ |
| | $v_f$ |

If we use the image where the diver has one foot on the cliff as $t = 0$, and subsequent images are at 0.25-s intervals, then the splash into the water occurs at

$$t_{tot} = 6 \cdot t_{image} = 1.5\,\mathrm{s}$$

To find the height of the cliff, we can separate the $\hat{x}$ & $\hat{y}$ components of position:

$$x = x_0 + v_{0x} \cdot t + \frac{1}{2}a_x \cdot t^2$$

$$y = y_0 + v_{0y} \cdot t + \frac{1}{2}a_y \cdot t^2$$

In free-fall, the only force acting on an object is gravity, so as long as the object is near the surface of the earth $\vec{a} = -g\,\hat{y}$. So the horizontal and vertical position of an object in free-fall are…

$$\boxed{x = x_0 + v_{0x} \cdot t} \tag{6.1}$$

and

If you are on the same train, holding the same ball and dropping it in the same way, but the train is moving at a constant velocity, from your reference frame the ball will behave in exactly the same way. And it will hit the floor in exactly the same amount of time.

In the reference frame of somebody watching the train go by, the ball starts in your hand not at rest but moving with the same horizontal velocity as the train. When you drop the ball, it will continue to move horizontally along with the train, but it will drop vertically. Each person sees the ball following a different path in their reference frame, but both see the ball in the air for the same amount of time.

The horizontal displacement that the ball travels while falling in the reference frame of the person on the train is zero, because the horizontal velocity is zero in that reference frame. But in the reference frame of the person watching the train go by, the horizontal displacement of the ball would be the velocity of the train multiplied by the time that the ball is in the air, since velocity is displacement over time.

In Figure 6.3, the horizontal spacing of the images are almost equally spaced–the slight decrease in spacing on the left is most likely due to the angle of the camera. The vertical spacing, on the other hand, increases with each successive image. This shows acceleration in the vertical direction, due to the force of gravity. It is also clear from the lengths of the arrows that the diver's speed is increasing during the fall.
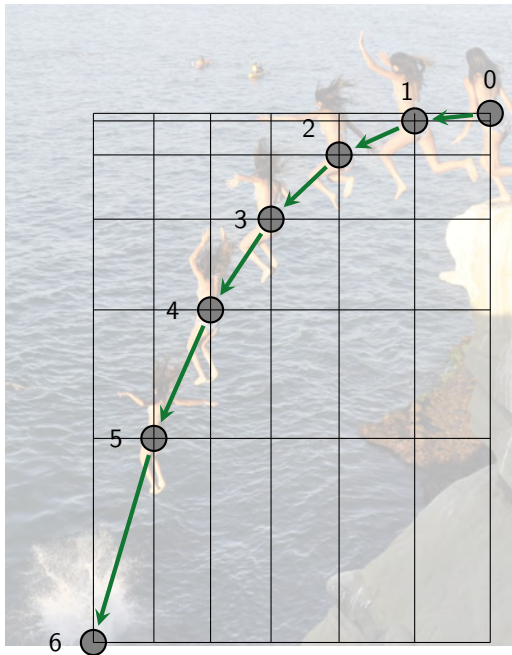


Figure 6.3: 2-D motion map of the diver[31]

$$y = y_0 + v_{0y} \cdot t - \frac{1}{2} g \cdot t^2 \qquad (6.2)$$

Note that the only connection between the $\hat{x}$ & $\hat{y}$ directions is the time. Often questions about two-dimensional physical scenarios can be answered by using information known about one direction to solve for time, and then using time to solve for information about the other direction. In this case we are given the time and can use it to solve for information about both directions.

The height of the cliff is the opposite of the displacement in the y direction, $y_0 - y$, since the diver starts at the top and ends at the bottom. So...

$$h_{cliff} = \frac{1}{2} g \cdot t_{tot}^2 - \underbrace{v_{0y} \cdot t_{tot}}_{0} = 11\,\text{m}$$

The diver's horizontal displacement comes from Equation 6.1:

$$\Delta x = v_{0x} \cdot t = -4.5\,\text{m}$$

Now we can find the speed of the diver by calculating $v_{x,f}$ and $v_{y,f}$ and using the Pythagorean theorem. There is no horizontal acceleration, so $v_{x,f} = v_{0x}$. In the vertical direction...

$$v_{f,y} = v_{0y} + a_y \cdot t_{tot} = 0 - g \cdot t_{tot} = -14.7\,\text{m/s}$$

$$v_f = \sqrt{v_{f,x}^2 + v_{f,y}^2} = 15\,\text{m/s}$$

## 6.2  Basketball Bounce

### Words

Figure 6.4 shows a ball bouncing, with images captured using a stroboscopic light source that flashed 25 times per second. We will assume that the photograph was taken somewhere near the surface of the earth.

We will analyze the photograph using all of our tools to see what we can learn just from the images.

This photograph looks like a 2-D map of the motion of the ball, but it doesn't contain arrows like a normal motion map. Can we determine which direction the ball is moving? The biggest indicator is the height of the peaks. The first peak is much higher than the second, and the speed of the ball when at the peaks looks like it is probably about the same for each peak.

So the ball has more mechanical energy at the top of the left peak than it has at the top of the right peak. Most likely some of the initial energy was transformed to thermal energy during an inelastic collision with the floor, so the peak on the left must be the first one, and the ball is moving to the right.

The image of the ball on the far right is considerably smaller than the image of the ball on the left, so it must also be moving away from the camera. That means any measurements of angles or distances will not be exact.

### Graphics



Figure 6.4: This single photograph of a bouncing ball was taken using a stroboscopic light source that flashed 25 times per second .[32]



Figure 6.5: The same photograph as above, with the images numbered for reference.[32]

### Numbers

**Assumptions:** $\theta$ is measured up from the horizontal direction; gravity near the surface of the earth; air resistance is negligible

| Knowns | Unknowns |
|---|---|
| $t_{image} = 0.04\,\text{s}$ | ??? |
| $g = 9.8\,\text{m/s}^2$ | |

There are 15 spaces between the images of the ball for the left bounce so…

$$t_{left} = 15 \cdot t_{image} = 0.6\,\text{s}$$

Similarly, $t_{right} = 0.52\,\text{s}$. These "times of flight" can be used to find initial and final vertical velocities and heights. Measuring the angles gives information about the horizontal direction. In this situation, *when the initial and final heights are the same* and the object is in free-fall, there are three "equations of projectile motion" that can be used. They are derived from the equations of motion that we have already been using.

$$t_{flight} = \frac{2\,v_0 \cdot \sin\theta}{g}$$

$$h_{max} = \frac{v_0^2 \cdot \sin^2\theta}{g}$$

$$R = \frac{v_0^2 \cdot \sin\left(2\theta\right)}{g}$$

The path of motion that the ball follows is in the shape of a parabola. This is the expected shape whenever an object is in free-fall near the surface of the earth. Other names for this type of motion are "projectile motion" and "ballistic motion."

If we neglect air resistance, which is typically very small at low speeds, then the only force that acts on the ball while it is in the air is the force of gravity, acting downward. When the ball is moving upward, gravity is doing negative work on the ball, slowing it down. When the ball is moving downward, gravity is doing positive work on the ball, speeding it up.

When the ball hits the floor, the ball applies a large normal force downward onto the floor and the floor pushes up on the ball with an equally large normal force. During the collision with the floor, the vertical part of the velocity of the ball changes drastically, from a large downward speed to a large upward speed. Looking at the horizontal direction, however, shows that the horizontal velocity did not change much. The ball continues to move to the right at a fairly uniform rate for the whole time that it was being imaged.

ball

$F_g$
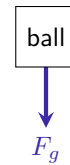
Figure 6.6: FBD of the ball for any time when it is in the air, images 2-15 and 17-28 in Figure 6.5.[1]

$F_n$

ball

$F_g$

Figure 6.7: FBD of the ball when it is in contact with the ground, images 16 & 29 in Figure 6.5. Lengths of the arrows are not exact, but are intended to signify that $F_g$ is the same as in Figure 6.6 and that $F_n > F_g$.[1]

...where $\theta$ is the angle between $v_0$ and the horizontal direction, approximately $70°$ in this case for both bounces; $t_{flight}$ is the total time in free-fall; $h_{max}$ is the vertical distance from the initial position to the peak; and $R$ is the horizontal range. It is important to remember that these are not "magic" equations that always give correct answers when you don't know what to do. They are specific to the situation of free-fall near the surface of the earth when initial height is equal to final height. They will not work in any other situation.

Using these equations, we find:

|  | Left Bounce | Right Bounce |
|---|---|---|
| $v_0$ | $3.1\,\mathrm{m/s}$ | $2.7\,\mathrm{m/s}$ |
| $h_{max}$ | $0.88\,\mathrm{m}$ | $0.66\,\mathrm{m}$ |
| $R$ | $0.64\,\mathrm{m}$ | $0.48\,\mathrm{m}$ |

Some of these numbers are surprising, if you know something about the size of a basketball. The range for the left bounce is 0.64 m, and appears in the photograph to be roughly 6 diameters of the ball, so the ball diameter is between 10 and 11 cm, half the size of a basketball! In fact, this is not a real basketball in the photograph, but a child's toy ball that is made to look like a basketball.

Since we don't know anything about the mass of the toy ball, we cannot calculate force, momentum, or energy.

## 6.3   At the Peak

### Words

Let's take a closer look at what is happening just at the peak of the flight of the ball from Section 6.2. We know that the path followed by the ball is parabolic in shape. At the top of the parabola, the motion of the ball is horizontal.

At every point on the parabola, there is a constant net force, which is simply the gravitational force, pointing downward. As a result, at every point on the parabola the ball has a constant downward acceleration. And since acceleration is a change in velocity over time, the velocity of the ball is changing at a constant rate everywhere on the parabola.

The ball is moving upward on the left side of the peak, and since the acceleration is downward that means it is slowing down. We can say that the gravitational force is doing negative work, decreasing the kinetic energy as the ball is moving upward. The ball is moving downward on the right side of the peak, and since the acceleration is downward that means it is speeding up. We can say that the gravitational force is doing positive work, increasing the kinetic energy as the ball is moving downward.

When the ball is right at the peak, its velocity is perpendicular to the net force, so no work is being done on it, positive or negative. That means its *speed is not changing*. But, it is still accelerating the same as everywhere else, so the *velocity is changing*!

### Graphics



Figure 6.8: A close-up of the top of the first peak for the bouncing ball from Section 6.2.[32]
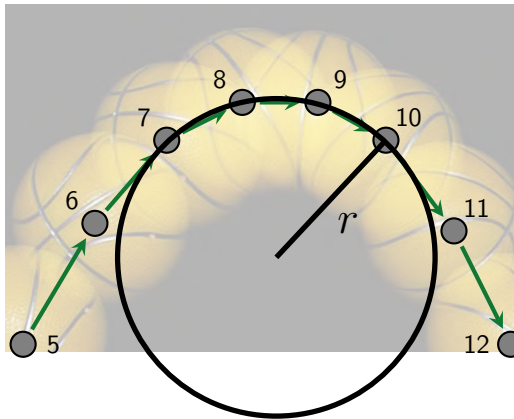


Figure 6.9: At the very peak of the ball's motion, the path briefly follows a circular path with radius of curvature $r$.[32]

### Numbers

**Assumptions:** gravity near the surface of the earth

| Knowns | Unknowns |
|---|---|
| $t_{image} = 0.04 \text{ s}$ | ??? |
| $g = 9.8 \text{ m/s}^2$ | |

The work that is done on the ball at the top of the parabola is given by:

$$W_{net} = F_{net} \cdot \Delta x \cdot \cos\theta = F_g \cdot \Delta x \cdot \cos 90°$$

$\cos 90° = 0$, so...

$$W_{net} = 0$$

...at the top of the parabola where the force is perpendicular to the direction of motion. There are no energy transformations happening at that point, so the speed $v$ is constant at the top of the parabola. And yet the acceleration due to gravity is constant everywhere on the parabolic arc, so $\vec{v}$ is changing. The only way to change $\vec{v}$ while keeping $v$ constant is to change the direction of $\vec{v}$.

How can the velocity change while the speed stays the same? Velocity is a vector, which includes speed and direction. When the net force (and therefore the acceleration) is perpendicular to the velocity, it causes a change in direction but not a change in speed. The path curves along a circular arc with a radius that is called the "radius of curvature."

This perpendicular net force is called the "centripetal" force, from the Latin words for "toward the center." The centripetal force is not a new type of force; it is simply a name for whatever force is causing an object to follow a curved path. In the case of the ball at the top of the arc, the centripetal force is the force of gravity. In other situations it could be a tension force, a friction force, a normal force, a combination of these, or any other type of applied force.

Imagine now a force that changes direction as the velocity changes direction. If a constant net force were kept always perpendicular to the direction of motion, the object would move in a complete circle. This is what happens, for example, in an Olympic hammer throw. The "hammer," a heavy ball on the end of a flexible cord, is spun in a circle. The tension in the cord supplies the centripetal force radially inward, and the velocity of the ball is in the "tangential" direction, perpendicular to the radius.
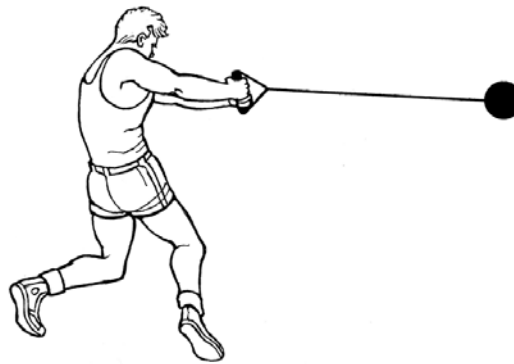


Figure 6.10: An Olympic hammer throw.[34]
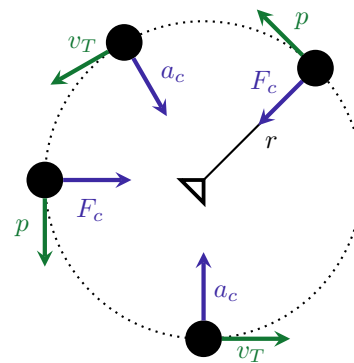


Figure 6.11: Looking down on the hammer throw. The ball is following the dotted line counterclockwise. $\overrightarrow{F_c}$, $\overrightarrow{a_c}$, $\overrightarrow{p}$, & $\overrightarrow{v}$ are shown at various positions.[1]

The radius of curvature $r$ in Figure 6.9 depends on two things: the acceleration due to gravity and the horizontal speed at the top. The radius would increase if gravity were weaker, allowing the ball to stay up longer, and the radius would also increase if the speed were higher, allowing the ball to move farther horizontally in the time needed for gravity to pull the ball down.

The magnitude of the centripetal acceleration $a_c$, the tangential (perpendicular to the radius) velocity $v_T$, and the radius of curvature $r$ are related by:

$$a_c = \frac{v_T^2}{r} \tag{6.3}$$

In Section 6.2, the speed and initial direction of the ball were found, which give us the horizontal velocity using the cosine. The horizontal velocity is the tangential velocity at the top of the parabola, so we can use it to find the radius of curvature.

$$r = \frac{v_T^2}{a_c} = \frac{(v_0 \cdot \cos\theta)^2}{g} = 0.12\,\text{m}$$

That is roughly the diameter of the ball, and in Figure 6.9 we can see that in fact the radius of curvature is very close to the diameter of the ball. Making this type of comparison can give us confidence that our work is correct. Using the relationship between force and acceleration given by Newton's Second Law, we could also find the centripetal force if we knew the mass:

$$F_c = m \cdot a_c = \frac{m \cdot v_T^2}{r} \tag{6.4}$$

## 6.4   Earth's Orbit

### Words

The earth orbits the sun at a distance of approximately 150 million kilometers, and it takes approximately 365 days to make one complete orbit.

Use this information to find the speed of the earth as it orbits around the sun and also the mass of the sun and any other information that can be found.

The speed of the earth can be found by considering that the earth makes one complete circuit around the sun every year. The speed of the earth is simply the path length traveled around the circular path divided by one year.

It seems surprising to think that we would not need to know the mass of the earth in order to find the mass of the sun for this question, but the force of the earth's gravity gives objects at the earth's surface a constant acceleration regardless of their mass. Just as the mass of the earth is much larger than the mass of anything on the surface of the earth, the mass of the sun is much larger than the mass of the earth. So it should make sense that the sun's gravitational force causes a constant acceleration at a given distance, at least as long as the other object's mass is much less than the mass of the sun.

### Graphics
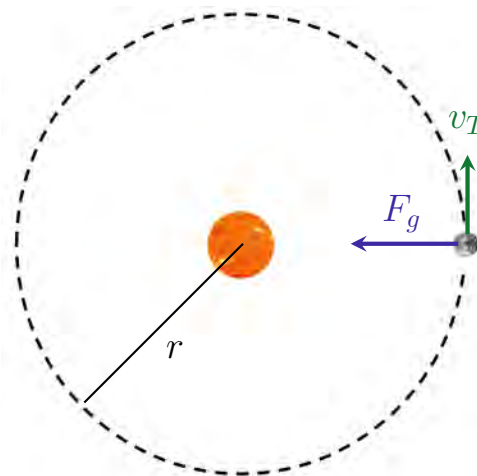


Figure 6.12: The earth's orbit around the sun is nearly circular.[35]

Note that the earth's speed is nearly constant as it orbits the sun; but its velocity, which includes direction, is always changing. It is more convenient to use *angular* velocity when describing a situation like this, because the earth's angular velocity, like its speed, is nearly constant.

### Numbers

**Assumptions:** circular orbit; $m_{earth} \ll m_{sun}$

| Knowns | Unknowns |
|---|---|
| $r = 1.5 \times 10^{11}$ m | $v_T$ |
| $t_{orbit} = 3.15 \times 10^7$ s | $m_{sun}$ |
| | ??? |

In 365 days, the earth orbits the sun once, a path length that is the circumference of a circle whose radius is the distance from the earth to the sun:

$$s_{orbit} = 2\pi \cdot r = 9.4 \times 10^{11} \text{ m}$$

To find the tangential speed of the earth in its orbit we need to consider the path length traveled:

$$v_T = \frac{s_{orbit}}{t_{orbit}} = 3 \times 10^4 \text{ m/s}$$

Now that we know the tangential speed of the earth, we can use that to determine the centripetal acceleration, or with mass the centripetal force. At these distances, we need to use Newton's Law of Universal Gravitation. It is this force which is the centripetal force that keeps the earth in a circular path, so we can use Equation 6.4 as well.

$$F_g = F_c$$
$$\frac{G \cdot m_{sun} \cdot m_{earth}}{r^2} = \frac{m_{earth} \cdot v_T^2}{r}$$
$$\frac{G \cdot m_{sun}}{r} = v_T^2$$

While it is interesting to know the tangential speed of the earth as it goes around the sun, that knowledge doesn't have a great impact on our daily lives. The thing that most affects our lives about the orbit of the earth around the sun is the passing of the seasons, which is related to angular position. This is an example when expressing position in terms of an angle, and how the angle changes over time, is more important than expressing position in terms of distances and how the distance changes over time.

There are also many other situations where angular changes are more important and easier to think about than changes in distance. For example, many vehicles have tachometers that display the angular speed of the engine measured in rpm (revolutions, sometimes called rotations, per minute).

When discussing angular motion, the units used for angles are radians. One radian is the angle created by an arc whose arc length is equal to the radius of the circle. So one full circle contains $2\pi$ radians. The angle then becomes a ratio of an arc length over a radius. Length per length. In other words, the unit "radian" is in some sense dimensionless. Expressing angles in radians greatly simplifies problem solving precisely because of this dimensionless quality of the angle measurement.

Just as we have learned about position, velocity, acceleration, momentum, etc. in linear form, we will now start to learn about angular position, angular velocity, angular acceleration, angular momentum, etc.
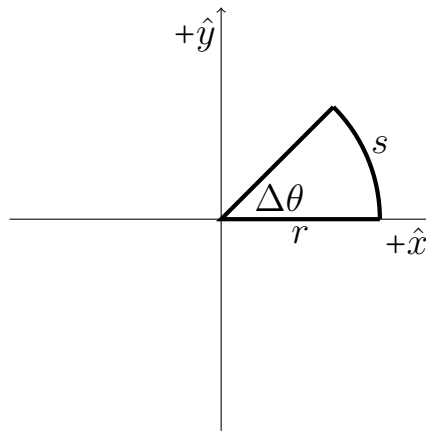


Figure 6.13: The angle $\Delta\theta$ in radians is the ratio of the arc length $s$ to the radius $r$.[1]
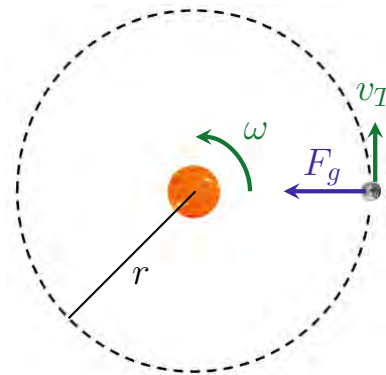


Figure 6.14: The earth's speed as it orbits the sun can be described in terms of angular velocity $\omega$. "$\omega$" is the lower-case Greek letter "omega."[35]

$$m_{sun} = \frac{v_T^2 \cdot r}{G} = 2.0 \times 10^{30}\ \mathrm{kg}$$

It is often useful to describe a system in terms of angles, so for the earth we could give an angular velocity around the sun instead of a tangential velocity. The SI unit for angles is the radian [rad], illustrated in Figure 6.13.

$$\boxed{\Delta\theta = \frac{s}{r}} \tag{6.5}$$

...where $s$ is the arc length and $r$ is the radius of a circle. $\theta = 0$ is normally at the positive x axis and the positive $\theta$ direction is counter-clockwise.

Similarly, we can define angular velocity $\omega$ in terms of tangential speed and radius:

$$\boxed{\omega = \frac{v_T}{r}} \tag{6.6}$$

The angular velocity of the earth around the sun can then be described as:

$$\omega_{earth} = \frac{3 \times 10^4\ \mathrm{m/s}}{1.5 \times 10^{11}\ \mathrm{m}} = 2 \times 10^{-7}\ \mathrm{rad/s}$$

The SI unit for angular velocity is therefore [rad/s]. We can also find the period $T$ of the earth's orbit around the sun, the amount of time for one complete revolution. When $\omega$ is constant,

$$\boxed{T = \frac{2\pi}{\omega}} \tag{6.7}$$

So for the earth, $T = 3.14 \times 10^7\ \mathrm{s}$.

## 6.5 Puck on a String

### Words

For situations where angular velocity is easier to think about than linear velocity, it would be convenient to also have angular versions of kinetic energy and momentum, which are strongly connected to velocity. Take, for example, a 0.2 kg frictionless puck that is connected by a 1.4-m-long string to a bolt that is solidly mounted in the floor.

If the puck is moving at constant angular speed of 4 rad/s in a circle around the bolt, its kinetic energy, which depends only on mass and speed, but not on direction, is constant.

We now have two different ways to look at the motion and the kinetic energy of the puck:

- We can see the puck as a moving object with a linear (tangential) velocity that is always changing direction because of the centripetal acceleration caused by the tension in the string. In this view, the puck has the same kind of kinetic energy that we are used to dealing with, which is usually called "translational" kinetic energy.

- Or we can see the puck as part of a puck-bolt system that does not have any linear velocity (since the bolt is solidly fixed in place), but that is spinning about an axis at a constant angular speed. In this view, the puck-bolt system has *rotational* kinetic energy, but *no translational* kinetic energy.

### Graphics
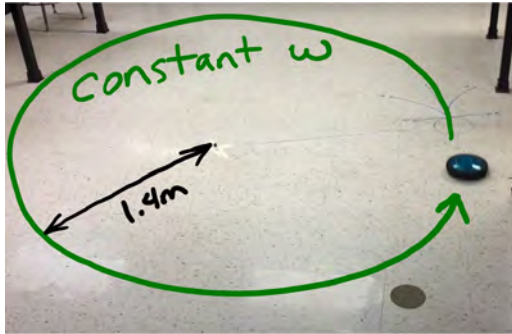


Figure 6.15: A frictionless puck moving in a circle at constant angular speed.[33]

Figures 6.15 & 6.16 are based on a screenshot from a YouTube video:
https://youtu.be/mNdLRySeh9o.

Something really interesting happens in the video at a place where the string falls off of the bolt. The puck stops moving in a circle with constant angular velocity, since the source of the centripetal force has disappeared. With no forces in the horizontal direction, the puck instead begins to move in a straight line with constant velocity.

### Numbers

| Knowns | Unknowns |
|---|---|
| $r = 1.4\,\mathrm{m}$ | $E_k$ |
| $\omega = 4\,\mathrm{rad/s}$ | rotational kinetic energy? |
| $m = 0.2\,\mathrm{kg}$ | angular momentum? |
| $F_f = 0$ | |

We can use $E_\mathrm{k} = \frac{1}{2} m \cdot v^2$ to find the kinetic energy of the puck, recognizing that its speed is simply $v_T$. But we don't have $v_T$, so we will also need to use Equation 6.6:

$$E_k = \frac{1}{2} m \cdot v_T^2 = \frac{1}{2} m \cdot (r \cdot \omega)^2$$

This is referred to as the rotational kinetic energy $E_{k,r}$ of a point mass, and the expression is normally grouped in a slightly different way:

$$E_{k,r} = \frac{1}{2} \left( m \cdot r^2 \right) \cdot \omega^2$$

For the puck in this example…

$$E_{k,r} = \frac{1}{2} \left( 0.2\,\mathrm{kg} \cdot (1.4\,\mathrm{m})^2 \right) \cdot (4\,\mathrm{rad/s})^2 = 3.14\,\mathrm{J}$$

The kinetic energy could also be found for the puck by finding $v_T$ and calculating kinetic energy as we have done before. Whether we choose to consider the puck as having only translational kinetic energy in the tangential direction or only rotational kinetic energy around the pivot, we will get the same result for its kinetic energy.

This type of thinking also works for momentum.

- We can see the puck as a moving object with a linear momentum that is always changing direction because of the tension in the string.

- Or we can see the puck as part of a system that does not have any linear momentum, but that has constant angular momentum around a pivot point at the location of the bolt.

Angular momentum, like linear momentum, is conserved for any isolated system. That is what makes it such a useful concept. An isolated system, remember, is one that does not interact with anything outside of the system.

By watching the video associated with the figures in this section, we can learn something surprising about angular momentum: even an object that is moving in a straight line can have angular momentum! When the string falls off of the bolt, there is no force applied to the puck, so its angular momentum can't change at that time—it has to keep the same amount of angular momentum that it had just before the string fell off.

For a pointlike object like this puck, the angular momentum depends on its mass, angular velocity, and distance from the pivot; or, if it is moving in a straight line then it depends on the mass, linear velocity, and the "lever arm," which is the perpendicular distance from the pivot to the line along which it is traveling.
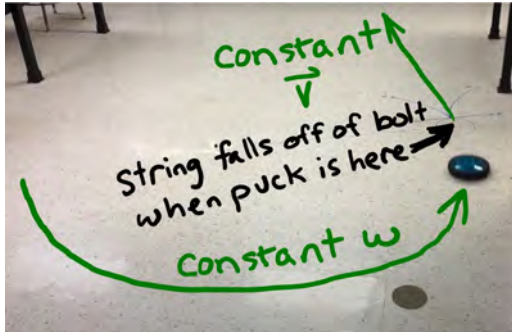


Figure 6.16: When the string falls off of the bolt, the puck begins to move in a straight line.[33]
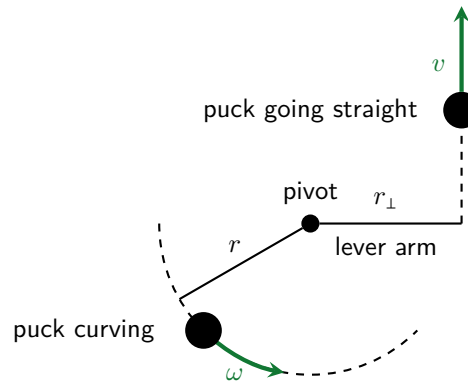


Figure 6.17: Motion of the puck, showing a lever arm, $r_\perp$, for an object moving in a direction that is not aligned with a pivot point.[1]

Angular momentum $L$ is conserved for an isolated system, so…

$$L_f = L_i \qquad (6.8)$$

For an object moving in a straight line that is not aligned with a pivot point, as shown when the puck is going straight in Figure 6.17, $L$ is given by…

$$L = m \cdot v \cdot r_\perp \qquad (6.9)$$

…where $r_\perp$ is the lever arm.

From Figure 6.17, we can see that in the case of a small object traveling on a circular path, the lever arm is simply $r$ and the speed is simply $v_T$. So for that situation, angular momentum can be described in angular terms using…

$$
\begin{aligned}
L &= m \cdot v_T \cdot r \\
&= m \cdot (\omega \cdot r) \cdot r \\
&= \left( m \cdot r^2 \right) \cdot \omega
\end{aligned}
$$

Notice that $\left( m \cdot r^2 \right)$ has made another appearance. And as we explore rotation we will see more of this same type of grouping of mass and radius. That is why they are usually grouped together in expressions for rotational motion.

Now we can find the angular momentum of the puck:

$$
\begin{aligned}
L &= \left( m \cdot r^2 \right) \cdot \omega \\
&= \left( 0.2 \, \text{kg} \cdot (1.4 \, \text{m})^2 \right) \cdot 4 \, \text{rad/s} \\
&= 15.7 \, \text{kg} \cdot \text{m}^2/\text{s}
\end{aligned}
$$

## 6.6   Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

### General

- When an object is following a curved path, the radial direction is in the direction of the radius of curvature and the tangential direction is perpendicular to the radius of curvature.
- The time needed for an object to move in a complete circle is called the period.
- The positive direction of angular quantities is defined as the counter-clockwise direction.

### Forces

- A net force applied perpendicular to an object's velocity causes the object to change direction but maintain constant speed.
- Centripetal force is always directed in the radial direction toward the center of the circular path.
- Centripetal force is not a new kind of force, but is used to describe the force that is causing circular motion.

### Motion

- Motion in the horizontal direction and motion in the vertical direction are independent of each other. The thing that connects them is time.
- An object that is in free-fall near the surface of the earth follows a path that is shaped like a parabola.
- Free-fall motion is sometimes called projectile motion or ballistic motion.
- When an object's acceleration is perpendicular to its velocity, the direction of the object's motion changes. It follows a path described by an arc of a circle with a "radius of curvature" $r$.
- Centripetal acceleration is always directed in, toward the center of circular path.
- Speed is path length traveled over time.
- The SI unit for angle is the radian [rad]. Zero radians is usually at the positive x axis.
- The SI unit for angular velocity is [rad/s].
- A "lever arm" is the perpendicular distance from a straight line to a pivot point.

### Momentum

- Angular momentum is conserved for any isolated system.
- The angular momentum of an object moving in a straight line depends on its mass, linear velocity, and lever arm.
- The SI unit for angular momentum is $[\mathrm{kg} \cdot \mathrm{m/s}]$.

## Energy

- Objects can have translational kinetic energy, rotational kinetic energy, or both.

## Mathematical Models

| equation | | restrictions on the validity of the equation |
|---|---|---|
| $x = x_0 + v_{0x} \cdot t$ | (6.1) | For objects in free-fall near the surface of the earth |
| $y = y_0 + v_{0y} \cdot t - \frac{1}{2} g \cdot t^2$ | (6.2) | For objects in free-fall near the surface of the earth |
| $a_c = \frac{v_T^2}{r}$ | (6.3) | -none- |
| $F_c = m \cdot a_c = \frac{m \cdot v_T^2}{r}$ | (6.4) | -none- |
| $\Delta\theta = \frac{s}{r}$ | (6.5) | -none- |
| $\omega = \frac{v_T}{r}$ | (6.6) | -none- |
| $T = \frac{2\pi}{\omega}$ | (6.7) | When $\omega$ is constant |
| $L_f = L_i$ | (6.8) | For an isolated system |
| $L = m \cdot v \cdot r_\perp$ | (6.9) | For pointlike objects with linear velocity |

## 6.7   Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

### Level 1 - Remember

6.1 **[W]** What is the difference between free-fall, projectile motion, and ballistic motion?

6.2 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

### Level 2 - Understand

6.3 **[W]** Explain how the photograph in Figure 6.5 shows that the normal force is much larger than the gravitational force when the ball is touching the ground in image 16.

6.4 **[W]** Describe how the momentum of the ball in Figure 6.5 changes over the course of time during which it is imaged.

6.5 **[W & G]** What type of force creates the centripetal force that keeps the ball moving in a circle in Figure 6.11?

6.6 **[W, G, & N]** In Section 6.5, the lever arm for momentum is described in the "Words" column as being a distance that is related to the momentum vector but in the "Numbers" column it is instead described as being related to the velocity vector. How can both of these be correct, or was this an error?

### Level 3 - Apply

6.7 **[N]** If the mass of the ball in Section 6.2 were doubled, how would that affect the radius of curvature at the top of the left peak that was found in Section 6.3? Explain your reasoning.

6.8 **[W & G]** If the hammer in Figure 6.11 were released when the cord was to the South compared to the thrower, in what direction would the hammer travel?

6.9 **[N]** Use the same analysis that was used in Section 6.4 to determine which planets in our solar system have a higher tangential velocity than the earth. You will need to look up information about the periods and radii of the orbits of of planets in our solar system to answer this question.

### Level 4 - Analyze

6.10 **[W, G, & N]** Describe the energy of the diver in Section 6.1 at the top of the cliff and just before they hit the water.

6.11 **[G]** Make energy bar graphs for the ball in Figure 6.5 for images 1 (where the ball is close to the ground but not touching the ground), 4, 8, 13, 16 (where the ball is on the ground and not moving), 22, and 29 (where the ball is on the ground and not moving). Without a mass for the ball, it is not possible to calculate the actual energies, so just make the relative heights of the bars as accurate as possible.

6.12 **[W & N]** An angular velocity of the earth around the sun is calculated at the end of Section 6.4. Convert that number into revolutions per year. Does your answer make sense? Explain.

6.13 **[W & G]** Look at the photograph of a fire poi dance at the beginning of Chapter 6. Assuming that the balls of fire are moving at constant speed, identify places in the balls' paths where the centripetal acceleration has a large magnitude and places where it has a small magnitude.

6.14 **[N]** The earth's mass is approximately $6 \times 10^{24}$ kg. Use this information and the information in Section 6.4 to find the amount of angular momentum that the earth has due to its orbit around the sun. Use the position of the sun as the pivot point. Do the calculation in two different ways: One using the angular speed of the earth and the other using its tangential speed. Verify that the result is the same either way.

6.15 **[N]** The earth's mass is approximately $6 \times 10^{24}$ kg. Use this information and the information in Section 6.4 to find the amount of kinetic energy the earth has due to its orbit around the sun. Do the calculation in two different ways: One using the angular speed of the earth and the other using its tangential speed. Verify that the result is the same either way.

## Level 5 - Evaluate

6.16 **[G & N]** What effect would each of the follow changes have on the total time, final speed, and horizontal displacement of the diver in Section 6.1?

   (a) Doubling the initial velocity of the diver and keeping everything else the same

   (b) Doubling the height of the cliff and keeping everything else the same

   (c) Keeping the same initial speed but jumping up and out instead of just horizontally out from the cliff, and keeping everything else the same

## Level 6 - Create

6.17 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Add the main ideas from this chapter to the concept map that you began for the question at the end of Chapter 1.

6.18 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

6.19 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

# Chapter 7

# Rotation

Interesting things happen when objects are allowed to rotate. Ice skaters start a spin and then go faster and faster seemingly without any effort. Children's tops fall over unless they are spinning.

At a playground or amusement park virtually every piece of equipment involves rotation in one way or another. Spinning wheels are used in almost every form of modern transportation. Even transportation that doesn't involve wheels, like flying in a helicopter or walking, involves rotation around a joint or axle.

This chapter focuses on a new set of tools–still forces, motion, momentum, and energy–but specifically applied to rotating objects.

Many things do not change. We will see that angular force (called torque) changes angular momentum, causes angular acceleration, and does work, much like force does in linear systems. Energy and angular momentum are still conserved in an isolated system. And most of the mathematical models that we have used will apply equally well to rotation after just a few small changes.

Figure 7.1: Sasha Cohen performs an I-spin at the 2009 Stars on Ice in Halifax, Nova Scotia.[36]

## 7.1 Kind of the Same

**Words**

To look at the similarities between linear and rotational motion, forces, and momentum, we will consider a thin, light rod with a small, massive ball on one end, rotating around a fixed pivot location at the opposite end of the rod. A "light" rod means that the mass of the rod itself is negligible.

Linear motion is described by position, velocity, and acceleration. Velocity is a change in position (also called displacement) over time, and acceleration is a change in velocity over time. Angular motion will be described in exactly the same way. The position is replaced by angular position, or simply angle. Angular velocity is a change in angular position over time, and angular acceleration is a change in angular velocity over time.

Each of these angular quantities is also directly related to its corresponding linear quantity in the tangential direction. For example, if the length of the rod were doubled but the angular velocity stayed the same, the ball would trace out a circle with twice the radius in the same amount of time, so the tangential velocity would double; if the length of the rod stayed the same but the angular velocity (rate of spin) was doubled, the ball would trace out a circle of the same size in half the time, so again the tangential velocity would double. So the angular quantities are a combination of the linear quantities and the radius.

**Graphics**
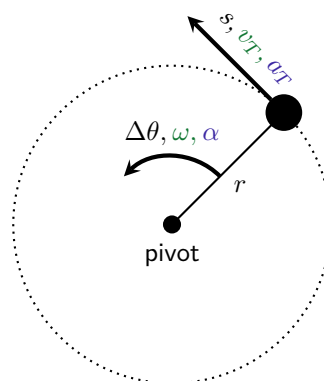


Figure 7.2: Looking down on the rod and ball. The ball is following the dotted line counterclockwise. $s$, $v_T$, & $a_T$ are the linear quantities corresponding to the angular quantities $\Delta\theta$, $\omega$, & $\alpha$, respectively. "$\alpha$" is the lowercase Greek letter "alpha."[1]

**Numbers**

$\Delta\theta$ and $\omega$ were already given in terms of $s$ and $v_T$ in Section 6.4. Angular acceleration $\alpha$, whose SI unit is $[\mathrm{rad/s^2}]$, is similarly related to $a_T$.

$$\alpha = \frac{a_T}{r} \qquad (7.1)$$

Every equation of motion that we have learned works equally well for angular motion, simply by replacing the linear quantities with their angular counterparts. So, for example...

$$\vec{x} = \vec{x_0} + \vec{v_0} \cdot t + \frac{1}{2}\vec{a} \cdot t^2$$

...becomes...

$$\theta = \theta_0 + \omega_0 \cdot t + \frac{1}{2}\alpha \cdot t^2 \qquad (7.2)$$

Average acceleration becomes...

$$\alpha_{avg} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{\Delta t} \qquad (7.3)$$

...and average velocity becomes...

$$\omega_{avg} = \frac{\Delta\theta}{\Delta t} \qquad (7.4)$$

...and...

$$\omega_{avg} = \frac{1}{2}\left(\omega_i + \omega_f\right) \qquad (7.5)$$

When we looked at linear motion, we saw that an acceleration only exists if there is an unbalanced force. So we would again expect that to be true for rotational motion. An angular acceleration only exists if there is an unbalanced angular force, which is called a "torque." The units of torque are $[\mathrm{N} \cdot \mathrm{m}]$.

Several forces are shown in Figure 7.3, all with the same magnitude. Which of them would create the largest angular acceleration?

Two of the forces, $F_1$ and $F_2$, are acting right at the pivot point. Since this physical scenario states that the pivot point is at a fixed location, it can't move. That means neither of those two forces can create any angular acceleration around the pivot.

$F_3$ looks like a very good choice, and in fact this is where you would probably apply the force if you were in this situation trying to accelerate the ball. Of all of the forces in Figure 7.3, $F_3$ is most effective at creating angular acceleration, because it creates the greatest amount of torque. $F_4$ and $F_5$ also create torque and will cause angular acceleration of the rod an ball, but not as much as $F_3$. That is because $F_3$ has the largest lever arm.

Just as forces cause a change in linear momentum over time, torques cause a change in angular momentum over time.
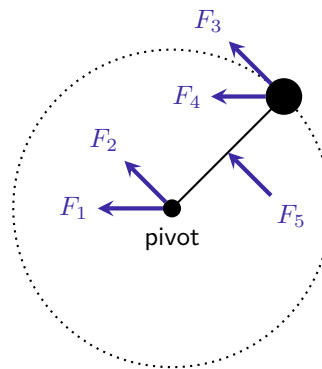


Figure 7.3: Several forces, all with the same magnitude, applied to a ball attached to a pivot by a light rod.[1]

The lever arms ($r_\perp$'s) are zero for $F_1$ & $F_2$ since they are both at the pivot. The lever arm for $F_3$ is the length of the rod, and the lever arm for $F_5$ is half the length of the rod. The lever arm for $F_4$ is a bit more difficult to find. It is longer than the lever arm for $F_5$ but shorter than the lever arm for $F_3$.
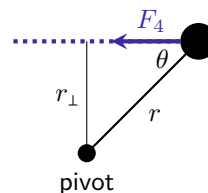


Figure 7.4: Finding the lever arm for $F_4$ from Figure 7.3.[1]

And, finally, the relationship between acceleration, displacement, and velocity becomes...

$$2\alpha \cdot \Delta\theta = \omega_f^2 - \omega_i^2 \qquad (7.6)$$

The angular version of force is torque $\tau$. "$\tau$" is the lowercase Greek letter "tau." Torque increases with the length of the lever arm:

$$\tau = F \cdot r_\perp \qquad (7.7)$$

As seen in Figure 7.4, the lever arm can be found from the distance $r$ to the pivot and the angle $\theta$ between the vector being considered (in this case a force vector) and the vector $\overrightarrow{r}$. Using trigonometry gives...

$$r_\perp = r \cdot \sin\theta \qquad (7.8)$$

Mathematically, torque affects rotating systems in the same way that force affects linear systems. That includes the ability to change angular momentum over time:

$$\tau = \frac{\Delta L}{\Delta t} \qquad (7.9)$$

141

## 7.2 Points and Hoops

### Words

In Section 7.1, we considered a ball whose mass is concentrated into a small point some distance away from the pivot. Now we will explore what happens to rotational kinetic energy when the mass is distributed across a larger area.

In each of these examples of different mass distributions, we will assume that the total mass of the system $m$ is the same, and that the mass is first distributed and then the system is rotated at the same angular speed $\omega$.

To begin, instead of a single mass $m$ at the end of a light rod with length $r$, we will consider two masses, each with a mass of $m/2$, connected by a light rod, still keeping each mass at a distance $r$ from the center pivot point, rotating at a constant angular speed $\omega$.

For each of these masses, the tangential speed will be the same as it was for the original single mass. We will need to combine their kinetic energies. Instead of a single mass at a certain speed, we have two half-masses at the same speed. The kinetic energy is the same, whether the mass is in one location or split into two locations.

What if we split the mass into three, or four, or six, or ten, or even more pieces, and kept all of them the same distance from the pivot?

### Graphics
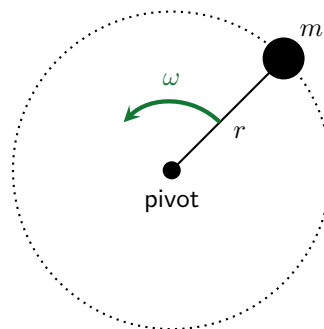


Figure 7.5: The original situation, with a single point mass rotating around a pivot at angular speed $\omega$.[1]
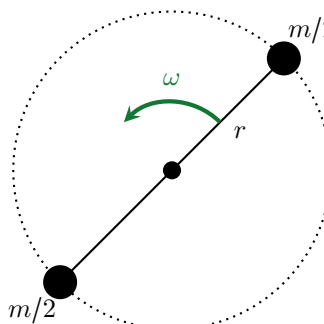


Figure 7.6: The second situation, with the same mass split into two parts, with both parts at the original distance from the pivot and the whole system rotating at the original angular speed $\omega$.[1]

### Numbers

**Assumptions:** pointlike masses

From Section 6.5 the rotational kinetic energy for a mass distribution like that in Figure 7.5 is...

$$E_{k,r} = \frac{1}{2}\left(m \cdot r^2\right) \cdot \omega^2$$

For the mass distribution shown in Figure 7.6, we need to consider each mass separately and add their energies to find the total rotational kinetic energy.

$$E_{k,r} = \frac{1}{2} \cdot \frac{m}{2} \cdot r^2 \cdot \omega^2 + \frac{1}{2} \cdot \frac{m}{2} \cdot r^2 \cdot \omega^2$$
$$= \frac{1}{2}\left(m \cdot r^2\right) \cdot \omega^2$$

The result is the same as if all of the mass were at a single point.

Using the same analysis for a mass distribution where the same mass is divided into a huge number $N$ of small pieces, all kept at the same distance from the pivot, we have to add up $N$ energies:

$$E_{k,r} = N \cdot \left(\frac{1}{2} \cdot \frac{m}{N} \cdot r^2 \cdot \omega^2\right) = \frac{1}{2}\left(m \cdot r^2\right) \cdot \omega^2$$

If $N$ is large enough, all of the mass could be spread out to be completely touching all of the way around the circle, creating a hoop, without affecting the rotational kinetic energy.

If the angular speed remains the same, each piece has the same tangential speed as the original single mass. When we combine their kinetic energies, the kinetic energy of the whole system is the same whether the mass is in one location or split into many locations. In fact, the kinetic energy would be the same even if you spread the mass out into a thin hoop at the same distance $r$ from the pivot.

Let's try one more thing. We will split the mass into five equal parts, space them out equally in a line so that the ones on the ends are at the original distance $r$ and the middle one is at the center, and then spin them at the same angular velocity as before.

The two pieces on the ends still move at the same tangential speed as before, but the pieces that are farther in move at a slower speed and the piece in the center doesn't move at all. So this time, when we add up the kinetic energies for the five pieces we will end up with less energy than before.

This time the way the mass is distributed has reduced the "rotational inertia" of the system of masses. If an object has a large rotational inertia, that means it is difficult to change its angular velocity. This is analogous to saying that if an object has a large mass, it is difficult to change its linear velocity. Rotational inertia is the angular counterpart of mass.

The closer the mass is to the pivot, the smaller the rotational inertia will be.
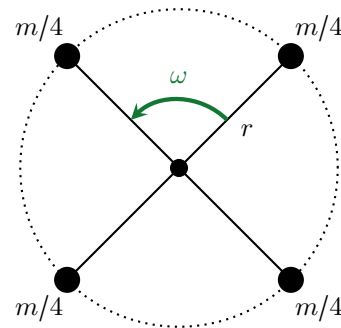


Figure 7.7: The third situation, with the same mass split into four parts, with all parts at the original distance from the pivot and rotating at the original angular speed $\omega$.[1]
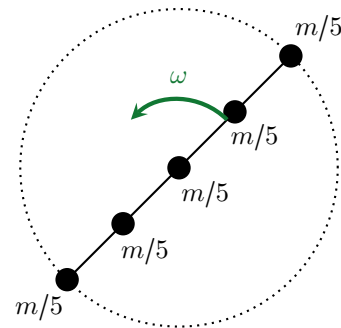


Figure 7.8: The fourth situation, with the same mass split into five parts at varying distances from the pivot and rotating at the original angular speed $\omega$.[1]

For the situation shown in Figure 7.8, we have two masses at radius $r$, two at $r/2$, and one at the pivot point, so radius zero. When we add their kinetic energies we get:

$$E_{k,r} = 2 \cdot \frac{1}{2} \cdot \frac{m}{5} \cdot r^2 \cdot \omega^2 + 2 \cdot \frac{1}{2} \cdot \frac{m}{5} \cdot (r/2)^2 \cdot \omega^2 + 0$$
$$= \frac{1}{2} \cdot \left( \frac{1}{2} m \cdot r^2 \right) \cdot \omega^2$$

This is a different result! If we want to keep the form of our equation for rotational kinetic energy, the expression in the parentheses is not always going to be $m \cdot r^2$.

The expression in the parentheses is called the "rotational inertia," $I$, and it is the angular counterpart to mass. There will always be a mass and a distance squared in $I$, but it also contains a multiplier that depends upon the distribution of the mass. The multiplier gets smaller as the mass is brought closer to the pivot.

Replacing linear variables with their angular counterparts, we now have a few more mathematical models for rotating systems:

$$E_{k,r} = \frac{1}{2} I \cdot \omega^2 \qquad (7.10)$$

$$L = I \cdot \omega \qquad (7.11)$$

$$\tau = I \cdot \alpha \qquad (7.12)$$

## 7.3   Helicopter Blades

### Words

A typical helicopter blade has a mass of approximately 60 kg and a length of approximately 10 m, and the blades typically rotate at approximately 500 rpm. Assume that the blades of the two-blade helicopter shown in Figure 7.9 can go from motionless to full angular speed in 5 seconds, with constant angular acceleration over that time. What can we find from this information?

Since we are dealing with rotational motion, we should be thinking in terms of angular quantities: angular position, angular velocity, angular acceleration, angular momentum, torque, and rotational kinetic energy.

Initially the blade is not moving, so no angular velocity, no angular momentum, and no rotational kinetic energy. But once the blade is spinning it has all three of those, so that means that a torque must have been acting on the blade, coming from the engine inside the helicopter.

The torque does several things. For one, it causes angular acceleration, increasing the angular velocity over time. The amount of angular acceleration depends not only on the torque but also on the rotational inertia of the blades. Increasing torque will increase the angular acceleration, but increasing the rotational inertia will decrease the angular acceleration.

### Graphics



Figure 7.9: A helicopter with two blades.[37]
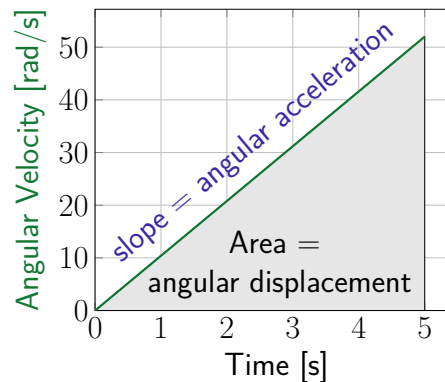


Figure 7.10: Similarly to linear velocity and linear acceleration, the slope of an angular velocity vs time graph is angular acceleration.[1]

### Numbers

**Assumptions:** $\alpha$ is constant

| Knowns | Unknowns |
|---|---|
| $m_{blade} = 60\,\mathrm{kg}$ | ??? |
| $l_{blade} = 10\,\mathrm{m}$ | |
| $\omega_i = 0\,\mathrm{rad/s}$ | |
| $\omega_f = 52\,\mathrm{rad/s}$ | |
| $t = 5\,\mathrm{s}$ | |
| 2 blades | |

The value for $\omega_f$ in the table above was found by converting from rotations per minute (rpm):

$$\frac{500\,\text{rotations}}{1\,\text{min}} \cdot \frac{2\pi\,\text{rad}}{\text{rotation}} \cdot \frac{1\,\text{min}}{60\,\text{s}} = 52\,\text{rad/s}$$

We can find the average angular acceleration using Equation 7.1:

$$\alpha_{avg} = \frac{\omega_f - \omega_i}{\Delta t} = \frac{(52-0)\ \text{rad/s}}{5\,\text{s}} = 10.4\,\text{rad/s}^2$$

We can also find the total angle that the blades rotate through as they come up to full speed by using Equations 7.4 & 7.5:

$$\Delta\theta = \omega_{avg} \cdot \Delta t = \frac{1}{2}\left(\omega_i + \omega_f\right) \cdot \Delta t$$

$$= \frac{1}{2}(0+52)\ \text{rad/s} \cdot (5\,\text{s}) = 130\,\text{rad}$$

The torque also changes the angular momentum of the blades. The change in angular momentum depends only on the net torque and the time over which that torque is applied.

And the torque also changes the rotational kinetic energy of the blades. That is because the torque is doing work on the blades, just as a force can do work on an object that has linear velocity. Since the blades are moving in the same direction as the torque, the work is positive, increasing the kinetic energy.

point mass $\qquad$ $I = m \cdot r^2$

solid cylinder or disk $\qquad$ $I = \dfrac{1}{2} m \cdot r^2$

hollow cylinder or hoop $\qquad$ $I = m \cdot r^2$

thin rod around center $\qquad$ $I = \dfrac{1}{12} m \cdot l^2$

thin rod around end $\qquad$ $I = \dfrac{1}{3} m \cdot l^2$

solid sphere $\qquad$ $I = \dfrac{2}{5} m \cdot r^2$

hollow sphere $\qquad$ $I = \dfrac{2}{3} m \cdot r^2$
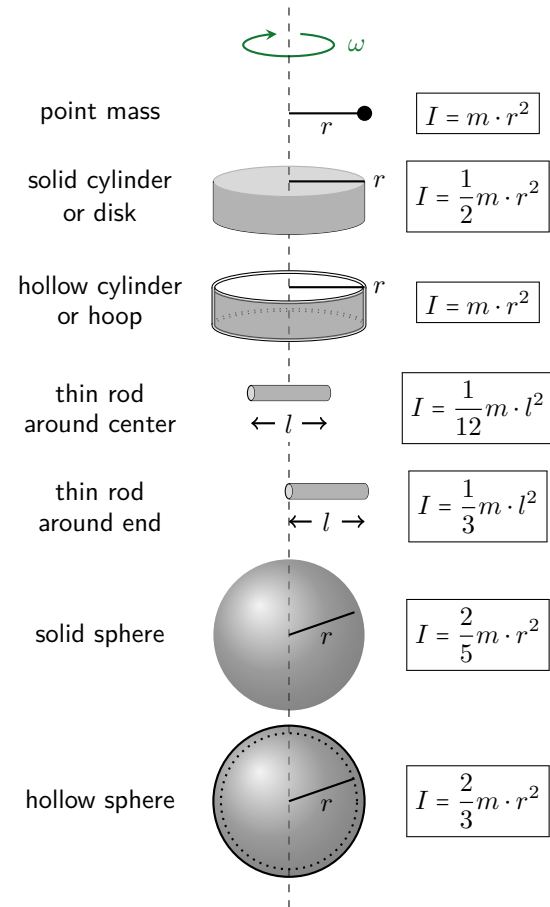
Figure 7.11: Rotational inertia for various shapes, each with mass $m$ and the same radius $r$ or length $l$, when rotated around the vertical dashed line.[1]

Since we are dealing with rotation, it will be helpful to find the rotational inertia. The rotational inertia for several shapes are shown in Figure 7.11. We need to look for the shape that is most similar to the blades.

In this case, if we consider the two blades as a single object with twice the length of one blade we can use the "thin rod around center" from Figure 7.11.

$$
\begin{aligned}
I &= \frac{1}{12} m_{tot} \cdot l_{tot}^2 \\
&= \frac{1}{12} \left(120\,\text{kg}\right) \cdot \left(20\,\text{m}\right)^2 \\
&= 4\,000\,\text{kg} \cdot \text{m}^2
\end{aligned}
$$

Now we can use Equation 7.7 to find the net torque applied to the blades:

$$
\begin{aligned}
\tau = I \cdot \alpha &= \left(4\,000\,\text{kg} \cdot \text{m}^2\right) \cdot 10.4\,\text{rad/s}^2 \\
&= 41\,600\,\text{N} \cdot \text{m}
\end{aligned}
$$

We can also use Equation 7.11 to find the final angular momentum of the blades:

$$
\begin{aligned}
L_f = I \cdot \omega_f &= \left(4\,000\,\text{kg} \cdot \text{m}^2\right) \cdot 52\,\text{rad/s} \\
&= 208\,000\,\text{kg} \cdot \text{m}^2/\text{s}
\end{aligned}
$$

And finally, we can use Equation 7.10 to find the final rotational kinetic energy:

$$
\begin{aligned}
E_{k,r,f} = \frac{1}{2} I \cdot \omega_f^2 &= \frac{1}{2} \cdot \left(4\,000\,\text{kg} \cdot \text{m}^2\right) \cdot \left(52\,\text{rad/s}\right)^2 \\
&= 5.4 \times 10^6\,\text{J}
\end{aligned}
$$

145

# 7.4 Figure Skating

## Words

Describe what happens if a 60-kg skater who is 1.7 m tall with a torso diameter of 0.3 m is spinning at a rate of 1.5 rad/s in the position shown in Figure 7.12 and then brings themselves into a vertical position, as in Figure 7.13.

When we are dealing with linear motion, we know that the more mass something has, the more difficult it is to move. But once you get it moving, it will have a large amount of momentum and kinetic energy. For rotational motion, it is not just the mass but how the mass is distributed that is important. This combination of mass and distance is called the rotational inertia. Figure skaters are experts at changing their rotational inertia, which is what is happening when they go from a horizontal position to a vertical one.

For someone who has observed a spinning figure skater, it is hard to forget what happens when they change their rotational inertia. The figure skater can start a spin, and then spin faster and faster, apparently without any effort, without even pushing off again on the ice.

The only forces on the skater are gravity and the normal force from the ice, exactly the same as for the motionless rock we considered so long ago. The rock didn't suddenly start spinning faster and faster. So how can a figure skater's angular speed increase when there is no torque?

## Graphics



Figure 7.12: Tangxu Li at Lillehammer.[38]
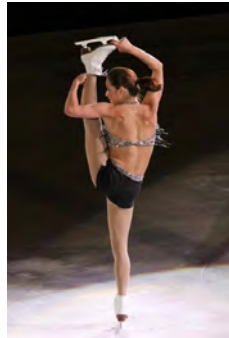


Figure 7.13: Sasha Cohen skating at Stars on Ice in 2009, and an approximation of her shape for finding her rotational inertia.[36]
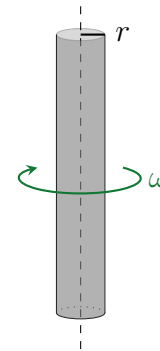
## Numbers

**Assumptions:** No horizontal forces; $\tau = 0$

| Knowns | Unknowns |
|---|---|
| $m = 60\,\text{kg}$ | ??? |
| height $l = 1.7\,\text{m}$ | |
| radius $r = 0.15\,\text{m}$ | |
| $\omega_i = 1.5\,\text{rad/s}$ | |

We need to find the shape from Figure 7.11 that is most similar to the skater, and use the rotational inertia for that shape. In this case, we have two different shapes to consider.

The final position shown in Figure 7.13 is simpler, so we will start there. Sasha Cohen's body does not form a perfect solid cylinder, but there is no better option among the shapes in Figure 7.11. It should be noted that the location and direction of the axis of rotation is important. The "thin rod" in Figure 7.11 is rotating in such a way that its ends go around, or one end stays in position while the other end goes around. The "solid cylinder" is rotating along the axis of the cylinder, which is the direction in which Sasha is rotating.

Approximating Sasha as a solid cylinder, her rotational inertia is

$$I_f = \frac{1}{2} m \cdot r^2 = \frac{1}{2} 60\,\text{kg} \cdot (0.15\,\text{m})^2 = 0.675\,\text{kg} \cdot \text{m}^2$$

For an isolated system, angular momentum is conserved. That is the key to what is happening to the figure skater. The figure skater has no linear velocity, but they have angular velocity. Angular momentum of a rotating object is the rotational inertia times the angular velocity, just as linear momentum is the mass times the velocity.

So when the skater is spinning with their body in a horizontal direction, their mass is spread out far from the vertical axis around which they are spinning. That gives them a large rotational inertia. Then, when they pull themselves into a vertical position they are bringing their mass in close to the axis around which they are spinning. This reduces their rotational inertia. Since angular momentum is conserved, reducing the rotational inertia creates a corresponding increase in angular velocity.

This increase in angular velocity also results in an increase in the rotational kinetic energy of the skater, so work was done on the skater. Where did the work come from? The skater does work on their own body. A centripetal force, toward the axis of rotation, is needed to keep something moving in a circle. In order to move something closer to the axis, the displacement is in the direction of that force, so the skater has to do work to come to a vertical position.
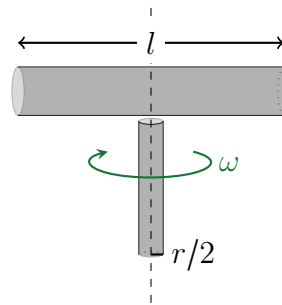


Figure 7.14: An approximation of the shape of Tangxu Li for finding his rotational inertia.[1]

The position that Tangxu Li is in is more difficult to analyze, because his body is going in two directions. His left leg is vertical, and so could be approximated as a cylinder just as we did with Sasha. But the rest of his body is horizontal and spinning end-to-end, which is much more like the "thin rod around center." To find Tangxu Li's rotational inertia, we can make both of these approximations, and add their rotational inertia. If we assume that his left leg contains $1/4$ of the mass of his whole body and is $1/2$ the radius of his torso, we have...

$$I_i = I_{leg} + I_{body} = \frac{1}{2}\left(\frac{1}{4}m\right)\cdot\left(\frac{r}{2}\right)^2 + \frac{1}{12}\left(\frac{3}{4}m\right)\cdot l^2$$
$$= (0.04 + 10.8)\,\mathrm{kg\cdot m^2} = 10.8\,\mathrm{kg\cdot m^2}$$

Apparently we really didn't even need to take the vertical leg into account. It is so close to the axis compared to the rest of the body that it doesn't significantly affect the rotational inertia. Now we can use conservation of angular momentum to find the final angular velocity of the skater:

$$L_f = L_i$$
$$I_f \cdot \omega_f = I_i \cdot \omega_i$$

$$\omega_f = \frac{10.8}{0.675}\cdot 1.5\,\mathrm{rad/s} = 24\,\mathrm{rad/s}$$

The angular velocity increases by more than a factor of 10! Now, using the Work-Energy Theorem with Equation 7.10...

$$W_{net} = \Delta E = \frac{1}{2}I_f \cdot \omega_f^2 - \frac{1}{2}I_i \cdot \omega_i^2 = 182\,\mathrm{J}$$

So 182 J of work was done by the skater to change position while spinning.

147

## 7.5   Charging a Radio

### Words

In areas with limited access to electrical power, hand-cranked devices are often used. Figure 7.15 shows a boy cranking a radio to charge its battery. He has to push on the end of the 0.08-m-long handle of the crank with a force of 5 N to get it to move. If he cranks at a constant rate of 10 rad/s, how much time will it take him to store 150 J of energy in the battery, assuming that the charging system is 100% efficient?

In this situation, we are given a force that is applied a certain distance away from a pivot, so the boy is applying a torque to the crank. With linear motion, a force that is applied through a distance does work (and so can store energy). The rotational corollary of this is that a torque that is applied through an angle does work.

The amount of work that the boy does will be proportional to the torque and also proportional to the angle. Since he is cranking at a constant rate, the angle will change linearly in time. The force he applies will have to be constant at 5 N so that there is no acceleration. So the applied torque is also constant. Since the angle changes linearly in time and the force is constant, the energy stored will also increase linearly in time. And since power is energy per time, the power the boy produces and stores is constant while he is cranking.

### Graphics



Figure 7.15: A boy charging his hand-cranked radio[41]

### Numbers

**Assumptions:** 100% efficiency; $+\hat{\theta}$ is counterclockwise; viewing image from the left

| Knowns | Unknowns |
|---|---|
| $r = 0.08\,\mathrm{m}$ | $t$ |
| $F_{applied} = -5\,\mathrm{N}\,\hat{\theta}$ | |
| $\omega = -10\,\mathrm{rad/s}$ | |
| $W = 150\,\mathrm{J}$ | |

The torque applied to the handle, as shown in Figure 7.16, is

$$\tau = F_{applied} \cdot r \cdot \sin\theta = -0.4\,\mathrm{N \cdot m}$$

The angular version of our expression for work done by a force is…

$$\boxed{W = \tau \cdot \Delta\theta} \tag{7.13}$$

…when the torque is constant. And since power is work per time, it can also be expressed as…

$$\boxed{P = \tau \cdot \omega} \tag{7.14}$$

From Equation 7.13 the total angle that the boy has to crank through to charge the radio is:

$$\Delta\theta = \frac{W}{\tau} = \frac{150\,\mathrm{J}}{-0.4\,\mathrm{N \cdot m}} = -375\,\mathrm{rad}$$

…or approximately 60 full revolutions.

$F_{applied}$

$\theta$ $r_\perp$

pivot
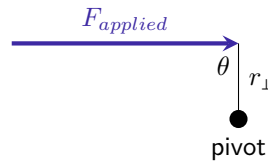
Figure 7.16: The boy applies a clockwise torque on the device, as seen from the left in Figure 7.15. Note that this theta refers to the angle between the force and the lever arm, not the rotation of the crank.[1]
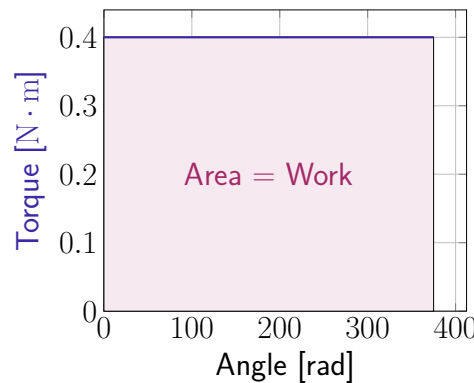


Figure 7.17: A torque in the direction of angular motion does an amount of work equal to the area under the curve in a Torque-vs-Angle graph. [1]

Torque and angle are taken to be positive for this graph. Using negative values as described in the "Numbers" section, the area would be negative (so under the x axis) but the angle would go right-to-left instead of left-to-right, which creates another negative. Looking at Figure 7.15 from the right instead of from the left would have made both torque and angle positive.

With the angular displacement we can find the time using Equation 7.4.

$$\Delta t = \frac{\Delta\theta}{\omega} = 37.5\,\text{s}$$

This question could also have been answered using a linear analysis of work done in the tangential direction:

$$s = \frac{W}{F_T} = \frac{150\,\text{J}}{-5\,\text{N}} = -30\,\text{m}$$

We aren't given a tangential velocity, but we can find it from the angular velocity:

$$v_T = \omega \cdot r = -0.8\,\text{m/s}$$

Then we can find the time:

$$\Delta t = \frac{s}{v_T} = 37.5\,\text{s}$$

Whichever way the analysis is done, the result is the same. The power generated by the boy is also the same either way:

$$P = \frac{W}{\Delta t} = \frac{150\,\text{J}}{37.5\,\text{s}} = 4\,\text{W}$$

It should be noted that there are two different kinds of "W" in the expression above. The italicized one in the numerator ($W$) is Work; the one at the end that is not italicized is watts.

149

## 7.6 Hoop Rolling

### Words

In the children's game "hoop rolling," a stick is used to start a hoop moving and then to keep it going. Let's say that the hoop starts at rest and accelerates uniformly across level ground to a speed of 5 m/s to the right in 2.5 seconds. The hoop has a mass of 0.4 kg and a radius of 0.2 m, and it rolls without slipping. Describe everything you can about this situation.

We know from considering other situations that velocity changes in the direction of acceleration, so acceleration is to the right. Momentum starts at zero and increases to the right along with the velocity, so there is a force to the right. Since the ground is level, we don't need to consider gravitational potential energy, and there also is no spring potential energy to consider. The hoop rolls without slipping, and usually thermal energy is generated by surfaces sliding together or by collisions. In this case, we don't have either. So the hoop starts with no kinetic energy and ends with kinetic energy. This means that positive work was done on the hoop, which makes sense because we have already established that there is a force to the right and the motion is to the right. Force in the direction of motion does positive work.

We have done all of that before. But that's not all that there is to the story. We haven't taken into consideration the fact that the hoop is rolling. This is a physical scenario that involves both translation (linear movement) and rotation.

### Graphics



Figure 7.18: Statue of a boy playing "hoop rolling."[39]
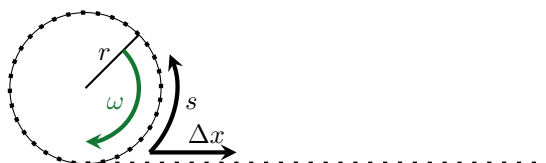


Figure 7.19: As the hoop rolls, it does not slide, so the distance it moves has the same magnitude as the path length on the circumference of the hoop, $\Delta x = s$. As the hoop rolls, each mark on the hoop will fall on its corresponding, equally-spaced mark on the ground.[1]

### Numbers

**Assumptions:** $a$ is constant; rolls without slipping

| Knowns | Unknowns |
|---|---|
| $\vec{v_0} = 0$ | ??? |
| $\vec{v_f} = 5\,\text{m/s}\,\hat{x}$ | |
| $\Delta t = 2.5\,\text{s}$ | |
| $m = 0.4\,\text{kg}$ | |
| $r = 0.2\,\text{m}$ | |

It is possible to approach this question just as we would have after one or two chapters, using a linear analysis of motion, forces, *etc*. But this analysis will be done by analyzing it from an angular perspective. Our knowns are all linear, but they are connected to angular motion through tangential velocity. In the reference frame of a person sitting at the center of the hoop, the ground goes to the left at the speed at which the hoop is moving, and the bottom edge of the hoop moves along with it.

$$\omega_0 = -\frac{v_0}{r} = 0$$

$$\omega_f = -\frac{v_f}{r} = -25\,\text{rad/s}$$

Why the minus signs? It is possible to keep all signs consistent mathematically, but it is often easier just to look at a sketch and see whether the motion ( or angular momentum, or torque) is counterclockwise (+) or clockwise (-) and use the appropriate

At first, the hoop is not rolling, so zero angular velocity, but at the end it is rolling to the right, which means that it is turning in the clockwise direction. The standard "positive" angular direction in physics is counter-clockwise, so the final angular velocity is negative. That means that the angular acceleration is also negative.

Angular momentum starts at zero and increases along with angular velocity, in the negative direction, so there is a negative torque.

Torque is created by an off-center force. In this case, the child is pushing to the right at some point near the middle of the hoop, and friction with the ground prevents it from rolling by pushing left on the bottom of the hoop. It is the combination of these two forces that creates the torque.

We also have one additional type of energy to consider: rotational kinetic energy. Not all of the work that the child did in pushing the hoop went into linear kinetic energy; some went into rotational kinetic energy.
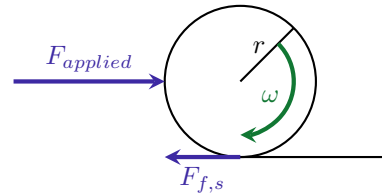


Figure 7.20: We know that the net force is to the right, because that is the direction of acceleration. But we also know that there is a force of friction to the left, because the hoop would slide to the right without this friction.[1]
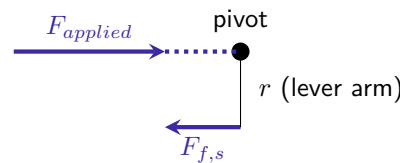


Figure 7.21: Force locations when considering torque.[1]

If we use the center of the hoop as the reference frame, the only motion of the hoop is rotational, around the center of the hoop. So we can use the center of the hoop as our pivot point. In this reference frame, it is the frictional force that is not aligned with the pivot but is separated by a lever arm distance. So it is the frictional force that creates the torque, making the hoop roll instead of sliding.

sign. In this example, we can see that the hoop is spinning clockwise.

Knowing the time, we can use Equation 7.3 to find the angular acceleration:

$$\alpha = \frac{\Delta\omega}{\Delta t} = -10 \,\text{rad/s}^2$$

We can also find the total angle that the hoop rolls through as it comes up to speed in the 2.5 s.

$$\Delta\theta = \theta - \theta_0 = \cancel{\omega_0}^{\,0} \cdot t + \frac{1}{2}\alpha \cdot t^2 = 31.3 \,\text{rad}$$

It should be clear that the appropriate rotational inertia is that of a hoop, so...

$$I = m \cdot r^2 = 0.016 \,\text{kg} \cdot \text{m}^2$$

With this information, we can find final angular momentum, torque, and final rotational kinetic energy:

$$L_f = I \cdot \omega_f = -0.4 \,\text{kg} \cdot \text{m}^2/\text{s}$$

$$\tau = I \cdot \alpha = -0.16 \,\text{N} \cdot \text{m}$$

$$E_{k,r,f} = \frac{1}{2}I \cdot \omega_f^2 = 5 \,\text{J}$$

So the total work done by the child on the hoop is...

$$W_{net} = \Delta E_{k,t} + \Delta E_{k,r} = \frac{1}{2}m \cdot v_f^2 + 5 \,\text{J} = 10 \,\text{J}$$

For a rolling hoop, half of its kinetic energy is rotational! We can use the Work-Energy Theorem to find the force applied by the child:

$$F_{applied} = \frac{W_{tot}}{\Delta x} = \frac{10 \,\text{J}}{r \cdot \Delta\theta} = 1.6 \,\text{N}$$

## 7.7  Center of Mass

### Words

The idea of "center of mass" is important for rotation. For one thing, the axis that an isolated, rotating object spins around goes through its center of mass. For another thing, when determining the torque created by an object's weight, all of the object's mass behaves as though it were sitting at the center of mass.

For some situations, it is easy to find the center of mass–it's just in the center!

This is the case for something like a bocce ball, which is a uniform, solid sphere. It is also the case for something like a tennis ball, which is a uniform, *hollow* sphere. Even though there is nothing actually at the center of the ball, that point is still the center of mass.

And even for objects with more unusual shapes, like a dumbbell, as long as the object is symmetrical, the center of mass will be at its center.

For objects with asymmetrical shapes or for a system of different objects, the center of mass is somewhere in the middle, shifted toward the side with more mass. In the photo of the tennis ball, bocce ball, and dumbbell, the tennis ball has a much smaller mass than the other two, so the center of mass of this group of three objects is somewhere inbetween the dumbbell and the bocci ball. Where exactly that center of mass is located depends upon the relative masses of the objects.

### Graphics
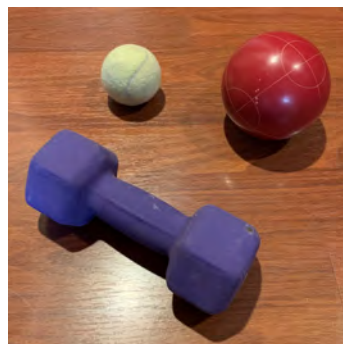


Figure 7.22: A tennis ball (top left), bocce ball (top right), and dumbbell[1]



Figure 7.23: A stack of books of varying widths and masses[1]

### Numbers

The center of mass of an object or system of objects is a position, so it is a vector quantity.

$$\overrightarrow{x_{com}} = \frac{m_1 \cdot \overrightarrow{x_1} + m_2 \cdot \overrightarrow{x_2} + \cdots}{m_1 + m_2 + \cdots} \tag{7.15}$$

...where $m_1$ and $\overrightarrow{x_1}$ refer to the mass and position of object 1, $m_2$ and $\overrightarrow{x_2}$ are for object 2, and every object in the system is included in the sums.

The easiest way to use this equation is to break the system up into pieces that have an obvious center of mass, and use those as objects 1, 2, *etc*.

Figure 7.23 shows a stack of four books. The center of mass of each individual book is at the center of the book. If we know the mass and position of each book we can find the center of mass of the stack:

| Book | Mass | Position |
|------|------|----------|
| 1 | 0.202 kg | 10.7 cm |
| 2 | 0.242 kg | 12.2 cm |
| 3 | 0.302 kg | 13.9 cm |
| 4 | 0.324 kg | 16.2 cm |

Combining this data using Equation 7.15 gives:

$$x_{com} = \frac{14.6\,\text{kg}\cdot\text{cm}}{1.07\,\text{kg}} = 13.6\,\text{cm}$$

152

To see how this works with rotation, we will consider a 3-m-long, 3-kg thin rod with a 4-kg point mass on the end. Imagine the rod connected by a hinge to a solid wall, and initially held out horizontally, as shown in Figure 7.24. If the rod-and-mass system is released from this position, what is its initial angular acceleration due to gravity? What is the initial tangential acceleration of the point mass due to gravity?

One end of the rod is held by the hinge, so the rod and point mass will rotate around the hinge because of the torque caused by gravity. Since we are dealing with rotation, we will need to find the rotational inertia of the system. For that, we can add up the rotational inertia of the rod rotating around the end and the point mass.

We will also need to find the torque created by gravity. To do this, we can first find the center of mass of the system and then consider the force of gravity of the whole rod-and-mass system to be acting at that single point.
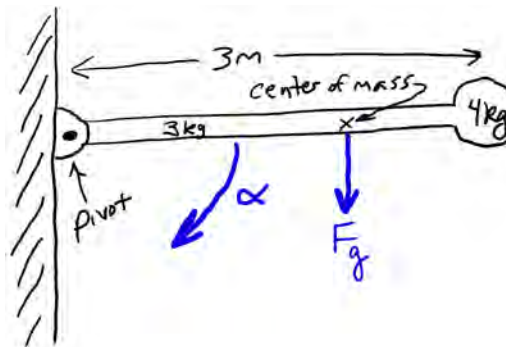


Figure 7.24: A long, thin rod with a mass on the end, connected to a wall with a hinge.[1]

| Knowns | Unknowns |
|---|---|
| $m_{rod} = 3\,\text{kg}$ | $\alpha_i$ |
| $l_{rod} = 3\,\text{m}$ | $a_{T,i,point}$ |
| $m_{point} = 4\,\text{kg}$ | |
| $g = 9.8\,\text{m/s}^2$ | |

We can add the rotational inertia of the rod and mass to find the rotational inertia of the system:

$$I_{system} = I_{rod} + I_{point}$$
$$= \frac{1}{3} m_{rod} \cdot l_{rod}^2 + m_{point} \cdot l_{rod}^2$$
$$= 9\,\text{kg} \cdot \text{m}^2 + 36\,\text{kg} \cdot \text{m}^2$$
$$= 45\,\text{kg} \cdot \text{m}^2$$

We can also use Equation 7.15 to find the center of mass in the $x$ direction, taking the "$x = 0$" position to be at the pivot:

$$x_{com} = \frac{m_{rod} \cdot x_{rod} + m_{point} \cdot x_{point}}{m_{rod} + m_{point}}$$
$$= \frac{(3\,\text{kg}) \cdot (1.5\text{m}) + (4\,\text{kg}) \cdot (3\text{m})}{(3\,\text{kg}) + (4\,\text{kg})}$$
$$= 2.36\,\text{m}$$

Now we can use Equations 7.12 and 7.7 to find the initial angular acceleration:

$$\alpha_i = \frac{\tau}{I_{system}}$$
$$= \frac{-F_g \cdot x_{com}}{I_{system}}$$
$$= \frac{-m_{system} \cdot g \cdot x_{com}}{I_{system}}$$
$$= -3.59\,\text{rad/s}^2$$

153

# 7.8 Off-Center Rotation

## Words

For most of the objects in Figure 7.11, the objects are assumed to be rotating around their center of mass. But sometimes we need to work with objects that are rotating around a point that isn't at their center of mass.

Let's take the example of the pendulum of a grandfather clock. The pivot is at the top of a 0.9-m-long rod, and the pendulum is composed of the rod itself with a 10-cm-radius disk attached near the bottom of the rod, centered at a position 0.78 m below the pivot. We will assume that the rod and the disk each have a mass of 0.3 kg. We can find the total rotational inertia of the pendulum around the pivot.

We already have expressions for a thin rod rotating around its end and a disk rotating around its center, but we don't have an expression for a disk rotating far away from its center. Since the mass is farther away from the axis of rotation in this case, we know the rotational inertia has to be larger.

It turns out that it is fairly simple to change to a different axis of rotation, as long as the new axis is parallel to an axis of rotation that you already know around the center of mass of the object. You just need to add one additional piece to the rotational inertia. This is called the "parallel axis theorem."

## Graphics



Figure 7.25: A grandfather clock. The pendulum with a dark rod and brass disk can be seen behind the glass.[40]

## Numbers

**Assumptions:** uniform rod and uniform disk

| Knowns | Unknowns |
|---|---|
| $m_{rod} = 0.3\,\text{kg}$ | $I_{\text{tot}}$ |
| $l_{\text{rod}} = 0.9\,\text{m}$ | |
| $m_{disk} = 0.3\,\text{kg}$ | |
| $r_{\text{disk}} = 0.1\,\text{m}$ | |
| $d_{\text{disk}} = 0.78\,\text{m}$ | |

The "parallel axis theorem" gives the rotational inertia for an object that is rotating around an axis other than its center of mass:

$$\boxed{I = I_{\text{com}} + m \cdot d^2} \qquad (7.16)$$

...where $d$ is the distance from the pivot to the center of mass of the object.
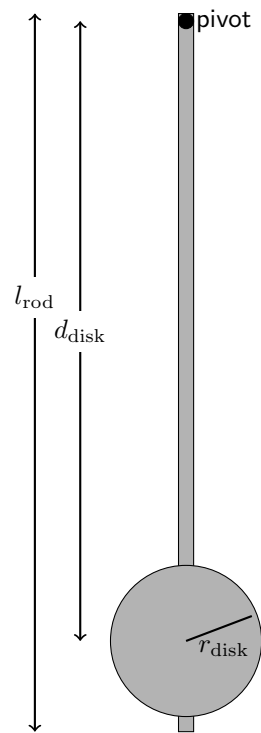
Figure 7.26: Schematic of the pendulum of the grandfather clock[1]

In this case, the center of the disk is 0.78 m from the pivot, so

$$
\begin{aligned}
I_{\text{disk}} &= \frac{1}{2} m_{\text{disk}} \cdot r_{\text{disk}}^2 + m_{\text{disk}} \cdot d_{\text{disk}}^2 \\
&= \frac{1}{2} \cdot (0.3\,\text{kg}) \cdot (0.1\,\text{m})^2 + (0.3\,\text{kg}) \cdot (0.78\,\text{m})^2 \\
&= 0.184\,\text{kg} \cdot \text{m}^2
\end{aligned}
$$

That only accounts for the disk, not the rod. So the total rotational inertia of the pendulum is…

$$
\begin{aligned}
I_{\text{tot}} &= I_{\text{rod}} + I_{\text{disk}} \\
&= \left(0.184\,\text{kg} \cdot \text{m}^2\right) + \frac{1}{3} m_{\text{rod}} \cdot l_{\text{rod}}^2 \\
&= \left(0.184\,\text{kg} \cdot \text{m}^2\right) + \frac{1}{3} \cdot (0.3\,\text{kg}) \cdot (0.9\,\text{m})^2 \\
&= \left(0.184\,\text{kg} \cdot \text{m}^2\right) + \left(0.081\,\text{kg} \cdot \text{m}^2\right) \\
&= 0.265\,\text{kg} \cdot \text{m}^2
\end{aligned}
$$

# 7.9 Balancing

## Words

Figure 7.27 shows a board and a brick balancing on the end of another board. The brick has a mass of 1.76 kg and its center of mass is 7.5 cm to the left of the upright board. The horizontal board has a length of 49 cm, and its left edge is 9.5 cm to the left of the upright board. We can use this information to find the mass of the horizontal board and the normal force that the upright board applies to the horizontal board.

The key to understanding this scenario is the rotational corollary to Newton's First Law. That law states that if the net force on a system is zero then the acceleration of that system is also zero. The rotational corollary of that statement would be that if the net torque on a system is zero then the angular acceleration of that system is also zero. Since the system is balancing motionless, the acceleration, both angular and linear, must be zero. That situation is called "static equilibrium."

There are actually two ways to approach this problem. One is by considering forces and torque as described above, and the other is by considering the center of mass. Let's first consider the center of mass.

Unless the brick, boards, and cement floor are somehow glued or bolted together (they aren't!), then to remain balanced they have to be supported at the center of mass. The center of mass of the brick is clearly above the horizontal board. If it weren't, for example if it were hanging more than

## Graphics



Figure 7.27: Two boards and a brick balancing[1]
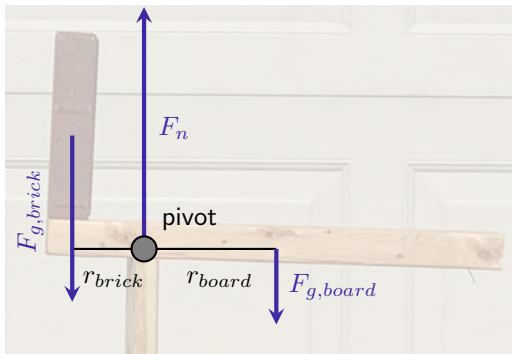


Figure 7.28: The forces and torques on the balancing brick and the horizontal board.[1]

## Numbers

| Knowns | Unknowns |
|---|---|
| $m_{brick} = 1.76\,\text{kg}$ | $m_{board}$ |
| $r_{brick} = 0.075\,\text{m}$ | $F_n$ |
| $r_{board} = 0.15\,\text{m}$ | |
| $g = 9.8\,\text{m/s}^2$ | |

$r_{board}$ was found by assuming that the center of mass of the board is at the center of the board, which is 0.245 m to the right of the left end of the board, and thus

$$r_{board} = (0.245\,\text{m} - 0.095\,\text{m}) = 0.15\,\text{m}$$

For an object that is in static equilibrium, the net force and the net torque are both zero:

$$\boxed{\Sigma \vec{F} = 0} \tag{7.17}$$

$$\boxed{\Sigma \tau = 0} \tag{7.18}$$

Calculation of the net torque around the pivot in Figure 7.28 gives

$$\Sigma \tau = \tau_{brick} + \tau_n^{\,0} - \tau_{board} = 0$$

$$F_{g,brick} \cdot r_{brick} \cdot \sin 90°^{\,1} = F_{g,board} \cdot r_{board} \cdot \sin 90°^{\,1}$$

$$m_{board} = m_{brick} \cdot \frac{r_{brick}}{r_{board}} = 0.88\,\text{kg}$$

We can also use Figure 7.28 as a free body diagram to find $F_n$, since the sum of forces is zero.

$$\Sigma F_y = -F_{g,brick} + F_n - F_{g,board} = 0$$

halfway off of the end of the board, the brick would fall. The combined center of mass of the brick and the horizontal board has to be above the vertical board for the same reason. If it weren't, the horizontal board would fall. The combined center of mass of both boards and the brick also has to be above the bottom of the vertical board. If it were not, the vertical board would fall.

Now, we will consider the situation from the perspective of forces and torque. Since we are not given information about the vertical board, and not asked any questions about it, we will only use torque to examine the brick and horizontal board. If they were to fall, they would tip around a pivot at the top of the vertical board. It doesn't tip, so that means the net torque around that pivot has to be zero.

There is a gravitational force acting downward on the brick, and since the brick is to the left of the pivot, it would cause counterclockwise (or positive) rotational motion. So the brick creates a positive torque. The horizontal board must create an equal but opposite torque to keep the system balanced. When considering torque, all of an object's mass acts as if it is at a single point: the center of mass of the object. The center of mass of the horizontal board should be near the center of the board, which is clearly to the right of the pivot. So the mass of the horizontal board has to be just enough that its gravitational force creates enough torque to balance the torque created by the brick.
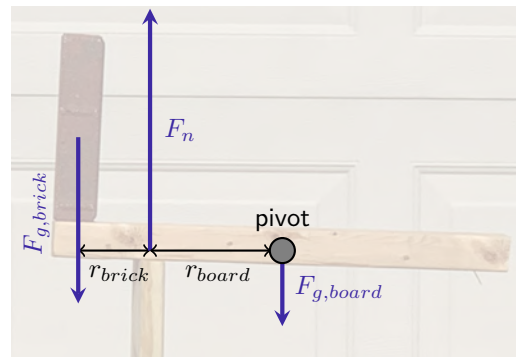


Figure 7.29: An alternate location for the pivot, allowing a solution for the normal force without knowing the mass of the board.[1]

For a system in static equilibrium, it isn't actually moving around any pivot point, so we are free to use any pivot point that is convenient. Notice that when we set the pivot at the point where the boards meet there was no torque generated by the normal force. If we instead chose a pivot at the center of mass of the horizontal board, its gravitational force will not generate any torque.

$$F_n = m_{brick} \cdot g + m_{board} \cdot g = 25.9 \,\text{N}$$

We could have found $m_{board}$ using center of mass instead. Since we are free to choose our zero position, it is convenient to put it at the location of the pivot in Figure 7.28, so that the center of mass of the system is at $r = 0$. Using Equation 7.15…

$$r_{com} = 0 = \frac{m_{brick} \cdot r_{brick} + m_{board} \cdot r_{board}}{m_{brick} + m_{board}}$$

$$m_{brick} \cdot r_{brick} = -m_{board} \cdot r_{board}$$

$$m_{board} = \frac{(-1.76\,\text{kg}) \cdot (-0.075\,\text{m})}{0.15\,\text{m}} = 0.88 \,\text{kg}$$

A negative value was assigned to $r_{brick}$ because it is to the left of the zero point at the pivot.

We could also have found $F_n$ without knowing $m_{board}$ simply by choosing a different pivot point. This is illustrated in Figure 7.29.

$$\Sigma \tau = \tau_{brick} - \tau_n + \cancel{\tau_{board}}^{0} = 0$$

$$F_{g,brick} \cdot (r_{brick} + r_{board}) = F_n \cdot r_{board}$$

$$F_n = \frac{m_{brick} \cdot g \cdot (0.075\,\text{m} + 0.15\,\text{m})}{0.15\,\text{m}} = 25.9 \,\text{N}$$

## 7.10   Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

### General

- Objects have a "rotational inertia" that increases with mass and size. The farther an object's mass is distributed from the axis of rotation, the larger its rotational inertia.

- The rotational inertia of multiple parts can be added together for a complex shape.

- An object can only balance when its center of mass is supported.

### Forces

- Torque $[\mathrm{N} \cdot \mathrm{m}]$ is the angular quantity that corresponds to force. It is a force applied in a direction that is not aligned with the pivot point, but separated from it by a lever arm distance.

- Torque causes angular acceleration, changes angular momentum over time, and does work through an angle.

- Static equilibrium is when an object is completely motionless. It can only occur when net force and net torque are both zero.

- When determining the torque created by an object's weight, all of the object's mass behaves as though it is at the center of mass.

### Motion

- Every equation of motion that we have learned works equally well for angular motion, simply by replacing the linear quantities with their angular counterparts.

- The SI unit for angular acceleration is $[\mathrm{rad/s^2}]$.

- The axis that an isolated, rotating object spins around goes through its center of mass.

### Momentum

- A point mass has angular momentum if it has momentum in a direction that is not aligned with the pivot point, but separated from it by a lever arm distance.

- Angular momentum is related to rotational inertia and angular velocity in the same way that linear momentum is related to mass and linear velocity.

- Angular momentum is conserved for an isolated system.

### Energy

- Rotating objects have rotational kinetic energy even if their center of mass is stationary.

## Mathematical Models

| equation | | restrictions on the validity of the equation |
|---|---|---|
| $\alpha = \frac{a_T}{r}$ | (7.1) | -none- |
| $\theta = \theta_0 + \omega_0 \cdot t + \frac{1}{2}\alpha \cdot t^2$ | (7.2) | only valid when the net torque is constant |
| $\alpha_{avg} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{\Delta t}$ | (7.3) | -none- |
| $\omega_{avg} = \frac{\Delta\theta}{\Delta t}$ | (7.4) | -none- |
| $\omega_{avg} = \frac{1}{2}\left(\omega_i + \omega_f\right)$ | (7.5) | only valid when the net torque is constant |
| $2\,\alpha \cdot \Delta\theta = \omega_f^2 - \omega_i^2$ | (7.6) | only valid when the net torque is constant |
| $\tau = F \cdot r_\perp$ | (7.7) | -none- |
| $r_\perp = r \cdot \sin\theta$ | (7.8) | -none- |
| $\tau = \frac{\Delta L}{\Delta t}$ | (7.9) | -none- |
| $E_{k,r} = \frac{1}{2}I \cdot \omega^2$ | (7.10) | -none- |
| $L = I \cdot \omega$ | (7.11) | -none- |
| $\tau = I \cdot \alpha$ | (7.12) | -none- |
| $W = \tau \cdot \Delta\theta$ | (7.13) | only valid when the torque is constant |
| $P = \tau \cdot \omega$ | (7.14) | only valid when the torque is constant |
| $\overrightarrow{x_{com}} = \frac{m_1 \cdot \overrightarrow{x_1} + m_2 \cdot \overrightarrow{x_2} + \cdots}{m_1 + m_2 + \cdots}$ | (7.15) | -none- |
| $I = I_{\text{com}} + m \cdot d^2$  "Parallel axis theorem" | (7.16) | -none- |
| $\Sigma\overrightarrow{F} = 0$ | (7.17) | constant velocity, static equilibrium |
| $\Sigma\tau = 0$ | (7.18) | static equilibrium |

# 7.11 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

## Level 1 - Remember

7.1 **[W & N]** Make a table of the names and symbols of the linear quantities and their corresponding angular quantities from the past two chapters.

7.2 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

## Level 2 - Understand

7.3 **[N]** Calculate the rotational inertia $I$ of the rod and ball as described in Section 7.1, if the length of the rod is 0.4 m and the mass of the ball is 1.2 kg.

7.4 **[W,G, & N]** Try to rank the objects in Figure 7.11 in order from those with the most mass concentrated near the axis of rotation to those with the most mass concentrated far from the axis of rotation. For the ones that have an $r$ instead of an $l$, how do the numbers multiplying $\left(m \cdot r^2\right)$ compare when you have put the shapes in order?

7.5 **[N]** At the end of Section 7.2 are three mathematical models with very little explanation. Replace the angular quantities in these three mathematical models with their linear counterparts, and verify that they are all valid mathematical models.

  (a) Equation 7.10 corresponds to...

  (b) Equation 7.11 corresponds to...

  (c) Equation 7.12 corresponds to...

## Level 3 - Apply

7.6 **[N]** Use dimensional analysis to show that Equation 7.11 and Equation 6.9 have the same SI units.

7.7 **[N]** If the ball described in Section 7.1 has a mass of 3.2 kg and is spinning at a constant rate of 2.5 rad/s on a rod with a length of 0.9 m, find its kinetic energy, its angular momentum, and the net torque that is being applied to keep it spinning at a constant rate.

7.8 **[N]** The analysis in Section 7.3 was done taking the two blades to be one object, a thin rod spinning about its center. That wouldn't work for a helicopter with three blades. Do the same analysis that was done in Section 7.3 but for the helicopter shown in the figure below. Take the mass of each blade, the length of each blade, the time, and the final angular speed to be the same as in Section 7.3.

Three-bladed helicopter.[42]

7.9 **[W, G & N]** If Sasha Cohen was spinning in the position shown in Figure 7.1 and then changed her position to be that like that of Elena Glebova in the image below, would her angular momentum, angular velocity, and rotational kinetic energy increase, decrease, or remain constant?
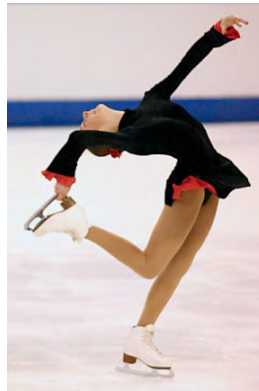

Elena Glebova in a spin.[43]

7.10 **[G & N]** Use Figure 7.21 and the torque that was found in Section 7.6 to find the magnitude of the force of friction between the hoop and the ground.

7.11 **[N]** Figure 7.11 includes only one object that is not rotating around its center of mass: the "thin rod around end." Use the parallel axis theorem and the rotational inertia of a thin rod around its center to verify the given rotational inertia for a thin rod around its end.

7.12 **[N]** Use the parallel axis theorem and one of the expressions for rotational inertia given in Figure 7.11 to find a simple expression for the rotational inertia of a solid sphere rotating around a point on its outer surface.

## Level 4 - Analyze

7.13 **[N]** Use dimensional analysis to find the SI unit for $E_{k,r}$. Is it the same unit that $E_k$ has for linear motion? Should it have the same unit? Explain your answer.

7.14 **[W, G, & N]** What is the total rotational inertia for two thin "half rods," each of mass $m/2$ and length $l/2$, rotated around their ends? Compare your answer to that for the rotational inertia of a single full rod with mass $m$ and length $l$, rotated around its center. What do you notice? Explain this result.

7.15 **[N]** How much work did the engine do on the blades in Section 7.3? Show your work or otherwise explain your answer.

7.16 **[W & N]** The initial tangential acceleration of the point mass in Section 7.7 was listed as an unknown, but it wasn't found. Find it. Consider the value that you found–is it surprising? Why or why not?

7.17 **[W & N]** The torque in Section 7.7 was calculated using the center of mass of the system as a whole. Try finding the torques created individually by the rod and by the point mass. Would using the individual torques have given the same answer for initial angular acceleration? Explain why or why not.

## Level 5 - Evaluate

7.18 **[N]** What effect does doubling angular velocity have on angular momentum and rotational kinetic energy?

7.19 **[W, G, & N]** In Section 7.6, a child is rolling a hoop. What would change if all of the knowns stayed the same, but the child was instead rolling a solid sphere?

7.20 **[W, G, & N]** In Section 7.6, a child is rolling a hoop. What would change if all of the knowns stayed the same, but the radius of the hoop doubled?

7.21 **[W, G, & N]** Try placing the pivot in Section 7.9 at the point where the brick is resting on the horizontal board. Does the equation for net torque also yield a valid solution at that location? Explain your answer.

7.22 **[W, G, & N]** In the analysis that was done in Section 6.5, it was noted that the angular momentum can't change when the string falls off of the bolt because there is no force to change the angular momentum. Now that we have learned more about angular momentum, we can see that the angular momentum was also not changing before the string fell off of the bolt. Why did the force of tension in the string, which was acting on the puck the whole time it was moving along a circular path, not change the angular momentum of the puck?

## Level 6 - Create

7.23 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Begin a new concept map just for rotation.

7.24 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

7.25 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

# Chapter 8

# Stability and Oscillations

An object that is in static equilibrium has zero net force and zero net torque acting on it. But what happens to such an object if a small unbalanced torque or unbalanced force is briefly applied?

In some situations, like that shown in the image of the acrobat balancing on a pile of chairs, that small torque will result in disaster. That's what makes acrobatic shows exciting to watch–we know that a tremendous amount of skill is needed to avert such a disaster.

In some situations, a small net torque or net force will cause an object's position to briefly shift, but then the object will return to its original position.

And in some situations, a small net torque or net force will cause an object to rock or swing back and forth, until eventually friction brings everything back into static equilibrium. This back-and-forth motion is called an "oscillation."

In this chapter we will explore what conditions determine whether an object in static equilibrium will experience disaster, return to its original state, or begin to oscillate when it is disturbed, and we will see how motion, momentum, forces, and energy interact during oscillations.



Figure 8.1: A performer with Peking Acrobats performing in Nashville, Tennessee.[44]

## 8.1 The Great Pyramid of Giza

### Words

The Great Pyramid of Giza is 137 m tall, 230 m long on each side, and has a mass of roughly 6 billion kg. Its center of mass is centered approximately 34 m above its base. For our purposes we will imagine that the sand it is sitting on is a hard, rough surface that can't be dented, and we will consider the pyramid to be one solid, unbreakable block, although in fact it is made up of roughly 2 million blocks of stone, and would break into roughly 2 million pieces if we attempted this with the actual pyramid!

First we will imagine briefly applying a force lifting one side of the pyramid to create a small net torque and see what happens to the pyramid. It would not be easy to apply such a force. When the pyramid is just sitting on the sand, there are two main forces acting on it: The force of gravity, which can be seen as acting downward on the center of mass of the pyramid, and the normal force which is pushing up from the sand all across the base of the pyramid but which is usually shown acting at a single point at the center.

To lift one side you would have to use a force that is half the weight of the pyramid, as shown in Figure 8.2. All of the normal force from the sand would be concentrated on the side opposite you, which is where the pyramid would pivot. So your force would have to create slightly more torque than the force of gravity, but your lever arm is twice as long as that for gravity.

### Graphics
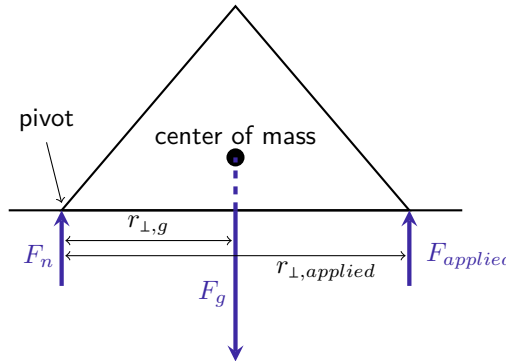


Figure 8.2: A huge applied force is needed to create enough torque around the pivot point to get the pyramid to move.[1]
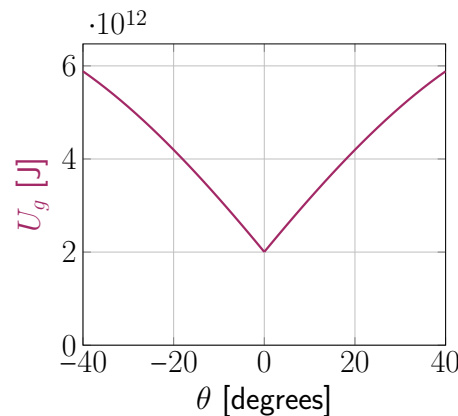


Figure 8.3: Gravitational potential energy is smallest with the pyramid flat on the ground.[1]

### Numbers

**Assumptions:** Pyramid is unbreakable; Sand is rough (high friction) and cannot be dented.

| Knowns | Unknowns |
|---|---|
| $m = 6 \times 10^9$ kg | $F_{applied,min}$ |
| $h_{pyramid} = 137$ m | |
| $h_{com} = 34$ m | |
| $r_{\perp,g} = 115$ m | |
| $r_{\perp,applied} = 230$ m | |

One thing we should be able to find in this situation is the minimum force needed to lift one side of the pyramid when it is sitting on its base, $F_{applied,min}$. To find that force, we need to analyze the situation shown in Figure 8.2 when it is in static equilibrium, just before the applied force is enough to lift the pyramid. The net torque around the pivot is zero in static equilibrium, so…

$$\tau_{net} = 0 = \tau_n + \tau_g + \tau_{applied}$$
$$= F_n \cdot r_{\perp,n}^{\;0} - F_g \cdot r_{\perp,g} + F_{applied} \cdot r_{\perp,applied}$$

Rearranging gives…

$$F_{applied} = \frac{F_g \cdot r_{\perp,g}}{r_{\perp,applied}} = \frac{F_g}{2}$$

$$= \frac{m \cdot g}{2} = 2.9 \times 10^{10} \text{ N}$$

When you release this force, the pyramid would just fall back into its original position. So it takes a huge force to create enough torque to move the pyramid, and when this force is released, the pyramid quickly goes back to its original state. It is extremely stable, which is why it is still standing forty-five centuries after it was built!

It is also possible to consider stability from an energy perspective. Objects like to go to the place that will give them the smallest possible amount of potential energy. In the case of the pyramid sitting on its base, there is no other position that has less gravitational potential energy, so it is very stable.

Now let's imagine flipping the pyramid upside-down. It can still be in static equilibrium if its center of mass is aligned perfectly above the tip, but if a small unbalanced torque is applied to the pyramid, the torque created by the force of gravity will continue to rotate the pyramid in the same direction as the initial torque, giving it more and more angular momentum,and it will come crashing down. This is an extremely unstable situation.

The center of mass in Figure 8.2 is above the base of the pyramid, but in Figure 8.4 it is outside of the "base" of the pyramid. In general, if the center of mass is outside of the base that holds the object up, it is in an unstable equilibrium unless there are other forces holding it in place.

From an energy perspective, any shift in the angle of the upside-down pyramid will give it less gravitational potential energy, which means more kinetic energy. So it will move away from the unstable equilibrium position at higher and higher speed.
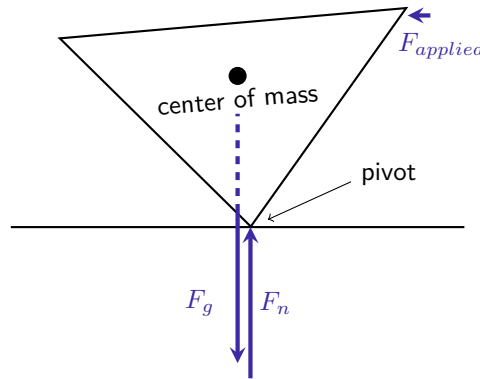


Figure 8.4: A small applied force creates torque around the pivot. This shifts the center of mass, so the gravitational force creates torque in the same direction.[1]
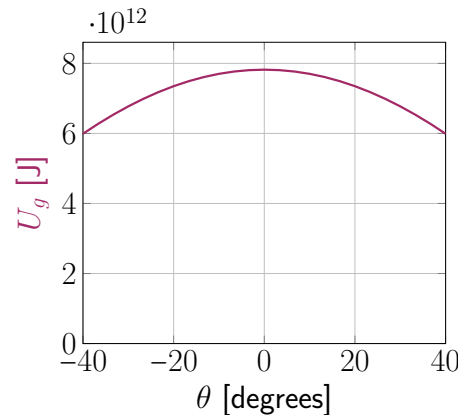


Figure 8.5: Gravitational potential energy is highest with the pyramid balanced on its tip. Here $0°$ is the upside-down position.[1]

This is the maximum force that keeps the pyramid in static equilibrium, which is equivalent to the minimum force to start moving it.

There is no minimum force needed to start the pyramid in motion when it is standing on its tip; any applied force at all is enough to start the pyramid in motion.

Gravitational potential energy in Figures 8.2 & 8.4 are calculated using ground level as zero. Since the center of mass of the pyramid is always above ground level, the gravitational potential energy is always positive. The height above the ground is calculated as a function of the angle of rotation, keeping one corner of the pyramid on the ground.

## 8.2 A Horizontal Spring and Mass

### Words

Now we will consider a horizontal system, ignoring any vertical forces because they are balanced and there is no vertical motion. A 2.4 kg block is motionless on a surface with negligible friction, connected to a solid wall by a spring with a spring constant of 90 N/m. Initially the block is resting at its equilibrium position, but then it is struck by an applied force that suddenly gives it a momentum of 1.2 kg·m/s to the left. What happens to the block?

If we look at this scenario in terms of energy, the block is given some kinetic energy and initially has no potential energy. But as the spring compresses, it stores potential energy, taking away the kinetic energy until the block completely stops moving. At that point, the unbalanced force from the spring is pushing the block back toward its starting point, so the spring potential energy transforms back into kinetic energy. The block passes through its equilibrium position where it again has kinetic energy but there is no potential energy, and continues with the spring pulling against the motion until again all kinetic energy has been removed from the block and stored as spring potential energy. The block will continue to oscillate back and forth in this way.

As the block oscillates, it is not only the position that is continually changing in time, but also its velocity, acceleration, and momentum, and the force that that spring is applying to it.

### Graphics
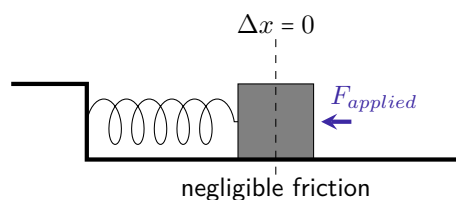


Figure 8.6: A mass on a spring is resting at its equilibrium position, and then is briefly hit with a horizontal applied force, giving it an initial momentum to the left.[1]
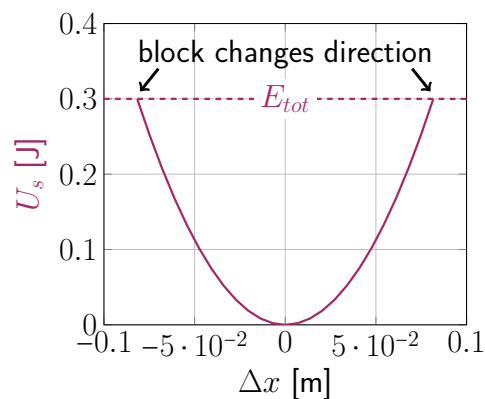


Figure 8.7: Spring potential energy is smallest at the equilibrium position, so the block oscillates around the equilibrium position.[1]

### Numbers

**Assumptions:** horizontal direction only; ideal spring; wall is immovable; friction is negligible

| Knowns | Unknowns |
|---|---|
| $m = 2.4\,\text{kg}$ | ??? |
| $k_s = 90\,\text{N/m}$ | |
| $\overrightarrow{p_i} = -1.2\,\text{kg}\cdot\text{m/s}\,\hat{x}$ | |
| $F_f = 0$ | |
| $\Delta x_0 = 0$ | |

We can use energy to analyze this situation, like we did for the pyramid. We are given the initial momentum and the mass, and we can use them to find the initial kinetic energy.

$$E_k = \frac{1}{2}m \cdot v^2 = \frac{1}{2}m \cdot \frac{p^2}{m^2} = 0.3\,\text{J}$$

Rearranging gives kinetic energy in terms of momentum, which is a useful mathematical model even when we aren't dealing with oscillation:

$$\boxed{E_k = \frac{p^2}{2m}} \qquad (8.1)$$

Now we can use conservation of energy to find the maximum displacement of the spring while it is bringing the block to a stop.

Figure 8.8 shows the position, momentum, and net force on the block as a function of time, although we don't know what the time scale should be. All three of these follow a sine-wave type of pattern, although they are shifted with respect to each other. Notice that when the position is at a positive maximum the momentum is at zero and the force is at a negative maximum. When the position is zero the force is also zero and the momentum is at a maximum, either positive or negative.

We could also create graphs for velocity and acceleration of the block, but they would be very similar to the graphs for momentum and force. That is because velocity is the momentum divided by the mass and acceleration is the unbalanced force divided by the mass.

The block and spring are in a stable equilibrium because the net force is always in a direction that pushes the block back toward the equilibrium position, never away from it. Or from an energy perspective we could say that the block and spring are in a stable equilibrium because the potential energy is always increasing and kinetic energy decreasing when the block is moving away from the equilibrium position.

This type of oscillation, with an unbalanced, conservative force that is proportional to the distance from the equilibrium position and no non-conservative forces, is called "simple harmonic motion."
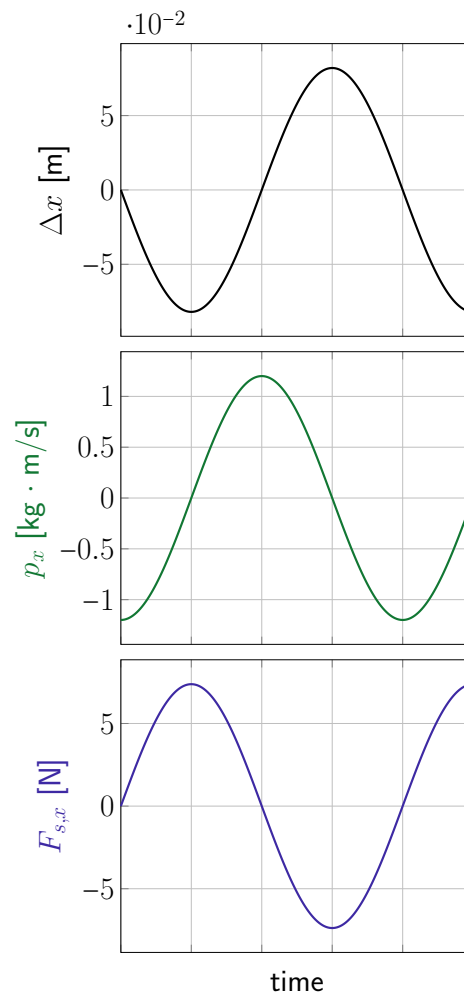


Figure 8.8: Position of the block, momentum of the block, and force on the block as a function of time. We don't yet know the time scale, but it is the same for all three graphs.[1]

Using the subscript "0" for the equilibrium position and "$max$" for the position where the spring is maximally stretched or compressed...

$$E_{max} = E_0$$

$$E_{k,max}{}^{\nearrow 0} + E_{th,max}{}^{\nearrow 0} + U_{s,max} = E_{k,0}{}^{\nearrow 0} + E_{th,0}{}^{\nearrow 0} + U_{s,0}{}^{\nearrow 0}$$

$$\frac{1}{2} k_s \cdot \Delta x_{max}^2 = \frac{p_0^2}{2m}$$

Solving for $\overrightarrow{\Delta x_{max}}$ gives...

$$\overrightarrow{\Delta x} = \pm \sqrt{\frac{p_0^2}{k_s \cdot m}}\, \hat{x}$$

$$= \pm \sqrt{\frac{(-1.2\,\text{kg} \cdot \text{m/s})^2}{90\,\text{N/m} \cdot 2.4\,\text{kg}}}\, \hat{x} = \pm 0.082\,\text{m}\,\hat{x}$$

Now that we know that the maximum displacement is ±0.082 m, we can use Hooke's Law to find the maximum force that is applied by the spring:

$$\overrightarrow{F_{s,max}} = -k_s \cdot \overrightarrow{\Delta x_{max}}$$

$$= -(90\,\text{N/m}) \cdot (\pm 0.082\,\text{m}\,\hat{x})$$

$$= \mp 7.38\,\text{N}\,\hat{x}$$

The significance of the upside-down "±" is that when the displacement is in the positive direction the force is in the negative direction, and vice-versa.

## 8.3  Jupiter's Moons

### Words

Circular motion has a lot in common with oscillation. If viewed from the side, an object moving in a uniform circular path appears to be oscillating in exactly the same way as a mass on a spring. For example, the planet Jupiter has four moons that are large enough to see from the earth with an amateur telescope or even a good pair of binoculars. These four moons have roughly circular orbits around Jupiter, but they always appear from earth to be oscillating back and forth across Jupiter in a straight line. We will call that line the parallel direction.

The moon named "Callisto" is on the right of figure 8.9. Callisto's orbit has a radius of $1.9 \times 10^9$ m, and it completes one orbit every 17 earth days. We can use this information to determine Callisto's position, velocity, and acceleration in the parallel direction as it orbits Jupiter.

If we take time $t = 0$ to be when Callisto is all of the way to the right as seen in Figure 8.10, then the initial position of Callisto is at its maximum positive value, $+r$. By the time it reaches the position at the top of Figure 8.10, it would be hidden from the earth, directly behind Jupiter. Its position in the parallel direction is zero at that point. Then it goes to $-r$ (all of the way to the left), then back to zero (in front of Jupiter as seen from the earth), and finally back to $+r$ again when it completes one full period.

### Graphics
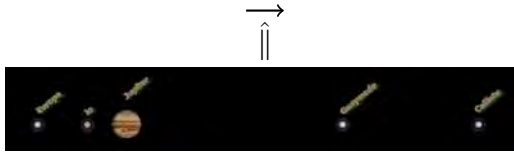


Figure 8.9: Jupiter and its four largest moons, as seen from Earth. The parallel direction $\hat{\parallel}$ is in the plane of the moons' orbits.[45]
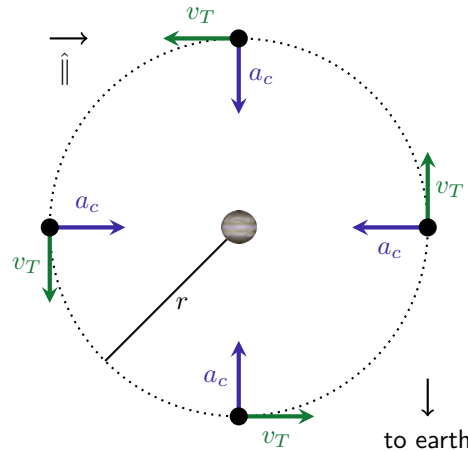


Figure 8.10: Callisto is shown at four different points in its near-circular orbit, but from earth we can only see it going back and forth in the horizontal $\left(\hat{\parallel}\right)$ direction, so we see only the parallel components of $\overrightarrow{x}$, $\overrightarrow{v}$, and $\overrightarrow{a}$.[46]

### Numbers

**Assumptions:** circular orbit; $+\hat{\parallel}$ is to the right in Figures 8.9 & 8.10

| Knowns | Unknowns |
|---|---|
| $r = 1.9 \times 10^9$ m | $x_\parallel$ |
| $T = 17\,\text{earth days}$ | $v_\parallel$ |
| | $a_\parallel$ |

First we should convert the period of the orbit to SI units: $17\,\text{earth days} = 1.5 \times 10^6$ s. We can also convert period into angular speed $\omega$, since we have used that more often more than period.

$$\omega = \frac{2\pi}{T} = 4.3 \times 10^{-6}\,\text{rad/s}$$

At all times in this full circle, its position in the parallel direction can be described by…

$$x_\parallel = r \cdot \cos\left(\omega \cdot t\right)$$

168

Velocity in the parallel direction also varies, but it starts at zero at $t = 0$, since it is only moving upward in Figure 8.10. Then it goes in the negative direction, goes back to zero on the left side, goes in the positive direction, and finally returns to zero as Callisto completes one period.

Centripetal acceleration is always pointed toward the center of the circle, so when the position is at its maximum positive value, acceleration is at its maximum negative value, and vice-versa.

When we analyzed the block on the spring, we considered momentum instead of velocity and net force instead of acceleration, but remember that momentum is proportional to velocity and net force is proportional to acceleration, so if we graph them then the shapes of their graphs will look the same, just with a different scale.

Comparing Figures 8.8 and 8.11, we can see that their shapes are very similar. Really there are only two major differences. Their scales are different, and the curves in Figure 8.8 are all shifted horizontally by the same amount compared to the curves in Figure 8.11. That is because the block on the spring was initially at zero and moving to the left while Callisto was initially at the far right and not moving at all in the parallel direction.
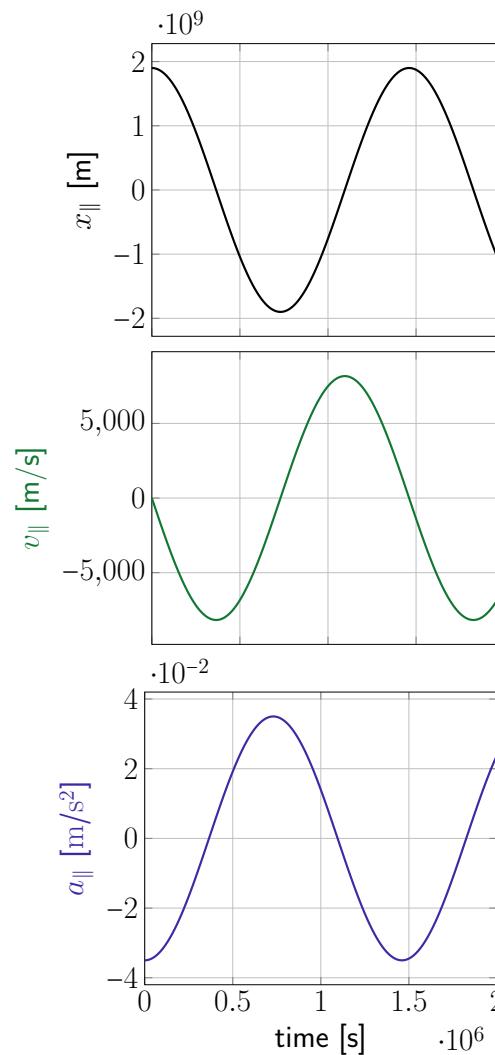


Figure 8.11: Parallel components of the position, velocity, and acceleration of Callisto.[1]

We know that $v_T = r \cdot \omega$, and the maximum value of $v_\parallel$ is $v_T$, so Callisto's velocity in the parallel direction can be described by…

$$v_\parallel = -r \cdot \omega \cdot \sin(\omega \cdot t)$$

Since our other mathematical models in this section depend on $\omega$, it would be convenient to have a mathematical model for centripetal acceleration that is also dependent upon $\omega$. We can create this by taking our expression for centripetal acceleration and converting linear velocity to angular velocity:

$$\boxed{a_c = r \cdot \omega^2} \tag{8.2}$$

Using this for the magnitude of the acceleration, Callisto's acceleration in the parallel direction is…

$$a_\parallel = -r \cdot \omega^2 \cdot \cos(\omega \cdot t)$$

Combining Equation 8.2 with our equation for centripetal force gives us an expression that can be used for centripetal force when we are given angular speed:

$$\boxed{F_c = m \cdot r \cdot \omega^2} \tag{8.3}$$

## 8.4 Simple Harmonic Motion

### Words

Simple harmonic motion was described in Section 8.2 as being caused by an unbalanced force that is proportional to the distance from the equilibrium position. Here we will consider more closely what simple harmonic motion (SHM) looks like.

The position of an object that is undergoing SHM moves back and forth in a regular, periodic way. This means that if you wait for the object to go through one complete cycle of its motion, it will follow exactly the same motion through the next complete cycle. The amount of time that you have to wait for the object to go through one complete cycle is called the period. This is exactly the same way that a period is used when talking about circular motion: it is the amount of time needed for an object to complete one full rotation.

For SHM, during one period the object never experiences any sudden changes in motion, but is gradually speeding up and slowing down. Its position, velocity, acceleration, momentum, and the net force applied to it all follow the same type of curve, which could be described as a sine shape or a cosine shape.

If you look at a sine curve and a cosine curve, you can see that they both have the same shape, but they are shifted with respect to each other. For simple harmonic motion, you could change one to the other just by timing the motion starting from a different point in the cycle, effectively moving the point where time is defined as zero.

### Graphics
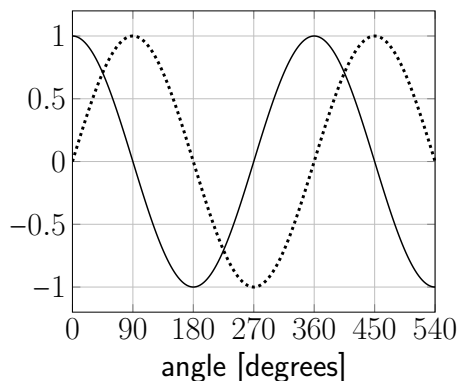


Figure 8.12: Cosine (solid) and sine (dotted) functions have the same shape but are shifted with respect to each other.[1]
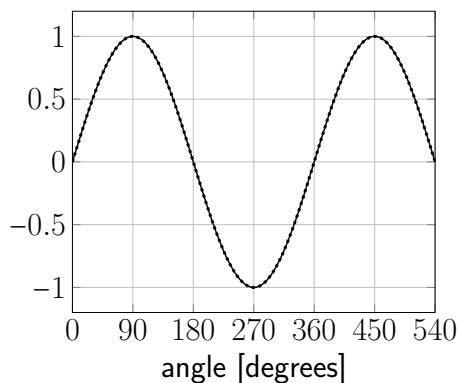


Figure 8.13: Sine of the angle (dotted) and cosine of the angle minus 90° (solid) are the same.[1]

### Numbers

We can create a mathematical model of simple harmonic motion that works for any starting conditions by introducing a "phase" $\phi$, which is the amount by which the object has to move to reach its maximum positive position. It makes sense to think of it in terms of an angle when the object is following a circular path like Callisto, but for an object like a spring on a block it may be easier to think of it as a shift in starting time to the time that the block reaches the far right position. Then the mathematical model for position during SHM becomes...

$$ x = \mathcal{A} \cdot \cos\left(\omega \cdot t - \phi\right) \qquad (8.4) $$

...where $\mathcal{A}$ is the amplitude of the motion, the maximum distance that the object travels from the equilibrium position. In the case of SHM $\mathcal{A}$ has units of length, but this same mathematical model also appears in other contexts where $\mathcal{A}$ can represent other physical quantities like an electric field.

If we consider SHM of an object that starts at equilibrium moving in the positive direction, the object travels through 1/4 of a period before reaching the maximum positive position. One period is $360°$, so $\phi$ is 1/4 of a period, or $90°$. Figure 8.13 shows a plot of the cosine of an angle minus $90°$, illustrating that a shift of $90°$ shifts the cosine function to match an object starting at zero (on the vertical axis) and moving in the positive direction as the angle increases. This is completely analogous to an object starting at zero position and moving in the positive direction as time increases.

170

We have already been introduced to the idea of a "period" as being the time for an object that is following a circular path to complete one revolution. For SHM, the period of the oscillation is the time required for the object to complete one full cycle of motion. That could be the time required for the object to move from the maximum positive position back to the maximum positive position. It is also the time time required to go from the minimum position back to the minimum position.

Looking at Figure 8.14, you can see that during one period $T$ the object passes through the equilibrium position twice, so if you want to measure one period from the equilibrium position then it is important to measure from the time the object passes the equilibrium position to the time when the object again passes through the equilibrium position *in the same direction*.

The maximum distance that the object travels from the equilibrium position is called the amplitude of the motion.
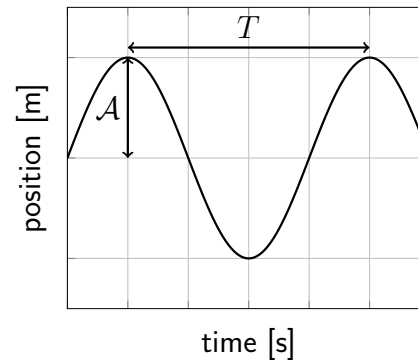


Figure 8.14: Period $T$ and amplitude $\mathcal{A}$. Note that $\mathcal{A}$ is measured only from the equilibrium position, not top-to-bottom.[1]

Equation 8.4 is similar to the expression found for $x_\parallel$ in Section 8.3, just replacing $r$ with $\mathcal{A}$ and including $\phi$ inside the cosine function. Following the same pattern of analysis that was used in Section 8.3, we can find mathematical models for the $\hat{x}$ components of velocity and acceleration for SHM:

$$v_x = -\mathcal{A} \cdot \omega \cdot \sin(\omega \cdot t - \phi) \qquad (8.5)$$

...and...

$$a_x = -\mathcal{A} \cdot \omega^2 \cdot \cos(\omega \cdot t - \phi) \qquad (8.6)$$

## 8.5 A Vertical Spring and Mass

### Words

Now we will consider a vertical system with no horizontal forces and no non-conservative forces. A spring with a spring constant of 4.5 N/m is hanging from the ceiling. You attach a 0.6 kg mass to the end, holding it so that the end of the spring is in the same position that it was in before you attached the mass, and then you suddenly release the mass. What happens?

Probably you can imagine what will happen when you release it. Because of gravity, the mass will fall, stretching the spring, and then will bounce back to the top, and it will continue repeating this same pattern until friction with the air and within the spring slows it to a stop, but we are assuming no friction so it would just continue to bounce forever.

Can we be more precise about the motion of the mass? If we start by looking at this scenario in terms of energy, the mass initially has no kinetic energy and there is no no potential energy stored in the spring, but it is being held up, so it has gravitational potential energy. As it falls, the mass loses gravitational potential energy but gains kinetic energy, and the spring also gains spring potential energy since it is stretching. When the mass reaches the lowest point in its bounce it stops moving, so at that point it has lost all of its kinetic energy. Since we know it will not go lower than this point we can call this height $y = 0$, so it also has no gravitational potential energy at that point. All of the energy has changed into spring potential energy.

### Graphics
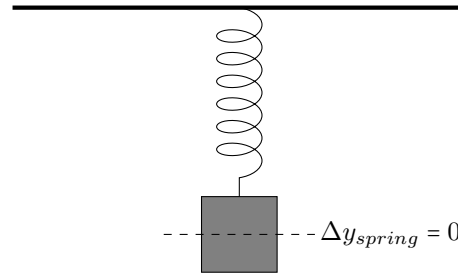


Figure 8.15: A mass is attached to a spring and held in place at the equilibrium position of the empty spring.[1]
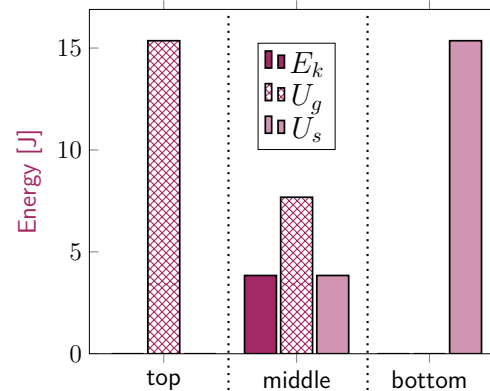


Figure 8.16: Energy bar graphs for the spring and mass.[1]

### Numbers

**Assumptions:** Ceiling is immovable; ideal spring; no friction; horizontal direction only; $+\hat{y}$ is upward

| Knowns | Unknowns |
|---|---|
| $m = 0.6\,\text{kg}$ | ??? |
| $k_s = 4.5\,\text{N/m}$ | |
| $\overrightarrow{p_i} = 0\,\text{kg}\cdot\text{m/s}$ | |
| $g = 9.8\,\text{m/s}^2$ | |

Let's begin with conservation of energy at three important heights: $y_{top}$ (at the top), $y_{bot}$ (at the bottom), and $y_{mid}$ (at the equilibrium position. To simplify the math we can make $y_{bot} = 0$. We know more about the energy at the top and bottom, so we will begin with those...

$$E_{bot} = E_{top}$$

$$\cancel{E_{k,bot}}^{0} + \cancel{U_{g,bot}}^{0} + U_{s,bot} = \cancel{E_{k,top}}^{0} + U_{g,top} + \cancel{U_{s,top}}^{0}$$

$$U_{s,bot} = U_{g,top}$$

$$\frac{1}{2}k_s \cdot \left(y_{top} - \cancel{y_{bot}}^{0}\right)^2 = m \cdot g \cdot \left(y_{top} - \cancel{y_{bot}}^{0}\right)$$

$$y_{top} = \frac{2m \cdot g}{k_s} = 2.61\,\text{m}$$

This means that the bottom position is 2.61 m below the point where the mass was released. The amplitude of the oscillation is therefore...

At the lowest point, the spring is able to pull the mass back up, so we know that the spring force must be larger than the force of gravity. The kinetic energy and the gravitational potential energy both increase as the mass starts to rise, while the spring potential energy decreases. The mass is not moving at the top or the bottom, but has non-zero velocity between the top and bottom, so there must be some point where the speed and the momentum are at a maximum. Momentum increases as long as the force is in the same direction as the velocity, so there must be some point in the path of the mass where the net force drops to zero and then changes direction. In other words, there must be a new equilibrium position along the path followed by the mass.

Since the force of gravity is constant throughout the path followed by the mass and the spring force increases proportionally with the distance the spring is stretched, the net force is also proportional to the distance from the equilibrium position. That means the oscillation of the mass is simple harmonic motion.

Remarkably, the period of the oscillation does not depend on the force of gravity. It also doesn't depend on the distance that the spring is stretched. The period gets longer (so a slower oscillation) as the mass increases, and the period gets shorter (so a faster oscillation) as the strength of the spring (the spring constant) increases.
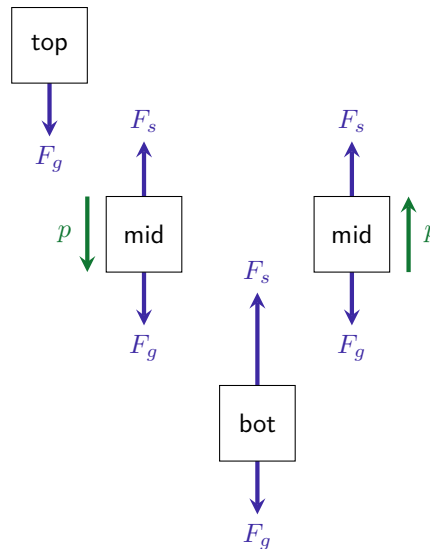


Figure 8.17: FBD of the mass in four different positions as it oscillates, including arrows showing the direction of the momentum when the mass is in the equilibrium position.[1]

$$\mathcal{A} = \frac{y_{top} - y_{bottom}}{2} = \frac{m \cdot g}{k_s} = 1.30\,\text{m}$$

From the perspective of forces, Figure 8.17 shows the two forces that act on the mass. The net force is zero at the equilibrium position and increases linearly with distance from the equilibrium position. The equilibrium position is the point where…

$$F_g = F_s$$
$$m \cdot g = k_s \cdot (y_{top} - y_{mid})$$

Solving for $y_{mid}$ gives…

$$y_{mid} = y_{top} - \frac{m \cdot g}{k_s} = 1.30\,\text{m}$$

…at the center of the oscillation.

When the mass is at the top position, the only force affecting it is gravity, so its acceleration is $-g\,\hat{y}$. The mass is also in SHM, so its acceleration is given by Equation 8.6, when the acceleration is at its maximum negative value, so $-\mathcal{A} \cdot \omega^2 = -g$. Solving for $\omega$ and using the expression we found earlier for $\mathcal{A}$ gives…

$$\omega = \sqrt{\frac{g}{\mathcal{A}}} = \sqrt{g \cdot \frac{k_s}{m \cdot g}} = \sqrt{\frac{k_s}{m}}$$

The angular velocity does not depend on gravity, so the result is valid for any spring-mass system, horizontal or vertical. It is usually expressed in terms of the period:

$$\boxed{T = 2\pi\sqrt{\frac{m}{k_s}}} \tag{8.7}$$

## 8.6 A Pendulum

### Words

We have considered stability for objects sitting on the ground and objects connected to springs. Now we will consider an object that is hanging from a rope. When the object is hanging straight down and not moving, it is in an equilibrium position. Is it a stable equilibrium?

If we think about it from the perspective of energy, pushing the object away from the equilibrium position in either direction will make it swing upward, increasing its gravitational potential energy. That means the equilibrium position is stable. The object would begin to swing back and forth. Is the oscillation simple harmonic motion? It depends...

We saw with the orbit of Callisto around Jupiter that uniform circular motion looks like SHM when viewed from the side. Let's try looking at this motion the same way, in the horizontal direction. If we start with the mass held out horizontally, that will be the maximum possible displacement. But at that point there are no forces in the horizontal direction, when for SHM the horizontal force would be at a maximum at that position. So a pendulum does not have SHM in the horizontal direction.

Let's try looking at the vertical direction. Here we see that the equilibrium position is at the bottom, but SHM always has the equilibrium position in the center. So no SHM in the vertical direction.

We still have another option–maybe there is SHM in the tangential direction?

### Graphics
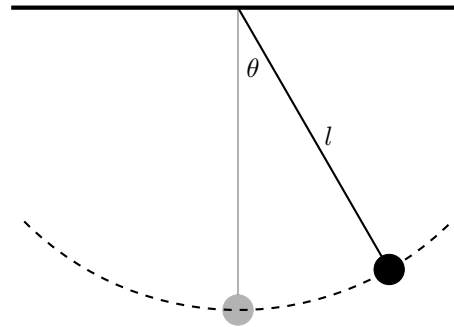


Figure 8.18: A mass hung from a light rope.[1]
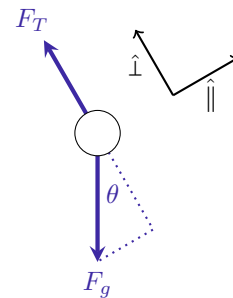


Figure 8.19: FBD of the mass on the string when at the position shown in Figure 8.18[1]

### Numbers

**Assumptions:** pointlike mass; negligible friction; negligible rope mass; near surface of earth

| Knowns | Unknowns |
|---|---|
| $m = 3\,\text{kg}$ | $\omega$ ? |
| $l = 1.5\,\text{m}$ | $T$ ? |
| $g = 9.8\,\text{m/s}^2$ | |

When the mass is a path length $s = l \cdot \theta$ (with $\theta$ measured in radians) away from the equilibrium position, the force in the tangential ($\hat{\parallel}$) direction as shown in Figure 8.19 would be $F_g \cdot \sin\theta$. So the position is proportional to $\theta$ but the force is proportional to $\sin\theta$.

Simple harmonic motion requires that the force be proportional to the displacement, so a pendulum does not display SHM in the tangential direction.

That is unfortunate, because SHM uses such easy mathematical models. This is a good time to remember that many of our mathematical models are really just approximations that work well in some situations. We've already used an approximation for gravitational force when analyzing the pendulum, one that is only valid for a limited range of heights at the surface of the earth.

Let's see if our mathematical models for SHM might also be valid in some limited range of angles for the pendulum.

To be able to work with the numbers, let's make the mass 3 kg, and hang it at the end of a light, 1.5-m-long rope.

The analysis shown in the "Numbers" column demonstrates that in fact there is not SHM in the tangential direction, but for small angles, when the pendulum is not swinging far away from the vertical position, the motion is *almost* the same as SHM, so this "small-angle approximation" is often used for analyzing the motion of the pendulum.

In the small-angle approximation, the period of the pendulum does not depend on the mass, but it does depend on the acceleration of gravity at the earth's surface and the length of the rope. The period increases (so it slows down) as the rope gets longer. If the pendulum were in a location where the acceleration caused by gravity is smaller, for example on the surface of the moon, the period of the pendulum would also increase.
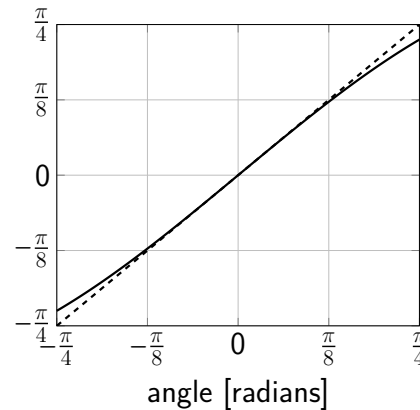


Figure 8.20: Sine of the angle (solid line) compared with the angle itself (dashed line), measured in radians.[1]

Figure 8.20 shows that for small angles (measured in radians), sine of the angle is roughly equal to the angle itself. So if we keep the angle small, the force is roughly proportional to the distance as measured along the path length. That allows us to use the mathematical models for SHM. Converting from $\hat{x}$ directions to tangential directions, the maximum positive value for position along the path length for SHM as given by Equation 8.4 is...

$$s_{max} = \mathcal{A} = l \cdot \theta_{max}$$

The maximum negative value for acceleration in the tangential direction, which occurs at the same position, is similarly given by Equation 8.6...

$$-a_{T,max} = -\mathcal{A} \cdot \omega^2 = -(l \cdot \theta_{max}) \cdot \omega^2$$

...where the expression found above for $\mathcal{A}$ has been substituted in. We can then use Newton's Second Law to go from $a_T$ to $F_g$...

$$-\frac{F_g}{m} \cdot \sin\theta_{max} = -\frac{\cancel{m} \cdot g}{\cancel{m}} \cdot \sin\theta_{max} = -(l \cdot \theta_{max}) \cdot \omega^2$$

Solving for $\omega$ and canceling $\theta_{max}$ with $\sin\theta_{max}$ since we are using the small-angle approximation gives...

$$\omega = \sqrt{\frac{g \cdot \sin\theta_{max}}{l \cdot \theta_{max}}} = \sqrt{\frac{g}{l}}$$

This expression is usually given in terms of the period, and is valid for small angles:

$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}} \tag{8.8}$$

## 8.7    Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

### General

- Simple harmonic motion looks exactly the same as one dimension of uniform circular motion.

- The time needed for an object to go through one complete cycle of simple harmonic motion is called the period.

- The period of a spring-mass system depends only on the mass and the spring constant. As the mass increases the period gets longer, and as the spring constant increases the period gets shorter.

- When the size of the swing is small, the period of a pendulum depends only on its length and the acceleration due to gravity. As the length increases, the period gets longer, and as the acceleration due to gravity increases the period gets shorter.

### Forces

- An object is in a stable equilibrium position if disturbing the object causes a force or torque that pushes the object back toward the equilibrium position.

- An object is in an unstable equilibrium position if disturbing the object causes a force or torque that pushes the object even further from the equilibrium position.

- Simple harmonic motion is caused by a net force that is proportional to the distance from the equilibrium position.

### Motion

- An oscillation is a back-and-forth motion.

- The maximum distance that an object travels from the equilibrium position during simple harmonic motion is called the amplitude of the motion.

### Momentum

- It is an object's momentum that causes it to continue moving through the equilibrium point during an oscillation.

### Energy

- Objects like to go to the position that will give them the smallest possible amount of potential energy.

- An object is in a stable equilibrium position if moving the object away from the equilibrium position causes the object to gain potential energy.

- An object is in an unstable equilibrium position if moving the object away from the equilibrium position causes the object to lose potential energy.

## Mathematical Models

| equation | | restrictions on the validity of the equation |
|---|---|---|
| $E_k = \frac{p^2}{2m}$ | (8.1) | -none- |
| $a_c = r \cdot \omega^2$ | (8.2) | -none- |
| $F_c = m \cdot r \cdot \omega^2$ | (8.3) | -none- |
| $x = \mathcal{A} \cdot \cos{(\omega \cdot t - \phi)}$ | (8.4) | Simple harmonic motion (SHM) |
| $v_x = -\mathcal{A} \cdot \omega \cdot \sin{(\omega \cdot t - \phi)}$ | (8.5) | Simple harmonic motion (SHM) |
| $a_x = -\mathcal{A} \cdot \omega^2 \cdot \cos{(\omega \cdot t - \phi)}$ | (8.6) | Simple harmonic motion (SHM) |
| $T = 2\pi\sqrt{\frac{m}{k_s}}$ | (8.7) | SHM of a spring-mass system |
| $T = 2\pi\sqrt{\frac{l}{g}}$ | (8.8) | Small-angle approximation, SHM of a pendulum |

## 8.8 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

### Level 1 - Remember

8.1 **[W]** What are the factors that affect the period of the simple harmonic motion of a spring-mass system?

8.2 **[W]** What are the factors that affect the period of the simple harmonic motion of a pendulum that is swinging at a small angle?

8.3 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

### Level 2 - Understand

8.4 **[W & G]** For an object that is experiencing simple harmonic motion to the left and to the right, in what direction is its velocity when its position is…

    (a) … at the far right?

    (b) … at the far left?

    (c) … at the equilibrium position?

8.5 **[W & G]** For an object that is experiencing simple harmonic motion to the left and to the right, in what direction is its acceleration when its position is…

    (a) … at the far right?

    (b) … at the far left?

    (c) … at the equilibrium position?

### Level 3 - Apply

8.6 **[W, & G]** Analyze the static equilibrium position of the acrobat in Figure 8.1:

    (a) Create a sketch that includes the main forces acting on the acrobat including an applied force that creates a torque that would shift the acrobat out of equilibrium, as was done with the pyramid in Section 8.1.

    (b) Use the sketch to explain whether the acrobat is in a stable equilibrium position or an unstable equilibrium position.

    (c) Create a graph of gravitational potential energy vs angle, as was done with the pyramid in Section 8.1. Without knowing mass or lengths, just draw the shape of the curve for the graph–no scale is needed.

    (d) Use the energy graph to explain whether the acrobat is in a stable equilibrium position or an unstable equilibrium position.

8.7 **[G]** Figure 8.6 shows the block and spring at the time when the applied force suddenly gives momentum to the block. This moment corresponds to the left sides of the graphs in figure 8.8, where the displacement and force from the spring are both zero, and the momentum is at a large negative value. Create your own sketches of the block and spring that correspond to the other times as marked in the displacement graph from figure 8.8 that is reproduced below. Include arrows in the sketches showing the direction in which the block is moving.
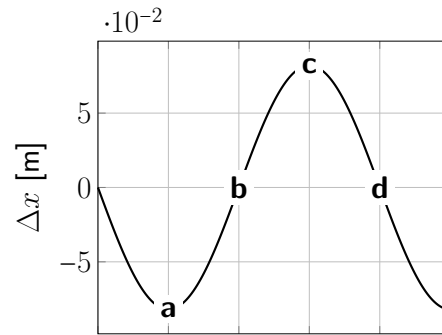


A part of the graph from Section 8.2 with positions marked for this question.[1]

(a) Position "**a**"

(b) Position "**b**"

(c) Position "**c**"

(d) Position "**d**"

8.8 **[G & N]** Figure 8.11 has numbers on all of the axes of each graph, but except at $t = 0$ none of the peaks, valleys or zero crossings occur on any of the gridlines. Calculate the values for…

(a) …the time when the position first crosses zero.

(b) …the time when the position first reaches its maximum negative value.

(c) …the time when the velocity first reaches its maximum positive value.

(d) …the time when the acceleration first reaches its maximum positive value.

8.9 **[G & N]** Figure 8.11 has numbers on all of the axes of each graph, but except at $t = 0$ none of the peaks, valleys or zero crossings occur on any of the gridlines. Calculate the values for…

(a) …the maximum positive value of the position.

(b) …the maximum negative value of the position.

(c) …the maximum positive value of the velocity.

(d) …the maximum negative value of the velocity.

(e) …the maximum positive value of the acceleration.

(f) …the maximum negative value of the acceleration.

8.10 **[N]** Callisto has a mass of $1 \times 10^{23}$ kg. Use that information along with the information provided in Section 8.3 to find the following:

(a) The angular momentum Callisto has due to its orbit around Jupiter.

(b) The magnitude of the force of gravity between Callisto and Jupiter.

(c) The magnitude of the linear momentum Callisto has in its orbit around Jupiter.

8.11 **[N]** Find the period of the simple harmonic motion of the spring and mass from Section 8.5.

8.12 **[N]** The unknowns listed in Section 8.6 were never actually calculated. Find their values.

## Level 4 - Analyze

8.13 **[W]** A statement is made in Section 8.1 that objects like to go to the place that will give them the smallest possible amount of potential energy. Does this agree with what you have learned in earlier chapters? Give at least two examples where this is the case. One example should be an object that is held above the ground and then released.

8.14 **[N & G]** Create two more graphs to go with the others in Figure 8.8. One should be for velocity of the block and one for acceleration of the block. The vertical scales should be correct for both graphs.

8.15 **[N]** Find the period of the simple harmonic motion of the spring and mass from Section 8.2.

8.16 **[G & N]** Calculate the heights of all of the energy bars in Figure 8.16.

8.17 **[N]** How could you change the pendulum in Section 8.6 so that its period would double?

## Level 5 - Evaluate

8.18 **[W, G, & N]** The pyramid sitting on its base in Figure 8.2 is said to be in a stable equilibrium position. But that is only true up to a certain point. How far would the pyramid have to be tilted from this position to put it into an unstable equilibrium?

   (a) Make a sketch of the pyramid when it reaches this unstable equilibrium position.
   (b) Explain in words what it is that makes this position an unstable equilibrium
   (c) Find the numerical value for the angle at which this unstable equilibrium occurs.

8.19 **[W, G, & N]** In Figure 8.16 the kinetic energy appears to be the same as the spring potential energy at the middle height. Is there any height for this physical system (whether shown in this energy bar graph or not) where...

   (a) ...the kinetic energy is equal to the gravitational potential energy?
   (b) ...the spring potential energy is equal to the gravitational potential energy?

   If so, find the height. If not, explain why not.

8.20 **[N]** The "small-angle approximation" doesn't actually say how small the angle needs to be. The critical factor in this approximation is whether $\sin(\theta)$ is reasonably close to $\theta$ when measured in radians. At what angle is $\sin(\theta)$ different from $\theta$ by...

   (a) ...0.1%?
   (b) ...1%?
   (c) ...10%?
   (d) ...a factor of 2?

## Level 6 - Create

8.21 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Begin a new concept map for stability, oscillations, and waves.

8.22 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

8.23 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

# Chapter 9

# Solids

Up to this point, almost every object that we have dealt with has been assumed to be completely rigid. In this chapter we will start allowing "solid" objects to bend, stretch, and break. When objects bend and stretch, they can transfer energy in waves. In this chapter we will explore two different types of waves in solids: *transverse waves* and *longitudinal waves.*

Transverse waves occur when a material bends, as with "battle ropes." A transverse wave occurs when each segment of the object, in this case each small section of rope, moves back and forth perpendicular to the direction that the wave is traveling. The waves travel horizontally along the length of the rope, but each individual section of rope is moving vertically, perpendicular to the direction in which the wave is moving.



Figure 9.1: Transverse waves traveling along battle ropes.[1]

Longitudinal waves occur when a material stretches and compresses. When a millipede walks, it moves its legs in longitudinal waves. A longitudinal wave occurs when each segment of the object, in this case the legs, moves back and forth parallel to the direction that the wave is traveling. The waves travel horizontally along the length of the millipede's body, and each individual leg also moves forward and back horizontally along the length of the millipede's body. This creates areas of compression, where the legs are closely spaced, and areas of "rarefaction" where the legs are widely spaced.



Figure 9.2: A millipede moving its legs in a longitudinal-wave-like fashion.[47]

## 9.1 Modeling a Rope

### Words

In the "Numbers" column we have been using "mathematical models" to describe physics with equations. These models are useful tools for describing physical situations, but often they are really only approximations that work under certain conditions. We can also create other types of models to help us simplify physical scenarios to make them easier to understand. Now that we are allowing solid objects to bend and stretch, it can be helpful to think of them not as single objects but as a system of many small, interconnected objects.

For example, a rope can be modeled as a line of small pieces of rope that are all held together. We want the rope to be able to bend and stretch, so we can think of all of the small pieces as being held together with springs. Ropes don't stretch very much, but they bend easily, so for our model of a rope we can assume that each individual piece in Figure 9.3 can move up and down, but not left or right.

What would happen if we started moving the rightmost piece of rope up and down with a period of 0.5 s, like the people are doing in Figure 9.1? Let's assume that the rope has a length of 6 m, a mass of 12 kg, and is being pulled horizontally with a tension of 150 N.

In our model of a rope, each piece of rope is connected to the pieces on either side of it, so they can pull on each other.

### Graphics
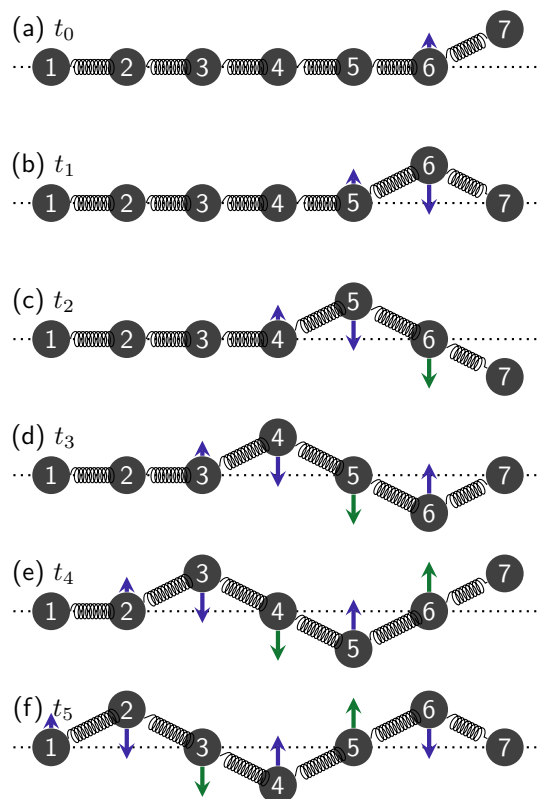


Figure 9.3: Parts (a) through (f) are diagrams of the same pieces of rope at different times. If section "7" of the rope is moved up and down, it creates a chain reaction of forces, momentum, and energy traveling left through the rope.[1]

### Numbers

| Knowns | Unknowns |
|---|---|
| $m = 12\,\text{kg}$ | ??? |
| $l = 6\,\text{m}$ | |
| $T = 0.5\,\text{s}$ | |
| $F_T = 150\,\text{N}$ | |

Each successive part in Figure 9.3 is a fixed time interval after the previous part. Close examination shows that the various "pieces" of the rope are all affected in the same way, but at slightly different times. Note that in part (b) of the figure, section 6 of the rope has no momentum (no green arrow) but is experiencing a large downward force (blue arrow). In part (c), section 5 of the rope has no momentum but it is experiencing a large downward force. In part (d) it is section 4 that is in the same condition, and this trend continues to the left at each successive time interval. There is a transverse wave moving to the left.

As long as the transverse displacement of the wave is small compared to the length of the wave, the speed of a transverse wave on a rope is given by:

$$v = \sqrt{\frac{F_T}{\mu_m}} \qquad (9.1)$$

…where $\mu_m$ is the linear mass density of the rope, which is defined as…

$$\mu_m \equiv \frac{m}{l} \qquad (9.2)$$

The forces and the resulting momenta of the pieces of rope are shown at different times in Figure 9.3. The force shown is the net vertical force caused by the two pieces on either side of each piece. The forces on piece 7 aren't shown because there are unknown forces coming from an external source–only the forces that are within the rope are shown. Each of the other pieces 1-6 are affected in exactly the same way, just at different points in time. Notice that whatever happens to piece 4, for example, next happens to piece 3. This creates a wave moving to the left, as shown in Figure 9.4.

The speed of the wave depends only on the tension in the rope and the "linear mass density" of the rope, where linear mass density is the mass of the rope per unit of length.

We can see a sine-wave-like pattern developing in the rope. We saw patterns like this in Chapter 8, but before it was normally in a graph with time as the horizontal axis. This time we are seeing the same pattern appearing not in time but in physical space. The time between successive peaks was called the period. The distance between successive peaks in space is called the wavelength.

Wavelength and period are related, because a period is the time required for one wavelength to pass by a given point in space. This can be seen by looking at piece 6 in Figure 9.4. It goes through one complete period from (b) $t_1$ to (f) $t_5$, during the time that one wavelength passes by at speed $v$.

Frequency, measured in hertz (Hz), is the inverse of the period, or the number of times a wavelength passes a given point per second.

(a) $t_0$

(b) $t_1$

(c) $t_2$

(d) $t_3$

(e) $t_4$

(f) $t_5$
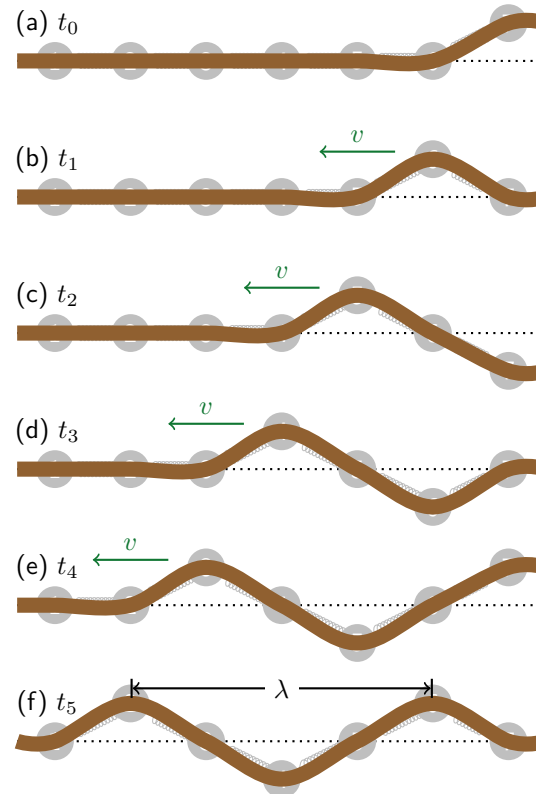


Figure 9.4: This figure is based on Figure 9.3, with the rope itself drawn over our model of the rope. The up-and-down motion of the pieces of rope result in a wave that moves to the left. The length of the wave is called the wavelength $\lambda$, and can be measured between successive peaks of the wave.[31]

So this wave travels to the left at a speed of…

$$v = \sqrt{\frac{150\,\text{N} \cdot 6\,\text{m}}{12\,\text{kg}}} = 8.66\,\text{m/s}$$

Note that the speed of the wave doesn't depend on the transverse speed of each individual piece!

Wavelength $\lambda$ is the physical distance between successive crests in a wave, and it is related to the speed and the period or the frequency $f$ of the wave:

$$\boxed{\lambda = v \cdot T = \frac{v}{f}} \tag{9.3}$$

…where…

$$\boxed{f = \frac{1}{T}} \tag{9.4}$$

So this wave has a wavelength of…

$$\lambda = 8.66\,\text{m/s} \cdot 0.5\,\text{s} = 4.33\,\text{m}$$

…and a frequency of…

$$f = \frac{1}{0.5\,\text{s}} = 2\,\text{Hz}$$

## 9.2   The End of the Rope

### Words

We have seen that a wave can travel along the length of a rope, but what happens when the rope ends? It depends. First we will consider what happens if the end of the rope is free to move.

Figure 9.5 uses the same model that we used before, but this time we are sending a single pulse to the right instead of constantly shaking one end up and down, and we are letting the end of the rope ("piece 7") move freely up and down, so it is only affected by the forces from "piece 6."

Notice what happens in the figure. Piece 7 actually goes much higher than the height of the original pulse, and as it comes down (because of the force from piece 6), it sends a pulse back to the left. Notice that no piece of the rope ever crosses the dotted equilibrium position line where the rope would be if there were no waves on it at all.

The wave pulse that was sent in bounces back from the free end of the rope.
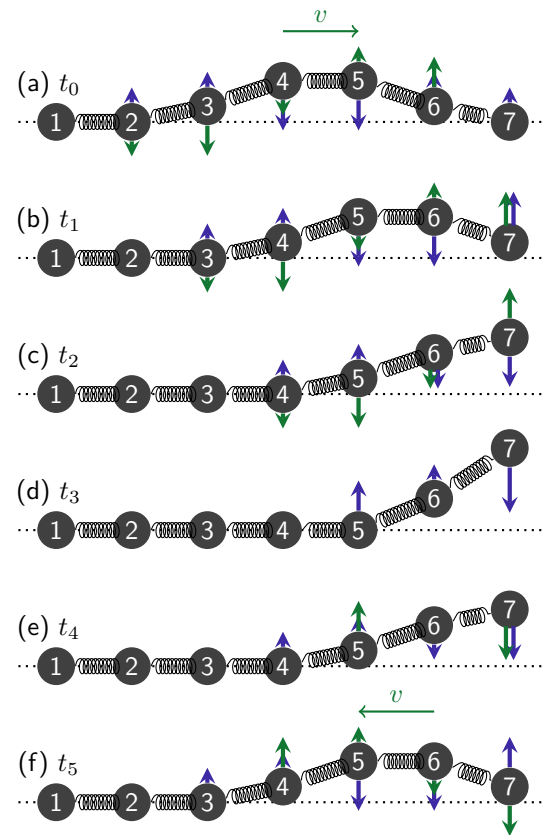
### Graphics

Figure 9.5: If piece "7" at the end of the rope is allowed to move freely up and down, a wave pulse moving to the right reflects back to the left when it reaches the end.[1]

### Numbers

Now we will consider what happens if a wave pulse reaches the end of a rope that is held firmly in place.

Figure 9.6 again uses the same model that we used before, but this time we are holding the end of the rope ("piece 7") firmly in place, so whatever force comes from piece 6 will be counteracted by whatever is holding the rope. Since this end of the rope doesn't move at all, we can say that it is fixed in place.

Notice what happens in the figure. Piece 7 stays fixed in place, and at $t_1$ and $t_4$ it is pulling very hard on the rope. In fact, it pulls so hard on the rope that the wave flips over when it reaches the end.

The wave pulse that was sent in flips over and bounces back from the fixed end of the rope. Notice that before it reaches the end of the rope, the wave pulse is completely above the equilibrium position, and when it bounces back it is completely below the equilibrium position.
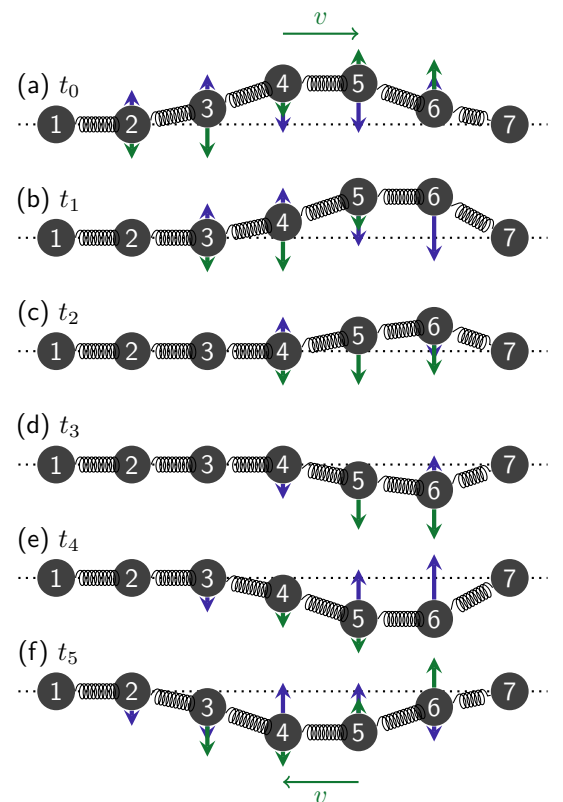


Figure 9.6: If piece "7" at the end of the rope is held firmly in place, a wave pulse moving to the right flips over and reflects back to the left when it reaches the end.[1]

## 9.3 Adding Waves

### Words

In Section 9.2 we saw waves that were reflected back from the end of a rope. During the time that the wave is reflecting, the pieces near the end of the rope are actually feeling the effects of both the wave that is traveling to the right and the wave that is bouncing off to the left, at the same time. Interesting things can happen when waves are traveling on a rope in two different directions.

Figure 9.7 shows two waves with equal amplitudes and a sine-wave shape moving in opposite directions on the same rope. On the left and right, in the regions where the waves aren't interfering with each other yet, each piece of the rope moves up and down just as before. But notice what happens in the center area once the waves meet.

The wave from the left is shown as a dashed line and the wave from the right is a dotted line. When they meet, they interfere with each other. At $t_3$, $t_5$, and $t_7$ the two waves are aligned with each other, so there is "constructive interference." The position of the rope is the sum of the positions of the two waves, so in this case it is a sine wave with twice the amplitude of each individual wave.

At $t_4$ and $t_6$ the two waves are completely misaligned with each other, so one is at a positive maximum at the same place that the other is at a negative maximum. This causes "destructive interference." The position of the rope is still the sum of the positions of the two waves, which is the equilibrium position at every point on the rope!

### Graphics

(a) $t_0$

(b) $t_1$

(c) $t_2$

(d) $t_3$

(e) $t_4$

(f) $t_5$

(g) $t_6$

(h) $t_7$
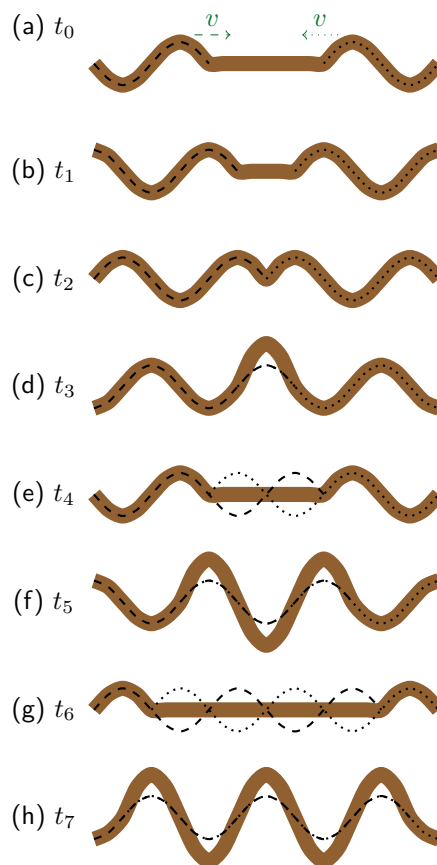
Figure 9.7: The two waves interfere with each other in the region where they overlap, creating a standing wave.[1]

### Numbers

We can do an example involving a musical instrument. The highest "E" string on a steel-string acoustic guitar has a linear mass density of $4 \times 10^{-4}$ kg/m and a length of 0.65 m. How much tension is needed for the string to be correctly tuned with a first harmonic frequency of 330 Hz? What is the frequency of the second harmonic?

**Assumptions:** the amplitudes of the transverse waves are small compared to the length of the string

| Knowns | Unknowns |
|---|---|
| $\mu_m = 4 \times 10^{-4}$ kg/m | $F_T$ |
| $l = 0.65$ m | $f_2$ |
| $f_1 = 330$ Hz | |

Figure 9.9 shows that for the first harmonic the length of the string is half of one wavelength, so $\lambda_1 = 2l$. Since the harmonic is created by traveling waves reflecting back and forth from the fixed ends of the string, we can use the same mathematical models that we used for traveling waves. Equation 9.3 relates the wavelength to the frequency and the speed of the wave, and Equation 9.1 relates the speed of the wave to the tension and linear mass density. Combining those mathematical models gives...

$$\lambda = \frac{v}{f} = \frac{1}{f} \cdot \sqrt{\frac{F_T}{\mu_m}}$$

The interference of these two identical waves coming from opposite directions creates some points on the rope, called "nodes," that don't move at all. The interference also creates some points on the rope, called "antinodes," where the amplitude of oscillation is very large. The nodes are equally spaced at intervals of half of the wavelength of the traveling waves.

Stringed instruments use this effect. If the ends of a string are held tightly in place and then the string is plucked, reflections of the resulting waves on the string create "harmonics" in the string. The first harmonic in such a string, shown at the top of Figure 9.9, has a node at each end (since the ends are held in place) and an antinode in the center. The wavelength of the first harmonic is twice the length of the string.

There are also other harmonics that are allowed on such a string. The second harmonic still has a node at each end, but also has a node in the middle. The wavelength of the second harmonic is equal to the length of the string. The third harmonic has two nodes in the middle; the fourth harmonic has three; and so forth. Higher harmonics have shorter wavelengths and faster frequencies.

The "pitch" we hear when listening to something is related to the frequency. We hear a higher pitch for a higher (*i.e.* faster) frequency.

(f) $t_5$

(g) $t_6$

(h) $t_7$

**N A N A N A N**
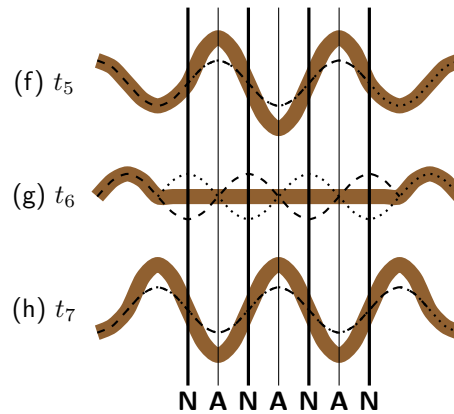
Figure 9.8: Looking at the parts of Figure 9.7 where the waves are interfering with each other, in places called nodes (thick lines) the rope doesn't move at all and in places called antinodes (thin lines) the motion of the rope is large.[1]

1st harmonic $\quad$ N $\quad$ A $\quad$ N $\qquad$ ($\lambda_1/2$)

2nd harmonic $\quad$ N A N A N $\quad$ ($\lambda_2$)

3rd harmonic $\quad$ N A N A N A N $\quad$ ($3\lambda_3/2$)
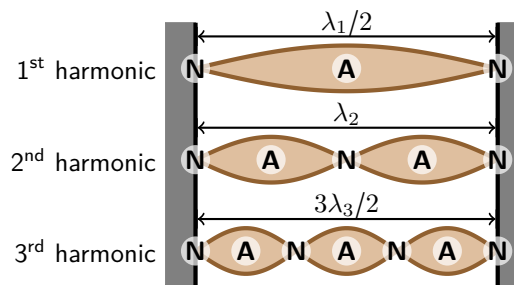
Figure 9.9: The first three harmonics on a string that is tightly anchored at each end. "**N**" indicates a node and "**A**" indicates an antinode.[1]

…which can be rearranged to solve for the tension force:

$$F_T = (\lambda \cdot f)^2 \cdot \mu_m$$

Substituting in for the first harmonic…

$$F_T = (\lambda_1 \cdot f_1)^2 \cdot \mu_m = (2l \cdot f_1)^2 \cdot \mu_m = 73 \text{ N}$$

We can again use Figure 9.9 to find information about other harmonics. Notice the pattern relating $\lambda$ to the length of the string when the string is held tightly at each end:

$$\boxed{\lambda_n = \frac{2l}{n}} \tag{9.5}$$

…for the nth harmonic.

Rearranging the first mathematical model in this section to solve for $f$ and using Equation 9.5 for $\lambda$ gives the frequency for the nth harmonic:

$$\boxed{f_n = \frac{n}{2l} \cdot \sqrt{\frac{F_T}{\mu_m}}} \tag{9.6}$$

Putting in the values we already know gives a second harmonic frequency $f_2 = 660 \text{ Hz}$.

## 9.4 Beat Frequencies

### Words

We have looked at two waves that interfere with each other while coming from different directions. Waves also interfere with each other when traveling in the same direction. If two waves have the same speed, the same frequency, and are in phase with each other, they interfere constructively, as shown in Figure 9.10. In this case, the amplitude of the combined wave is just the sum of the amplitudes of the other two waves.

On the other hand, if the two waves have the same speed, the same frequency, and are $180°$ out of phase with each other, they interfere destructively, as shown in Figure 9.10. In this case, the combined wave is zero everywhere. This is the principle used by noise-canceling headphones. The headphones detect sound waves and then create identical waves that are $180°$ out of phase with the detected waves, and feed the combined sound wave to the ear.

### Graphics
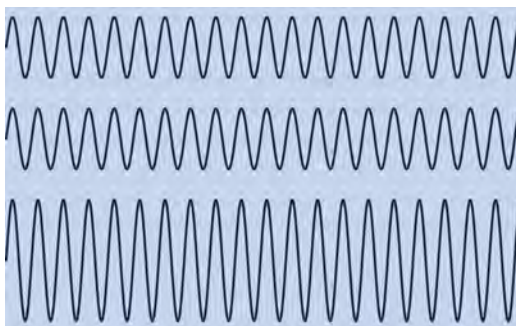


Figure 9.10: The top two patterns are waves traveling in phase in the same direction. The bottom pattern is the sum of the other two.[1]
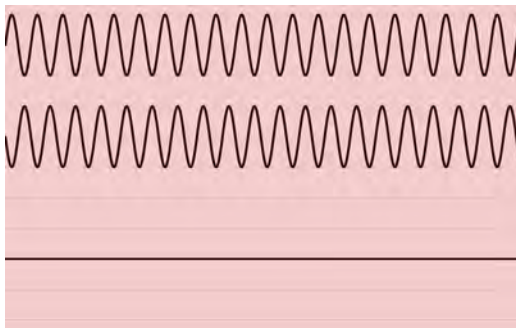


Figure 9.11: The top two patterns are waves traveling $180°$ out of phase in the same direction. The bottom pattern is the sum of the other two.[1]

### Numbers

When two waves with similar frequencies $f_1$ and $f_2$ interfere with each other, they produce a "beating" pattern. The frequency of the beating, corresponding to the dashed "envelope" in Figure 9.12, is...

$$f_{\text{beat}} = |f_1 - f_2| \qquad (9.7)$$

The "carrier frequency" of the beating wave, the frequency that is actually heard beating, is simply the average of the two frequencies that were added together:

$$f_{\text{carrier}} = \frac{f_1 + f_2}{2} \qquad (9.8)$$

When two waves of slightly different frequencies interfere with each other, they create "beats." At some times the waves are aligned in such a way that their amplitudes add, and at other times the waves are aligned in such a way that their amplitudes cancel. The result is a sound wave with a frequency that is the average of the frequency of the two slightly different original waves. But the sound is not constant–instead it slowly increases and decreases in amplitude, at a frequency that is equal to the difference between the two original waves.
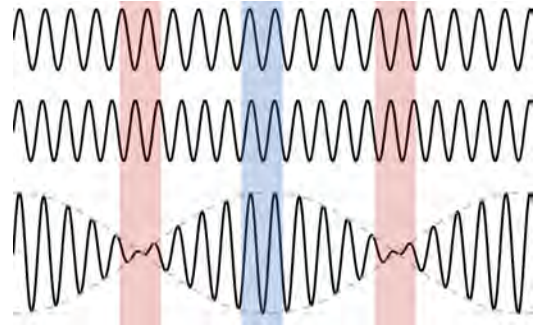


Figure 9.12: The top two patterns are waves with slightly different frequencies traveling in the same direction. They slowly go into phase and out of phase with each other, so they add constructively in some places and destructively in others. The "beat" pattern on the bottom is the sum of the other two. The dashed line is an "envelope" that contains the beat pattern. This envelope travels along with the wave. [1]

## 9.5 Longitudinal Waves

### Words

Longitudinal waves create areas of compression and rarefaction in a material. Figure 9.13 shows a compression wave pulse initially traveling to the right and then reflecting back to the left.

With longitudinal waves, monitoring the maximum and minimum locations of the pieces of material was a good way to find the location of the wave. But notice that in the figure the positions of the individual pieces do not indicate the location of the wave. Piece 2 shifts to the right when the wave passes, and then it stays in the same position until the wave passes that location again.

A better way to find the location of a longitudinal wave is to look for the places where the spacing between the pieces is different, either a compression or a rarefaction. In Figure 9.13, the location of the compressed spring is clearly moving to the right from $t_0$ to $t_3$ and moving to the left from $t_4$ to $t_6$.

The location of the pressure wave is also where the pieces of the rope are affected by the largest net force. In the case of a compression wave pulse, the net force on each piece is outward from the compression.

The speed of a longitudinal wave in a solid depends on the density and the elastic properties of the material. A higher density causes a lower speed, and a higher Young's modulus (a measure of elasticity) causes a higher speed.

### Graphics
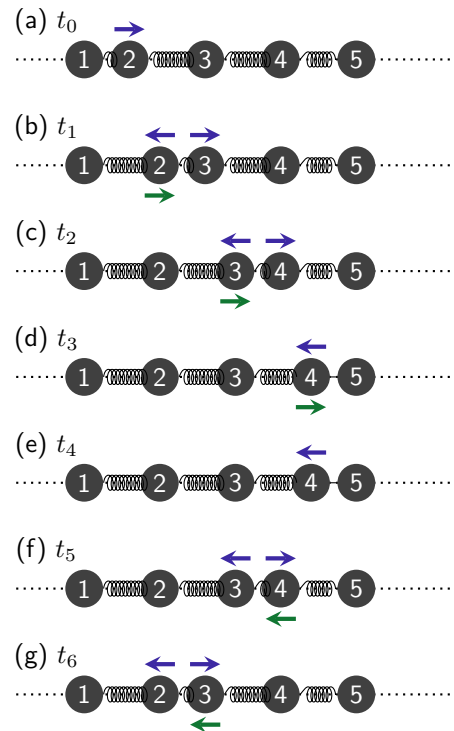


Figure 9.13: If piece "5" is held firmly in place, a compression wave pulse moving to the right reflects back to the left when it reaches the end.[1]

### Numbers

To find the speed of a longitudinal wave through a material, we need to know the Young's modulus $Y$ and the mass density $\rho_m$ of the material, where density is mass per unit volume:

$$\rho_m \equiv \frac{m}{V} \qquad (9.9)$$

…where $m$ is the mass of a piece of material and $V$ is the volume of the piece of material. The $m$ subscript on $\rho_m$ is to make it clear that we mean the *mass* density. The following table gives these values of $\rho_m$ and $Y$ for several different materials. Note that $Y$ varies by a factor of more than $100\,000$ across these different materials.

| Material | $\rho_m \left[\text{kg/m}^3\right]$ | $Y \left[\text{N/m}^2\right]$ |
|---|---|---|
| Rubber | $1\,500$ | $5 \times 10^6$ |
| Tendon | $1\,200$ | $5 \times 10^8$ |
| Nylon | $1\,200$ | $3 \times 10^9$ |
| Bone | $4\,000$ | $2 \times 10^{10}$ |
| Concrete | $2\,400$ | $3 \times 10^{10}$ |
| Steel | $8\,000$ | $2 \times 10^{11}$ |
| Diamond | $3\,500$ | $1 \times 10^{12}$ |

Figure 9.13 considered a compression wave pulse reflecting off of a fixed end of a piece of material. Figure 9.14 shows what happens when the same pulse reaches a free end of the material. With nothing pushing back on piece 5, it moves outward from its original position, pulling on piece 4 as it goes. This sends a *rarefaction* wave pulse back to the left.

With longitudinal waves, the wave reflected off of the end of the rope would be either flipped over or not depending on the conditions at the end of the rope. Longitudinal waves behave in a similar way– compression bounces back as compression if the end is firmly held in place but compression bounces back as rarefaction if the end is free to move.

Rarefaction in a solid material creates regions of tension, as can be seen by the stretched strings in our model and the resulting net force arrows that pull neighboring pieces of material toward each other. Compression creates areas of pressure in the material, where the springs in our model are compressed and the resulting net forces push neighboring pieces of material away from each other.

(a) $t_0$

(b) $t_1$

(c) $t_2$

(d) $t_3$

(e) $t_4$

(f) $t_5$

(g) $t_6$

(h) $t_7$
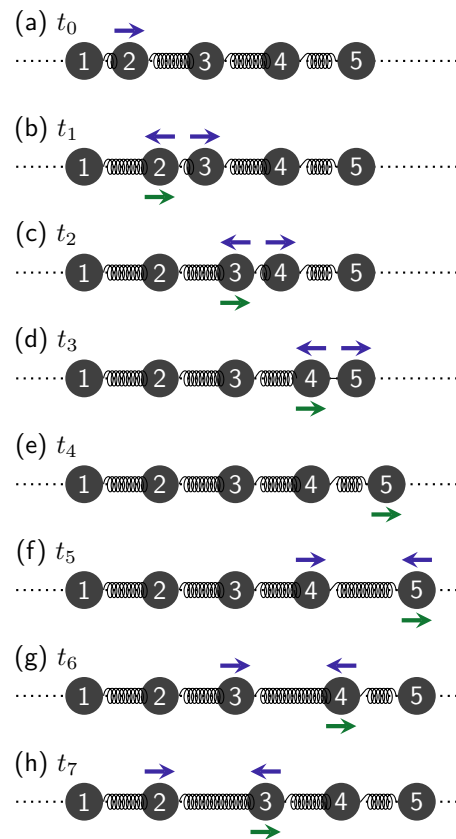
Figure 9.14: If piece "5" is allowed to move freely, a compression wave pulse moving to the right creates a rarefaction pulse that reflects back to the left.[1]

The speed of a longitudinal wave in a solid material is given by…

$$v = \sqrt{\frac{Y}{\rho_m}} \qquad (9.10)$$

Using the same guitar string example that we used in Section 9.3, we can find the speed of a longitudinal wave traveling along the length of the string:

$$v = \sqrt{\frac{Y}{\rho_m}} = \sqrt{\frac{2 \times 10^{11} \text{ N/m}^2}{8\,000 \text{ kg/m}^3}} = 5\,000 \text{ m/s}$$

193

## 9.6  Stretching and Breaking

### Words

The Young's modulus tells us more about a material than the speed of waves. It is a measure of how much the material stretches or compresses when a force is applied. The Young's modulus of a material is determined by its molecular structure–how tightly bound the atoms are to each other and also the configuration of bonds in the material.

There is also a limit to how much force can be applied to a given material before it breaks. Depending upon the complexity of the bonds, the Young's modulus and other measures of the material properties could be dependent upon direction or they could vary depending on whether the material is under tension or compression.

Some materials have structure in them that is not related to the molecules themselves but to larger variations in the material. Wood is a good example of this, because wood has a "grain," and when placed under tension, wood is much stronger across the grain than it is along the grain.

Materials can also have vastly different properties when they are under compression than when they are under tension. Concrete, for example, is very strong under compression but easily breaks under tension. That is why iron "rebar" is often placed inside concrete slabs in places where the concrete could be under tension. Iron is very strong under tension, so it holds the concrete together.

### Graphics
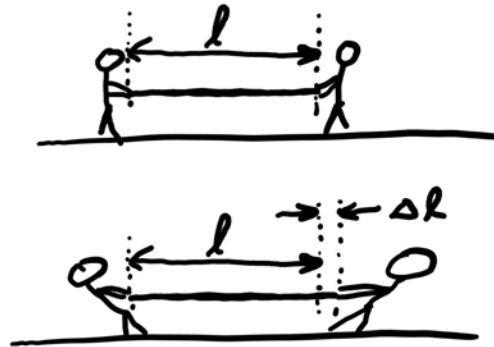


Figure 9.15: A rope of length $l$ stretches to a length $l + \Delta l$ when a force is applied.[1]



Figure 9.16: The board broke vertically into two pieces, along the grains visible in the board, when the black belt hit it with her foot. [48]

### Numbers

By how much does a 5-m-long nylon rope with a diameter of 2 cm stretch when it is pulled by a force of 150 N?

| Knowns | Unknowns |
|---|---|
| $r = 0.01\,\mathrm{m}$ | $\Delta l$ |
| $l = 5\,\mathrm{m}$ | |
| $F_T = 150\,\mathrm{N}$ | |
| nylon | |

The amount by which the length of a solid changes when a force is applied is given by…

$$\Delta l = \frac{l \cdot F}{Y \cdot A} \tag{9.11}$$

…where $l$ is the original length, $F$ is the force applied parallel to the length, $Y$ is Young's Modulus for the material, and $A$ is the cross-sectional area of the material.

So for this example…

$$\Delta l = \frac{5\,\mathrm{m} \cdot 150\,\mathrm{N}}{3 \times 10^9\,\mathrm{N/m^2} \cdot \pi \cdot (0.01\,\mathrm{m})^2}$$
$$= 8 \times 10^{-4}\,\mathrm{m}$$

Even for a relatively stretchy material like nylon, the change in length is quite small. Most solids are very difficult to stretch or compress.

The breaking point of a material is strongly affected by local factors. Materials often break not in the place where the largest force is applied, but in the place where the largest pressure is applied. Pressure is defined as force per area, and it has units of pascals [Pa]. Often tools that are made for breaking or cutting have sharp edges, so the force will be applied over a very small area, creating large pressure.

If force is applied evenly instead of being concentrated at a single location then breaks will usually start at an inside corner or a place where the material has a defect. Think of a chain, which is only as strong as its weakest link.

Glass workers take advantage of breaks starting at defects by intentionally scratching one side of a plane of glass in the place where they want it to break. This makes the glass break cleanly along the scratch when the glass is flexed in such a way that the scratched surface is under tension.



Figure 9.17: Rebar being added to cement.[49]



Figure 9.18: A thumbtack creates very little pressure on the thumb, but enough pressure at the bottom to punch into wood.[50]
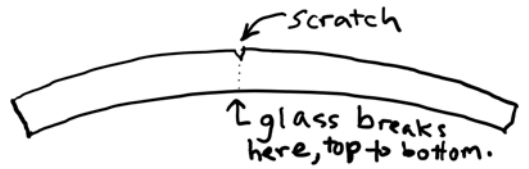


Figure 9.19: Flexing upward creates tension along the top and compression along the bottom.[1]

Consider the thumbtack in Figure 9.18. The top surface has a diameter of 6 mm and the point at the bottom narrows to 0.1 mm. How much pressure is applied to the thumb and to the wood if you push down on the thumbtack with a force of 40 N?

| Knowns | Unknowns |
|---|---|
| $r_{thumb} = 0.003\,\mathrm{m}$ | $P_{thumb}$ |
| $r_{point} = 5 \times 10^{-5}\,\mathrm{m}$ | $P_{point}$ |
| $F_{thumb} = 40\,\mathrm{N}$ | |

Pressure is defined as force per area:

$$P \equiv \frac{F}{A} \qquad (9.12)$$

…where the force is applied parallel to the normal vector; that is, perpendicular to the surface of A.

If we neglect the gravitational force and assume that this is a static situation then the net force on the thumbtack will be zero, so the force between the thumb and the thumbtack is equal to the force between the thumbtack and the wood. This allows us to find the pressures:

$$P_{thumb} = \frac{F_{thumb}}{A_{thumb}} = 1.4 \times 10^6\,\mathrm{Pa}$$

$$P_{point} = \frac{F_{thumb}}{A_{point}} = 5 \times 10^9\,\mathrm{Pa}$$

The pressure on the wood is more than 1000 times larger than the pressure on the thumb!

## 9.7 Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

### General

- In a transverse wave, each piece of an object is moving in a direction perpendicular to the direction in which the wave is moving.

- In a longitudinal wave, each piece of an object is moving in a direction parallel to the direction in which the wave is moving.

- Compression refers to a state in which pieces of a material are pressed more closely together than normal.

- Rarefaction refers to a state in which pieces of a material are separated more widely than normal.

- We can model a solid object as many small, interconnected objects to study how the object bends and stretches.

- Wavelength is the physical distance between successive peaks in a wave.

- Frequency is the number of times an object goes through an oscillation (or also the number of times a wavelength passes a given point) in a unit of time.

- Constructive interference occurs where two waves are completely aligned with each other, so their amplitudes add.

- Destructive interference occurs where two waves are completely misaligned with each other, so their amplitudes subtract.

### Forces

- Rarefaction in a solid material creates regions of tension, and compression in a solid creates areas of pressure.

- Pressure is defined as force per area, and it has units of pascals [Pa].

- Materials often break not in the area with the largest force but in the area with the largest pressure.

### Motion

- The speed of a transverse wave depends upon tension and linear mass density.

- If the end of a rope is allowed to move freely, transverse waves bounce back from the end of the rope.

- If the end of a rope is held firmly in place, transverse waves flip over and bounce back from the end of the rope.

- When two waves pass by each other, they interfere with each other so that the position of the object is given by the sum of the two waves.

- A node is a place on a vibrating object that does not move.

- An antinode is a place on a vibrating object where the motion is at a maximum.

- A rope that is held tightly at each end can vibrate at harmonic frequencies where the wavelength is some multiple of half the length of the rope.

- If the end of a solid is held firmly in place, longitudinal waves bounce back from the end of the rope.

- If the end of a solid is allowed to move freely, longitudinal waves change from compression to rarefaction (or rarefaction to compression) and bounce back from the end of the rope.

- The speed of longitudinal waves depends on density and elastic properties of the material.

## Momentum

- (Nothing!)

## Energy

- Waves can transfer energy.

## Mathematical Models

| equation | | restrictions on the validity of the equation |
|---|---|---|
| $v = \sqrt{\frac{F_T}{\mu_m}}$ | (9.1) | Transverse displacement is small compared to wavelength |
| $\mu_m \equiv \frac{m}{l}$ | (9.2) | -none- |
| $\lambda = v \cdot T = \frac{v}{f}$ | (9.3) | -none- |
| $f = \frac{1}{T}$ | (9.4) | -none- |
| $\lambda_n = \frac{2l}{n}$ | (9.5) | Vibrating string held firmly at each end |
| $f_n = \frac{n}{2l} \cdot \sqrt{\frac{F_T}{\mu_m}}$ | (9.6) | Vibrating string held firmly at each end |
| $f_{\text{beat}} = |f_1 - f_2|$ | (9.7) | -none- |
| $f_{\text{carrier}} = \frac{f_1 + f_2}{2}$ | (9.8) | -none- |
| $\rho_m \equiv \frac{m}{V}$ | (9.9) | -none- |
| $v = \sqrt{\frac{Y}{\rho_m}}$ | (9.10) | Longitudinal wave |
| $\Delta l = \frac{l \cdot F}{Y \cdot A}$ | (9.11) | -none- |
| $P \equiv \frac{F}{A}$ | (9.12) | -none- |

## 9.8 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

### Level 1 - Remember

9.1 **[W]** What is the name of a wave in which the pieces of the material don't actually move in the direction that the wave is moving?

9.2 **[W]** What is the name of a wave in which the pieces of the material move parallel to the direction that the wave is moving?

9.3 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

### Level 2 - Understand

9.4 **[W]** Can transverse waves travel through a solid? If so, give an example.

9.5 **[W]** Can longitudinal waves travel through a solid? If so, give an example.

9.6 **[W & G]** Can wavelength be measured from trough to trough instead of from peak to peak?

9.7 **[W & N]** How could you change the amount of force to increase the pressure on an object?

9.8 **[W & N]** How could you change the area over which a force is applied to increase the pressure on an object?

### Level 3 - Apply

9.9 **[W & N]** How would doubling the mass of the rope while keeping its length and the tension the same affect the speed of a longitudinal wave?

9.10 **[N]** Frequency is a useful concept not just for waves but also for circular motion and oscillations. What is the frequency of Callisto's orbit around Jupiter, which we considered in Section 8.3?

9.11 **[W & N]** The image below shows the end of the neck of a guitar. There are six strings, all made from steel, that are various thicknesses. Each of the strings is connected to a knob at the top that can be used to change the tension in the string. There are also horizontal metal "frets" along the neck of the guitar. The string can be held down tightly against these frets to effectively shorten the length of the string. Explain how each of these three things affects the frequency of the first harmonic of each string:

(a) The thicknesses of the strings?

(b) The knobs that change tension?

(c) The use of the frets?

The neck of a guitar, showing the different thicknesses of the strings, the knobs that change tension, and the frets.[1]

9.12 **[N]** The human range of hearing is roughly 20 Hz to 20 kHz. In addition, some humans can detect beats in sound waves with beat frequencies as high as 7 Hz. Which of the following pairs of frequencies would produce beats that could be audible to a person?

(a) 5 Hz & 7 Hz

(b) 50 Hz & 70 Hz

(c) 500 Hz & 700 Hz

(d) 5 kHz & 7 kHz

(e) 500 Hz & 500 Hz

(f) 500 Hz & 502 Hz

## Level 4 - Analyze

9.13 **[W & G]** In Figure 9.3, in what direction is the wave moving and in what direction is the piece of rope moving at piece 5...

(a) ...at time $t_2$?

(b) ...at time $t_3$?

(c) ...at time $t_4$?

(d) ...at time $t_5$?

9.14 **[W & G]** Compare the rope at time $t_1$ and $t_5$ in Figure 9.5. What do you notice about...

(a) ...the positions of the pieces of the rope?

(b) ...the forces on the pieces of the rope?

(c) ...the momenta of the pieces of the rope?

Explain why there are similarities or differences in the forces and momenta.

9.15 **[N]** Use the information in the table in Section 9.5 and the linear mass density of the steel string in Section 9.3 to find the diameter $\varnothing$ of the guitar string. The equation for the volume of a cylinder in Appendix D may be helpful.

9.16 **[W & G]** Compare the rope at time $t_0$ and $t_5$ in Figure 9.6. What do you notice about...

    (a) ...the positions of the pieces of the rope?

    (b) ...the forces on the pieces of the rope?

    (c) ...the momenta of the pieces of the rope?

Explain why there are similarities or differences in the forces and momenta.

9.17 **[N]** Find the speed of transverse waves on the guitar string studied in Section 9.3. Compare this to the speed of longitudinal waves on the same string studied in Section 9.5.

9.18 **[W, G, & N]** Explain using words, graphics, or numbers why lying down on thin ice is safer than standing on thin ice.


## Level 5 - Evaluate

9.19 **[W & G]** In the introduction to this chapter it mentions that waves can carry energy. Consider Figures 9.3 & 9.4. Is energy being carried by this wave? What kind of energy or energies are involved? If energy is being carried, in what direction is it moving on average?

9.20 **[W & G]** Explain how the forces shown by blue arrows affect the momentum shown by green arrows in Figure 9.3 for piece 6.

9.21 **[G]** Make a sketch showing what the position of the rope would look like in Figure 9.5 at time $t_6$.

9.22 **[G]** Make a sketch showing what the rope would look like in Figure 9.6 halfway between times $t_2$ and $t_3$. Include arrows showing the forces and momenta associated with each of the "pieces" of the rope at that time.

9.23 **[G]** In all of the figures in Section 9.4, the two waves that were being added together had the same amplitudes. Re-draw all three waves in each of the three figures if the middle wave had half the amplitude of the top wave. In each figure, the top wave should be unchanged, the middle wave should be half its current amplitude, and the bottom wave should be the new sum of the top two waves.

9.24 **[W & N]** The force of gravity was neglected when considering the pressures involved when using a thumbtack in section 9.6. Determine whether it was reasonable to neglect the force in this situation by determining what the mass of the thumbtack would have to be to create a 10% change in one of the calculated pressures.

9.25 **[W & G]** The image below shows a truck sitting on a bridge that is made from a simple slab of concrete, shown with light diagonal markings. Answer the following questions using words or drawings.

    (a) What side of the concrete is under tension?

    (b) What side of the concrete is under compression?

    (c) Where is the concrete most likely to break, considering that it is weaker under tension than it is under compression?

    (d) If you could add iron rebar to only one side of the concrete, where should you put it?
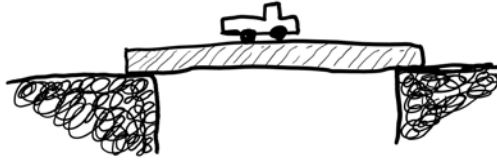
A truck on a bridge.[1]

9.26 **[W & G]** The image below shows a truck sitting on a cantilever bridge that is made from a simple slab of concrete, shown with light diagonal markings. Answer the following questions using words or drawings.

(a) What side of the concrete is under tension?

(b) What parts of the concrete are under compression?

(c) Where is the concrete most likely to break, considering that it is weaker under tension than it is under compression?

(d) If you could add iron rebar to only one part of the concrete, where should you put it?
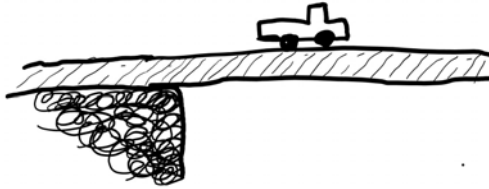


A truck on a cantilever bridge.[1]

## Level 6 - Create

9.27 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Add the main ideas from this chapter to the similar concept map that you began for the question at the end of Chapter 8.

9.28 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

9.29 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

# Chapter 10

# Liquids

We created a model of a solid that was made up of many small pieces all held together by springs. This model works all of the way down to the molecular scale, with the springs representing the forces holding individual molecules to each other. Now we will begin to explore liquids. The molecules of a liquid behave differently from those of a solid. They are closely spaced, much like a solid, but there are not strong bonds between the molecules to hold them in place; they are free to move around while still staying close together.

We can think of molecules in a liquid like people in a crowd–they are closely spaced, able to move around, and taking the shape of the available space.

Much like most solids, most liquids are not easily compressed into a smaller volume–we will consider them to be completely incompressible, so they will have a constant volume. But unlike solids, liquids can flow and change shape, usually to fit the shape of a container.

A material that can flow and change shape is called a *fluid*. Liquids are the first type of fluid that we will consider. Gases are another type of fluid that we will study later.



Figure 10.1: Molecules in a liquid behave like people in a crowd.[52]



Figure 10.2: Fluids (of which liquids are one example) take the shape of their container.[51]

## 10.1 Doric Temple of Athena Lindia

**Words**

The idea of pressure has already been introduced as a force per area. Let's explore that idea in a solid and a liquid. The ruins of the Doric Temple of Athena Lindia in Rhodes have sandstone columns that are 8 m tall and 1 m in diameter. How much pressure do these columns apply to the floor of the temple? What would the pressure be if the column were instead made of liquid water in a cylindrical tube?

To find the pressure on the floor, it seems like we should need to know the gravitational force and the area. To find the gravitational force we would need the mass, but we aren't given that. We need some additional information about sandstone: its density, that is its mass per volume. The density of sandstone is $2\,400\,\text{kg/m}^3$.

If we take the column to be a uniform cylinder, its volume is its height times its cross-sectional area. So as that area increases, so does the volume, which means a larger mass. But we also know that pressure decreases as area increases. So it turns out that while the force on the ground does depend on the diameter of the column, the pressure on the ground doesn't. It only depends on the density and the height.

It is also important to note that if the column is in static equilibrium then the ground is applying an equal-but-opposite upward force on the column, over the same area, so the earth also applies the same amount of pressure back on the column.

**Graphics**



Figure 10.3: The sandstone columns are 8 m tall and 1 m in diameter.[53]
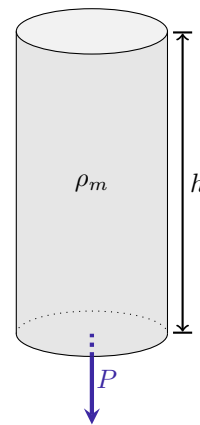


Figure 10.4: A solid column of height $h$ and density $\rho_m$ applies pressure on the ground.[1]

**Numbers**

**Assumptions:** a cylinder is a good approximation of the shape; near the surface of the earth; sandstone and water are incompressible

| Knowns | Unknowns |
|---|---|
| $h = 8\,\text{m}$ | $P_{sandstone}$ |
| $r = 0.5\,\text{m}$ | $P_{water}$ |
| $\rho_{m,sandstone} = 2\,400\,\text{kg/m}^3$ | |
| $\rho_{m,water} = 1\,000\,\text{kg/m}^3$ | |
| $g = 9.8\,\text{m/s}^2$ | |

The volume of a cylinder is the height times the cross-sectional area. Taking these to be perfectly round cylinders, that gives...

$$V_{cylinder} = h \cdot \left( \pi \cdot r^2 \right)$$

...where $\pi \cdot r^2$ is the cross-sectional area. Since pressure at the bottom of the column is the force per area, near the surface of the earth that would be...

$$P = \frac{F_g}{A} = \frac{m \cdot g}{\pi \cdot r^2}$$

We can rearrange our expression for mass density to solve for $m$, and then combine that with the volume of the cylinder:

$$m = \rho_m \cdot V = \rho_m \cdot h \cdot \pi \cdot r^2$$

Putting this together with our expression for the pressure at the bottom of the cylinder:

If instead of being solid the column were made of a liquid with the same density as sandstone, then the total force and total pressure on the ground would be exactly the same. Of course if the column were liquid then it wouldn't actually stand up on its own–it would flow downward, spreading all over the ground instead. So if we want to consider a column of water then we have to add a tube holding it in position.

In order to keep the liquid in place, the tube would have to supply a pressure inward that is exactly equal to the pressure with which the liquid pushes out on the tube. The pressure in a fluid (liquid is a type of fluid) pushes out in all directions. So at the bottom of the tube, the pressure on the sides of the tube would be the same as the downward pressure on the earth. But the pressure decreases as you go up the column, reaching zero at the top, because there is less and less liquid pushing down from above due to gravity. This is illustrated in Figure 10.5.

If the liquid were fresh water, then the downward pressure on the ground would be lower than the pressure for a sandstone column, because the density of water is $1\,000\,\mathrm{kg/m^3}$, which is lower than the density of sandstone.

"Specific gravity" is often used in the medical field as a unitless way to refer to density. Specific gravity is simply the density of some material compared to the density of fresh water. So fresh water has a specific gravity of 1. Sandstone, with density 2.4 times that of water has a specific gravity of 2.4.
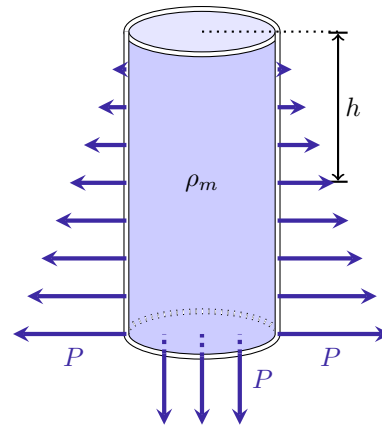


Figure 10.5: A column of liquid in a tube applies outward pressure on the ground and also on the sides of the tube. The amount of pressure increases with depth $h$.[1]

$$P = \frac{\rho_m \cdot h \cdot \cancel{\pi \cdot r^2} \cdot g}{\cancel{\pi \cdot r^2}}$$

$$\boxed{P = \rho_m \cdot g \cdot h} \qquad (10.1)$$

It is interesting that the pressure doesn't depend on the cross-sectional area of the pillar. There is also nothing that limits the analysis to a solid, so it is also valid for liquids. For a liquid, $h$ represents the height of the surface above the point of interest, or alternatively the depth of the point of interest below the surface.

If the column is sandstone, the pressure it applies to the earth is…

$$P_{sandstone} = \rho_{m,sandstone} \cdot g \cdot h = 188\,000\,\mathrm{Pa}$$

If the column is water, the pressure it applies at the bottom (to the earth and to the sides of the tube) is…

$$P_{water} = \rho_{m,water} \cdot g \cdot h = 78\,000\,\mathrm{Pa}$$

At a position higher up the column, for example 1 m from the ground, the pressure that the water applies to the sides of the cylinder is…

$$P_{water} = 1\,000\,\mathrm{kg/m^3} \cdot 9.8\,\mathrm{m/s^2} \cdot (8-1)\ \mathrm{m}$$
$$= 69\,000\,\mathrm{Pa}$$

## 10.2   Transfer of Pressure

### Words

We have already noted that liquid takes the shape of the container that it is in, but what happens to the pressure in the liquid if the container is oddly shaped, or if the container is squeezed with some additional applied pressure besides just the force of gravity on the liquid itself?

Many mechanical systems, like brakes in a car or the arms of cranes and backhoes, are controlled by hydraulic systems, in which a liquid fills a series of hoses and chambers, and pistons are used to apply pressure to the liquid in one location so the liquid applies pressure in another location. We will consider a simple system to understand how these hydraulic systems work: A pitcher with a narrow neck.

The pitcher is 16 cm tall, including a 2 cm diameter neck and an 8 cm diameter sphere at the bottom. We will assume that it contains water, and we will consider it when completely filled and when "half-filled" just to the bottom of the neck.

Since the pressure created in a liquid by gravity depends only on the density of the liquid and the distance below the surface, perhaps surprisingly, the actual shape of the container doesn't matter. If the pitcher is "half-filled," meaning up to half of the height, then the pressure at the bottom will be half what it would be if the container were filled.

### Graphics



Figure 10.6: A 4$^{th}$ century pitcher with a long, narrow neck. [54]
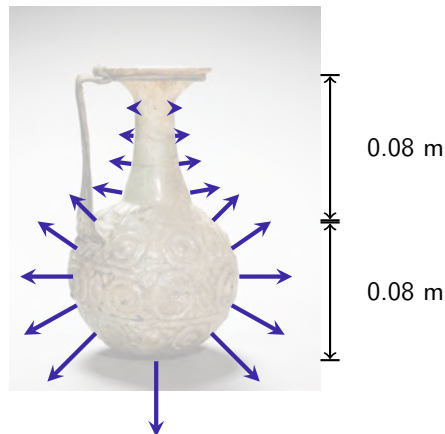


0.08 m

0.08 m

Figure 10.7: Pressure in the filled pitcher[55]

### Numbers

**Assumptions:** a cylinder and sphere is a good approximation of the shape; near the surface of the earth; water is incompressible

| Knowns | Unknowns |
|---|---|
| $h_{pitcher} = 0.16\,\mathrm{m}$ | $P_{filled}$ |
| $r_{neck} = 0.01\,\mathrm{m}$ | $P_{half-full}$ |
| $h_{neck} = 0.08\,\mathrm{m}$ | $V_{filled}$ |
| $r_{sphere} = 0.08\,\mathrm{m}$ | $V_{half-full}$ |
| $\rho_{m,water} = 1\,000\,\mathrm{kg/m^3}$ | |
| $g = 9.8\,\mathrm{m/s^2}$ | |

First, we will imagine the pitcher filled with water. The volume of the water would be the volume of the neck plus the volume of the sphere:

$$V_{filled} = \pi \cdot r_{neck}^2 \cdot h_{neck} + \frac{4}{3}\pi \cdot r_{sphere}^3$$
$$= \pi \cdot (0.01\,\mathrm{m})^2 \cdot 0.08\,\mathrm{m} + \frac{4}{3}\pi \cdot (0.08\,\mathrm{m})^3$$
$$= 2.5 \times 10^{-5}\,\mathrm{m^3} + 2.14 \times 10^{-3}\,\mathrm{m^3}$$
$$= 2.17 \times 10^{-3}\,\mathrm{m^3}$$

If the pitcher were filled only up to the bottom of the neck, the volume would be just the second term from the calculation above:

$$V_{half-full} = 2.14 \times 10^{-3}\,\mathrm{m^3}$$

Looking at the shape of the container, we can see that it takes very little additional water to completely fill the container once the sphere is full, and yet that small amount of water is enough to double the pressure at the bottom, because the neck is so tall and narrow.

Now let's consider another situation where the bottom half of the pitcher is filled, the neck of the pitcher is empty, and then a plug is pushed down to the base of the neck and held down with an external applied force. This force creates an external pressure on the water beneath it. That pressure is added equally to every part of the liquid. This is called "Pascal's Law."

This principle is used in hydraulic systems. If a force is applied to a small area of the liquid in a closed system, it creates a large pressure increase in all of the liquid. If the liquid goes to a movable part of the system with a relatively large area, a force is generated that is much larger than the force applied to the small area.

We have already seen that systems of pulleys can be used to create machines with a "mechanical advantage" larger than one. That means the output force is larger than the input force. Hydraulic systems can behave in the same way if the area is smaller for the input force than it is for the output force.
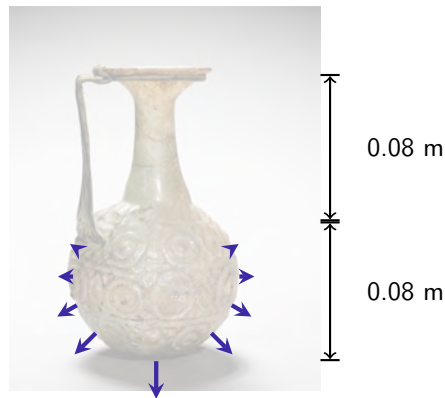


Figure 10.8: Pressure in the half-filled pitcher[55]
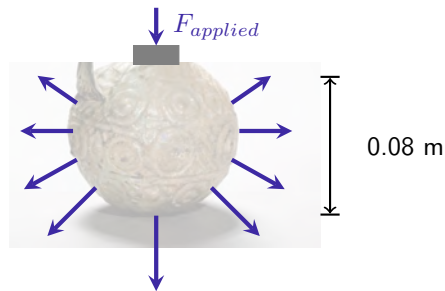


Figure 10.9: Applied pressure on half-filled pitcher[55]

Comparing these two volumes, we can see that the neck holds very little water compared to the sphere at the bottom.

According to Equation 10.1, the pressure created by gravity in a liquid depends only on gravity, the density of the liquid, and the height of the surface above the point of interest. When filled, the pressure at the bottom of the pitcher is:

$$
\begin{aligned}
P_{filled} &= \rho_m \cdot g \cdot h \\
&= 1\,000 \, \text{kg/m}^3 \cdot 9.8 \, \text{m/s}^2 \cdot 0.16 \, \text{m} \\
&= 1\,568 \, \text{Pa}
\end{aligned}
$$

Similarly, the pressure at the bottom of the pitcher when it only half-full is 784 Pa.

If an external force applies an additional pressure to a liquid, the force increases by that same amount of pressure, so that the total pressure becomes:

$$
\boxed{P_{tot} = \rho_m \cdot g \cdot h + P_{external}} \tag{10.2}
$$

To make the pressure at the bottom of the half-filled pitcher the same as that of the filled pitcher, an external pressure of 784 Pa would need to be applied to the liquid at the base of the neck. We could find the amount of force that would need to be applied to a plug in the neck of the pitcher to create such a pressure. Rearranging our expression for pressure…

$$
\begin{aligned}
F &= P \cdot A \\
&= 784 \, \text{Pa} \cdot \pi \cdot (0.01 \, \text{m})^2 \\
&= 0.246 \, \text{N}
\end{aligned}
$$

…which is exactly the weight of the water that fills the neck of the pitcher.

## 10.3 Floating and Sinking

### Words

We have looked at liquids placed into containers. Now we will look at solid objects placed into liquids. Some objects sink and other objects float; the determining factor for sinking or floating is density. Materials with lower density tend to rise up and those with higher density sink down.

The ingredients in the salad dressing shown in Figure 10.10 have separated themselves into layers based on their densities: sugar with the highest density on the bottom, then vinegar, garlic, oil, and finally dried herbs with the lowest density on the top. For our analysis we will use a simpler situation with just two materials each time: Either water and ice or water and aluminum.

First we will consider a cube of ice 20 cm on each side floating in fresh water. How much of the ice is submerged below the surface of the water and how much is above the surface? The densities of most materials can be found using an internet search.

The ice cube is in equilibrium in this scenario, so the net force must be zero. Gravitational force acts downward, so some other force must be acting upward to balance it. There is nothing like a rope or a spring, so it can't be tension force or spring force. It isn't sitting on a solid surface, so no normal force. It isn't a frictional force, either. This is a new force that we haven't considered yet, *buoyant force*. Buoyant force is caused by gravity creating pressure in the fluid around an object, creating a net upward force on the object.

### Graphics



Figure 10.10: Unmixed salad dressing, with ingredients separated according to their densities [1]
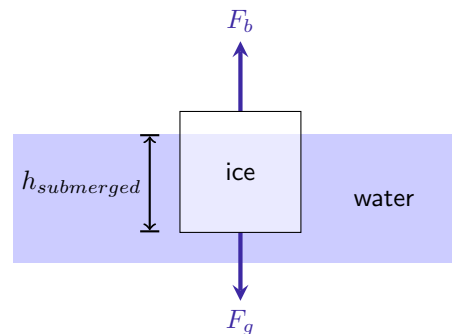


Figure 10.11: Free-body diagram of ice floating in water, with the water level shown for clarity.[1]

### Numbers

**Assumptions:** near the surface of the earth; water and ice are incompressible

| Knowns | Unknowns |
|---|---|
| $V_{ice} = 0.2 \times 0.2 \times 0.2 \,\text{m}^3$ | $V_{submerged}$ |
| $\rho_{m,water} = 1\,000 \,\text{kg/m}^3$ | $m_{ice}$ |
| $\rho_{m,ice} = 920 \,\text{kg/m}^3$ | |
| $g = 9.8 \,\text{m/s}^2$ | |

We can rearrange the definition of mass density to find the mass of the ice block:

$$m_{ice} = \rho_{m,ice} \cdot V_{ice}$$
$$= 920 \,\text{kg/m}^3 \cdot 0.2 \times 0.2 \times 0.2 \,\text{m}^3$$
$$= 7.36 \,\text{kg}$$

The ice is in equilibrium and we only need to consider the vertical direction for forces, so...

$$F_{net,y} = F_b - F_g = 0$$

$$F_b = F_g = m_{ice} \cdot g = 72.1 \,\text{N}$$

The buoyant force is created by the upward pressure of the water on the bottom of the ice, so using Equation 10.1...

$$F_b = P_{bottom} \cdot A_{bottom}$$
$$= (\rho_{m,water} \cdot g \cdot h_{submerged}) \cdot A_{bottom}$$
$$= \rho_{m,water} \cdot g \cdot V_{submerged}$$

...since the submerged volume is the submerged height times the cross-sectional area.

For a floating object, the buoyant force created by the pressure in the fluid exactly balances the gravitational force on the object itself. This happens when a volume of the liquid equal to the submerged volume of the object has the same mass as the object.

Now we will consider instead a 20 cm x 20 cm x 20 cm block of aluminum sinking in fresh water. What is the buoyant force on the aluminum? Again, the buoyant force is created by the pressure in the fluid, but this time there is fluid on all sides. Since pressure increases with depth, the force upward on the bottom is greater than the force downward on the top. Because of this, the buoyant force is always upward.

Much like in the case of the floating ice, the buoyant force is related to the submerged volume of the object (in this case the whole object), but this time the buoyant force is smaller than the gravitational force. That is because buoyant force is equal to the weight of a volume of water equal to the volume of the object, and water has a lower density than the object. As long as the object and the fluid around it are incompressible, the buoyant force on a completely submerged object does not change as it sinks deeper into the liquid.

If the object sinks to the bottom of the liquid, for example sitting on the bottom of a lake, the forces are again balanced but in this case the buoyant and normal forces together balance the gravitational force.
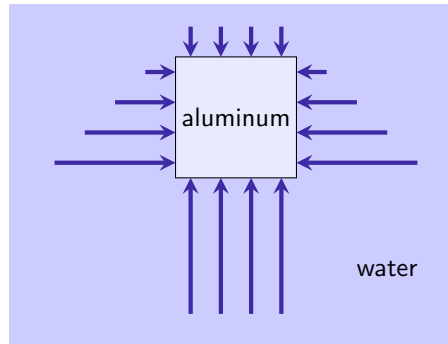


Figure 10.12: Pressure on a block of aluminum sinking in water.[1]

Looking at Figure 10.12, we can see that the pressure in the horizontal direction presses equally on both sides of the block, canceling out. But the pressure on the bottom of the block is larger than the pressure on the top.
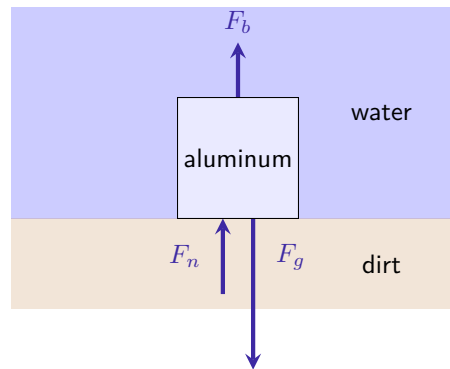


Figure 10.13: Free-body diagram of aluminum sitting at the bottom of a lake.[1]

**Assumptions:** near the surface of the earth; water and aluminum are incompressible

| Knowns | Unknowns |
|---|---|
| $V_{Al} = 0.2 \times 0.2 \times 0.2\,\mathrm{m}^3$ | $F_b$ |
| $\rho_{m,water} = 1\,000\,\mathrm{kg/m}^3$ | $m_{Al}$ |
| $\rho_{m,Al} = 2\,700\,\mathrm{kg/m}^3$ | |
| $g = 9.8\,\mathrm{m/s}^2$ | |

In the case of a block that is completely submerged, there is pressure at the top and the bottom. The pressure at the top can be considered an external pressure caused by the water above the block, and using Equation 10.2 the pressure in the water at the bottom becomes…

$$P_{bottom} = \rho_{m,water} \cdot g \cdot h + P_{top}$$

The buoyant force would be the combined force from the pressure on the top and the bottom. The areas of the top and the bottom are equal, so…

$$
\begin{aligned}
F_b &= P_{bottom} \cdot A_{bottom} - P_{top} \cdot A_{top} \\
&= (P_{bottom} - P_{top}) \cdot A \\
&= ((\rho_{m,water} \cdot g \cdot h + P_{top}) - P_{top}) \cdot A \\
&= \rho_{m,water} \cdot g \cdot h \cdot A \\
&= \rho_{m,water} \cdot g \cdot V_{submerged}
\end{aligned}
$$

This is the exact same result that we got for a floating object. It is generally true for any object in any fluid under the influence of gravity:

$$\boxed{F_b = \rho_{m,fluid} \cdot g \cdot V_{submerged}} \tag{10.3}$$

## 10.4 Flowing

### Words

In our previous physical scenarios with liquids, the liquid has always been stationary, so the focus has been primarily on forces. Now we will allow the liquids to flow, so we will begin to consider motion and energy. The scenario we will consider is water flowing through a horizontal pipe with varying diameter: first with a diameter of 10 cm, then narrowing to 5 cm, and finally returning to 10 cm. The water enters the first part of the pipe flowing to the right at 5 m/s at a pressure of $4 \times 10^5$ Pa in the center of the pipe.

We will assume for now that the water is an ideal fluid; that is, the flow is smooth and constant and there is no friction because only elastic collisions occur within the fluid. We will also assume that the fluid moves at the same speed across the entire cross-section of the pipe at any given horizontal position. What is the speed and pressure of the water at the center, top, and bottom of each section of the pipe?

If we consider just the first section of the pipe, the velocity of the water is constant, to the right. The pressure increases with depth, so the pressure at the bottom of the pipe would be higher than $4 \times 10^5$ Pa and the pressure at the top of the pipe would be lower than $4 \times 10^5$ Pa.

The flow of a liquid can be described in terms of the *volumetric flux*, that is, the volume of flow per time, $[\text{m}^3/\text{s}]$. In a pipe, the same volume of water flows through each section of pipe, so the volumet-

### Graphics
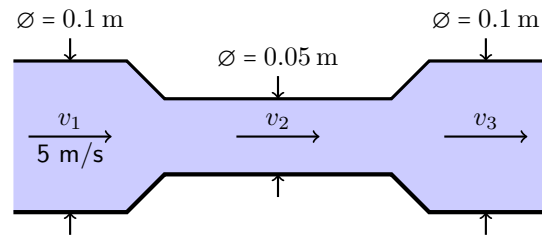


Figure 10.14: Water flowing through a pipe with varying diameter. "∅" is the symbol for diameter.[1]
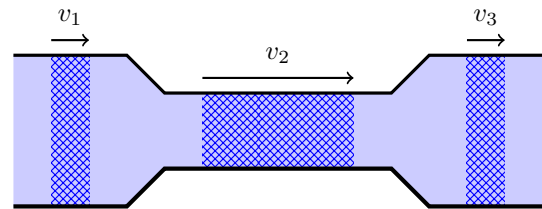


Figure 10.15: Water flowing through a pipe with varying diameter. Equal volumes are marked in each section of pipe. Since the volumetric flux is constant, it takes water the same amount of time to pass through each marked region. That means the speed has to be higher in the narrower part of the pipe. [1]

### Numbers

**Assumptions:** laminar flow; near the surface of the earth; water is incompressible; ideal fluid

| Knowns | Unknowns |
|---|---|
| $A_1 = 7.85 \times 10^{-3} \text{ m}^2$ | $v_2$ |
| $A_2 = 1.96 \times 10^{-3} \text{ m}^2$ | $v_3$ |
| $A_3 = 7.85 \times 10^{-3} \text{ m}^2$ | $P_{1,top}$, $P_{1,bottom}$ |
| $v_1 = 5 \text{ m/s}$ | $P_{2,top}$, $P_{2,center}$, $P_{2,bottom}$ |
| $P_{1,center} = 4 \times 10^5 \text{ Pa}$ | $P_{3,top}$, $P_{3,center}$, $P_{3,bottom}$ |
| $\rho_{m,water} = 1\,000 \text{ kg/m}^3$ | |
| $g = 9.8 \text{ m/s}^2$ | |

The areas were found using $A = \pi \cdot r^2$.

Volumetric flux is given by:

$$\boxed{\Phi_V = v \cdot A \cdot \cos \theta} \qquad (10.4)$$

…where $v$ is the speed of the fluid, $A$ is the area through which the fluid is flowing, and $\theta$ is the angle between the velocity and the normal vector of the area. For flow through a pipe this angle is zero. Volumetric flux is constant in all sections of the pipe…

$$\Phi_{V,1} = \Phi_{V,2} = \Phi_{V,3}$$

$$v_1 \cdot A_1 = v_2 \cdot A_2 = v_3 \cdot A_3$$

Substituting in our known values, $v_3$ is 5 m/s, the same as $v_1$, and $v_2$ is 20 m/s.

ric flux is the same for each. This means that the speed of the water must be faster through the narrower section of pipe, since the cross-sectional area of that pipe is smaller.

The change in the speed of the water tells us that the water accelerated to the right as it went from the wider section of pipe to the narrower section. So there must be a net force in the direction of the flow when the pipe narrows. The only forces in that direction are from the pressure in the fluid itself. So the pressure in the narrow part of the pipe must be lower than in the wider part of the pipe. Then when the water passes into the third part of the pipe, where it is wider again, the water experiences an acceleration to the left, returning to its original speed. So the pressure in the third part of the pipe is the same as in the first part.

This may seem counter-intuitive, because many people would tend to think that pressure must be highest where the speed is highest. But remember that pressure is related to force. And we have already learned that just because an object is moving at high speed does not necessarily mean that there is a large force acting on the object. It just means that for some period in the object's history there was a net force acting in the direction that the object is currently moving. The same is true here–a larger pressure from the first section of the pipe caused the higher speed in the second section of the pipe.

The reduction of pressure in places where a pipe narrows is referred to as the "Venturi effect."
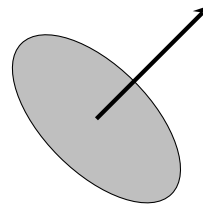


Figure 10.16: The normal vector of an area points directly out of the surface. [1]
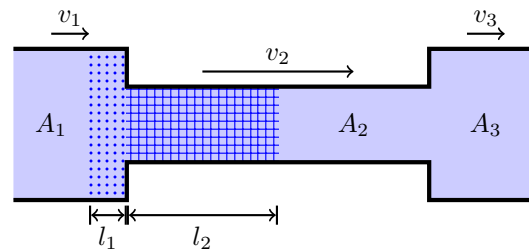


Figure 10.17: This simplified model of water flowing through a pipe makes it easier to find the change in pressure as the water flows through it. A volume of water, shaded with dots in the first section of the pipe, flows to the right, so the same water a short time later occupies the volume shaded with a grid in the second section of the pipe.[1]

Consider the volume of water in Figure 10.17 that starts in the first section of pipe and moves to the section section of pipe. Energy has to be conserved unless external work is done on the water.

$$W_{net} = \Delta E$$

The pressure in the first section of the pipe does work pushing to the right for a distance $l_1$ while the pressure in the second section of the pipe does work pushing to the left for a distance $l_2$. This increases the kinetic energy of the water. In addition to these changes in the horizontal direction, any change in the vertical direction creates a change in gravitational potential energy. Combining these three in an incompressible fluid gives "Bernoulli's Equation:"

$$\boxed{\left(P + \frac{1}{2}\rho_m \cdot v^2 + \rho_m \cdot g \cdot h\right) \text{ is constant}} \quad (10.5)$$

If we choose to set the height $h = 0$ at the center of the pipe, we can find the constant value given by Equation 10.5 for this system at the center of the first section of pipe:

$$P_{1,center} + \frac{1}{2}\rho_{m,water} \cdot v_1^2 + \rho_{m,water} \cdot g \cdot 0$$

$$= \left(4 \times 10^5 \,\text{Pa}\right) + \frac{1}{2}\left(1\,000 \,\text{kg/m}^3\right) \cdot \left(5 \,\text{m/s}\right)^2 + 0$$

$$= 412\,500 \,\text{Pa}$$

This constant is valid anywhere in the pipe. At the top of the second section of pipe, for example...

$$P_{2,top} + \frac{1}{2}\rho_{m,water} \cdot v_2^2 + \rho_{m,water} \cdot g \cdot 0.025 \,\text{m}$$

$$= 412\,500 \,\text{Pa}$$

## 10.5 Intermolecular Forces

### Words

Up to this point, we have only been dealing with "ideal fluids," in which the flow, if any, is smooth and constant and the only interactions between the molecules of the fluid or between the fluid and the walls of a container are like elastic collisions. Now we will introduce turbulent flow and *viscosity*, which comes from attractive forces between the molecules of the fluid and between the fluid and the walls of a container.

For fluids traveling at relatively low speeds and uninterrupted by obstacles, the fluid flow is *laminar*. Laminar flow is when the fluid is moving along smooth paths. At higher speeds or when obstacles are introduced, fluid flow becomes *turbulent*. Turbulent flow is when the flow is erratic, with mixing, rotation, and chaotic changes in magnitude and direction of the velocity. Laminar and turbulent flow can both be seen in Figure 10.18 and are drawn schematically in Figure 10.19.

Figure 10.19 shows laminar flow in a pipe for a fluid with non-zero viscosity (so not an ideal fluid). Viscosity can be thought of as how "thick" a fluid feels. Honey, for example, has a high viscosity. For this type of fluid, the speed is higher in the center of the pipe. That is because frictional forces between the pipe wall and the fluid cause the fluid at the edges to slow to a speed near zero. The speed of the fluid increases the further it is from the walls of the pipe, so it is at a maximum in the center.

### Graphics



Figure 10.18: The water is experiencing laminar flow as it comes from the left and starts to fall into the pool below. Once the water hits the pool below, the flow becomes turbulent. [56]
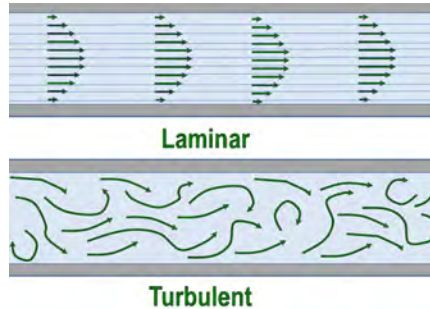


Figure 10.19: Laminar flow is shown in the upper pipe and turbulent flow in the lower pipe. [57]

### Numbers

**Assumptions:** laminar flow; near the surface of the earth; incompressible fluids

| Knowns | Unknowns |
|---|---|
| $r = 0.002\,\text{m}$ | $\Delta P$ |
| $l = 0.026\,\text{m}$ | |
| $\Phi_V = 0.06\,\text{l/min}$ | |
| $\mu_v = 0.005\,\text{Pa}\cdot\text{s}$ | |
| $\rho_m = 994\,\text{kg/m}^3$ | |
| $g = 9.8\,\text{m/s}^2$ | |

The volumetric flow rate for a viscous fluid in a pipe is highly dependent upon the exact conditions. If the flow in a pipe with a circular cross-section is laminar and the pipe is neither too short nor too wide then the volumetric flow rate is given by...

$$\Phi_V = \frac{\pi \cdot r^4 \cdot \Delta P}{8\mu_v \cdot l} \tag{10.6}$$

...where $r$ is the radius of the pipe, $l$ is the length of the pipe, and $\mu_v$ is the viscosity of the fluid. Before solving Equation 10.6 for the change in pressure, the flow rate needs to be converted to SI units:

$$\Phi_V = 0.06\,\text{l/min} \cdot \left(\frac{1\,\text{m}^3}{1\,000\,\text{l}}\right) \cdot \left(\frac{1\,\text{min}}{60\,\text{s}}\right)$$
$$= 1 \times 10^{-6}\,\text{m}^3/\text{s}$$

Because of the frictional force that constantly opposes the motion of the fluid, motion would stop unless there was a constant pressure difference between the two ends of the pipe. The volumetric flow rate in a pipe depends on the pressure difference between the two ends of the pipe, the viscosity of the fluid, the cross-sectional area of the pipe, and the length of the pipe.

We will consider this effect in a brachial artery, which carries blood to the arm. The brachial artery has a diameter of approximately 4 mm and a length of 26 cm. The flow rate is approximately 0.06 liters per minute. The density of blood is approximately 994 kg/m$^3$ and the the viscosity of blood is approximately 0.005 pascal-seconds [$Pa \cdot s$].

The analysis in the "Numbers" section shows that a pressure difference of only 20.7 Pa is enough for the blood to flow through the artery if the arm is horizontal, but the heart needs to provide more than ten times that amount if the arm is held straight up. In fact, the heart has to provide enough pressure for blood to flow through every artery, vein, and capillary in the body–more than 10,000 Pa of pressure. Blood pressure is usually measured in millimeters of mercury. One millimeter of mercury is enough pressure to raise a column of mercury against the force of gravity by 1 mm.

Another effect of attractive intermolecular forces between the molecules in a fluid is surface tension. A drop of water tends to pull itself together as much as possible into a shape like a ball. The actual details of the shape change depending upon the effects of gravity and the surfaces in contact with the drop.



Figure 10.20: The brachial artery is the large artery that runs through the upper arm. It is red in this sketch from Gray's Anatomy. [58]



Figure 10.21: Note how the smaller water droplets form nearly perfect spheres on this lotus leaf, while the larger drops are more deformed by gravity. [59]

Now we can find the change in pressure.

$$\Delta P = \frac{8\mu_v \cdot l \cdot \Phi_V}{\pi \cdot r^4}$$

$$= \frac{8 \cdot (0.005\,\text{Pa}\cdot\text{s})\,(0.026\,\text{m})\left(1 \times 10^{-6}\,\text{m}^3/\text{s}\right)}{\pi \cdot (0.002\,\text{m})^4}$$

$$= 20.7\,\text{Pa}$$

If the arm is horizontal, this change in pressure needs to be provided by the heart. If the arm is hanging down, then gravity provides a change pressure given by Equation 10.2, taking $P_{tot}$ to be the pressure at the bottom and $P_{external}$ to be the pressure at the top:

$$\Delta P = P_{tot} - P_{external}$$

$$= \rho_m \cdot g \cdot h$$

$$= \left(994\,\text{kg/m}^3\right) \cdot \left(9.8\,\text{m/s}^2\right) \cdot (0.026\,\text{m})$$

$$= 253\,\text{Pa}$$

…where the length of the artery is used as the height since the artery is vertical. This is more than enough pressure for the blood to flow through the artery even without the heart pumping. But if the arm is raised, the heart would have to provide enough additional pressure to make the blood flow upward against gravity.

## 10.6   Water Waves

### Words

If you have ever watched a wave move across a body of water, whether in a teacup or on the ocean, you have probably noticed that the water rises and falls. When the wave is caused by something falling into the water, the waves form rings that move outward. When the wave is caused by wind, it looks like long straight lines. Either way, the waves *look like* transverse waves rising and falling.

In fact, they are more complicated than transverse waves. Transverse waves travel easily through a solid because of the strong forces that hold each atom to all of its neighboring atoms. Liquids don't have such strong forces between atoms, so transverse waves do not travel easily through liquids.

What if instead a longitudinal wave were introduced into a body of water? This would cause compression and rarefaction of the water molecules, but water, like most solids and liquids, is nearly incompressible. Solids cannot flow, so the incompressibility makes longitudinal waves travel at very high speeds through the material. But liquids can flow, so if you try to compress the liquid in the horizontal direction it will flow upward instead of compressing. Then if you try to rarefy the liquid it will flow downward. This creates a circling motion in the water, as can be seen in Figure 10.23. The circling motion is a combination of transverse and longitudinal waves that is often simply referred to as a water wave. Waves tend to travel more slowly through liquids than through solids.

### Graphics
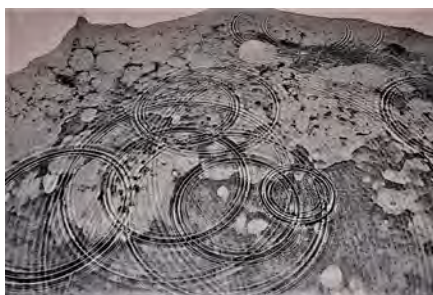


Figure 10.22: Waves move outward from places where water has dropped into the pool. Note that the waves can pass through each other. [60]
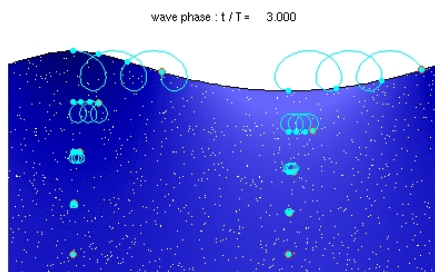


Figure 10.23: Individual particles in a water wave follow a near-circular path, moving both parallel and perpendicular to the wave itself, which is moving to the right in this image. [61]

A moving GIF of Figure 10.23 is viewable at https://commons.wikimedia.org/wiki/File:Deep_water_wave.gif.

### Numbers

| Knowns | Unknowns |
|---|---|
| $d = 0.02\,\text{m}$ | $\theta_{node\,0}$ |
| $\lambda = 0.01\,\text{m}$ | $\theta_{antinode\,1}$ |
| | $\theta_{node\,1}$ |

The angles $\theta_{node\,0}$, $\theta_{antinode\,1}$, and $\theta_{node\,1}$ are illustrated in Figure 10.25.

For interference between two point sources, the angles for the antinodes where destructive interference occurs can be found by using the following relationship:

$$\boxed{\mathbf{m} \cdot \lambda = d \cdot \sin\theta_{antinode\,\mathbf{m}}} \qquad (10.7)$$

…where $\mathbf{m}$ is an integer (0, ±1, ±2, *etc.*), $\lambda$ is the wavelength, and $d$ is the distance between the sources of the waves.

If we choose $\mathbf{m} = 0$, this gives an angle $\theta = 0$. That direction corresponds to directly in front of the bee in Figure 10.24 or directly above the point sources in Figure 10.25, and in both of these cases we can see that the waves are strong in those directions.

Solving Equation 10.7 for $\theta$ using $\mathbf{m} = 1$ gives the angle $\theta_{antinode\,1} = 30°$, which looks like a reasonable answer if we compare to Figure 10.25.

Using $\mathbf{m} = -1$ would give a negative angle–this would simply refer to the same antinode on the other side of the vertical line in Figure 10.25.

Like other kinds of waves, water waves can interfere with each other. In Figure 10.22, the waves pass through each other, as we have seen before with longitudinal and transverse waves in solids. In Figure 10.24, the waves are created in two places very close to each other and in a regular pattern. This creates a situation where the interference between the waves created by the bee's wings makes a complex pattern.

There are large ripples coming away from the bee in some directions (for example, the direction marked with a dotted line in Figure 10.24). These are the places where the waves created by the two wings are interfering *constructively*. But the water is almost smooth in other directions (for example, the direction marked with a solid line in the same figure). These are the places where the waves created by the two wings are interfering *destructively*.

For the analysis of this pattern around the bee, we will assume that the distance between the centers of the wings is 2 cm. Looking carefully at the image, that means the wavelength of the water waves is approximately 1 cm. This pattern is reproduced in Figure 10.25 by creating concentric circles of white and black around two closely spaced positions.

The result is a series of nodes (where the wave amplitude is near zero) and antinodes (where the wave amplitude is at a maximum) radiating from the center point between the two sources of the waves.



Figure 10.24: A bee beating its wings in water, showing constructive (dotted lines) and destructive (solid lines) interference. [62]
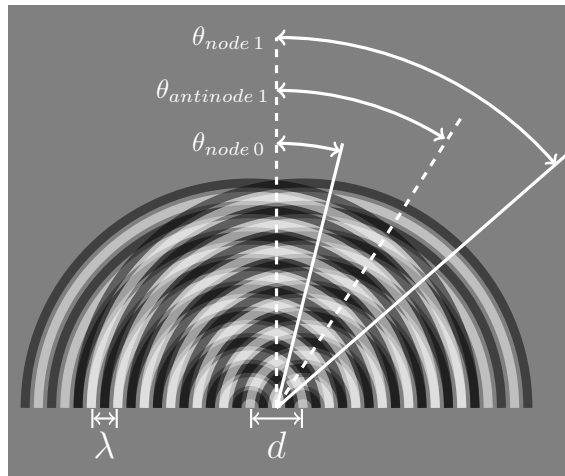


Figure 10.25: This image represents waves with wavelength $\lambda$ spreading out in circles from two starting positions separated by a distance $d$.[1]

For the nodes where destructive interference occurs, the mathematical relationship is very similar:

$$\left(\mathbf{m} + \frac{1}{2}\right) \cdot \lambda = d \cdot \sin\theta_{node\,\mathbf{m}} \qquad (10.8)$$

…where again $\mathbf{m}$ is an integer (0, ±1, ±2, *etc.*), $\lambda$ is the wavelength, and $d$ is the distance between the sources of the waves.

This time if we choose $\mathbf{m} = 0$ we find an angle for the "zeroth" node $\lambda_{node\,0} = 14.5°$, which again looks like a reasonable answer if we compare to Figure 10.25.

If we choose $\mathbf{m} = 1$, we find that $\lambda_{node\,1} = 48.6°$, which again looks like a reasonable answer.

Every configuration of point sources will have only a limited number of valid solutions for the variable $\mathbf{m}$ in the mathematical models given here. The number of valid solutions depends on the relative spacing of the point sources compared to the wavelength. In this situation, if we try to use $\mathbf{m} = 2$ to find the next node $\lambda_{node\,2}$, we find that $\sin\lambda_{node\,2} = 1.25$, but there is no angle for which sin is greater than one. But that's ok—on Figure 10.25 there isn't another node visible at an angle larger than $\lambda_{node\,1} = 48.6°$. There is no mathematical solution for another node because there is no other node.

215

## 10.7   Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

### General

- The molecules of a liquid are free to move around while still staying close together.

- Most liquids are not easily compressed, so we will consider them to be completely incompressible.

- Liquids are a type of fluid.

- Fluids can flow and change shape, usually to fit the shape of a container.

- Materials with lower density tend to rise up and those with higher density sink down.

- Ideal fluids are those for which the flow, if any, is smooth and constant and only elastic collisions occur between the molecules of the fluid or between the fluid and the walls of a container.

- Water waves can interfere with each other.

### Forces

- The pressure in a fluid depends on gravity, the density of the fluid, the depth, and the speed of the fluid.

- The pressure in a fluid pushes out in all directions.

- If an external force is applied to a fluid, the pressure caused is added equally to every part of the fluid.

- Buoyant force is caused by the pressure in a fluid around an object.

- Buoyant force always points upward.

- An increase in the speed of horizontal flow in a fluid happens along with a decrease in the pressure of the fluid.

- Attractive intermolecular forces in a fluid creates surface tension, pulling the liquid together as much as possible into the shape of a ball.

### Motion

- Volumetric flux is the volume flow per time.

- The volumetric flux in a pipe is constant throughout the length of the pipe.

- Water flows faster through narrower sections of a pipe.

- Laminar flow is when the fluid is moving along smooth paths.

- Turbulent flow is when the flow is erratic, with mixing, rotation, and chaotic changes in magnitude and direction of the velocity.

- For laminar flow, speed is highest in the center of the flow and lowest at the edges.

- Waves in a liquid are usually a combination of longitudinal and transverse waves, often referred to as "water waves."

## Momentum

- (Nothing!)

## Energy

- (Nothing!)

## Mathematical Models

| equation | | restrictions on the validity of the equation |
|---|---|---|
| $P = \rho_m \cdot g \cdot h$ | (10.1) | no flow, no external pressure, near the surface of the earth |
| $P_{tot} = \rho_m \cdot g \cdot h + P_{external}$ <br><br> "Pascal's Law" | (10.2) | no flow, near the surface of the earth |
| $F_b = \rho_{m,fluid} \cdot g \cdot V_{submerged}$ <br><br> "Archimedes' Principle" | (10.3) | near the surface of the earth |
| $\Phi_V = v \cdot A \cdot \cos\theta$ | (10.4) | -none- |
| $P + \frac{1}{2}\rho_m \cdot v^2 + \rho_m \cdot g \cdot h$ is constant <br><br> "Bernoulli's equation" | (10.5) | within a single, steady flow with no friction |
| $\Phi_V = \frac{\pi \cdot r^4 \cdot \Delta P}{8\mu_v \cdot l}$ <br><br> "Poiseuille Flow" | (10.6) | long, narrow, round pipe with laminar flow |
| $\mathbf{m} \cdot \lambda = d \cdot \sin\theta_{antinode\ \mathbf{m}}$ | (10.7) | -none- |
| $\left(\mathbf{m} + \frac{1}{2}\right) \cdot \lambda = d \cdot \sin\theta_{node\ \mathbf{m}}$ | (10.8) | -none- |

## 10.8 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

### Level 1 - Remember

10.1 **[W]** What are the main differences between a solid and a liquid at the molecular level?

10.2 **[W]** What is meant by "incompressible?"

10.3 **[W]** After a solid object is completely submerged in a liquid, how does buoyant force change as the object sinks deeper, assuming that the object and the fluid are both incompressible?

10.4 **[W]** What kind(s) of waves can travel easily through a liquid?

10.5 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

### Level 2 - Understand

10.6 **[W]** What happens to pressure as you move down deeper into a fluid?

10.7 **[W]** What happens to pressure in one part of a fluid if you increase the pressure on the fluid in another part of it?

10.8 **[W]** Is there any situation where buoyant force points downward? If so, give an example.

10.9 **[N]** A steady volumetric flow of 2 liters per second enters a garden hose with a diameter of 2.5 cm. Just before the water comes out of the other end of the hose, a nozzle reduces the diameter to 0.5 cm.

    (a) What is the magnitude of the volumetric flow out of the nozzle?

    (b) Is the pressure in the nozzle higher, lower, or equal to the pressure in the hose? Assume that the hose is horizontal.

### Level 3 - Apply

10.10 **[W & N]** Use the information in Section 10.3 to calculate the specific gravity of ice.

10.11 **[W]** What causes pressure to increase as you move down deeper into a fluid?

10.12 **[N]** What is the pressure 10 m below the surface of a 40-m-deep freshwater lake?

10.13 **[W & G]** The image below shows an oddly-shaped container filled with a liquid. Be specific about the locations in your answers.

    (a) Where in the liquid would the pressure be highest?

    (b) Where in the liquid would the pressure be lowest?
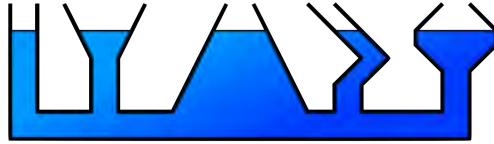
A series of "communicating vessels" that form a single, oddly-shaped container.[63]

10.14 **[N]** One of the unknowns in Section 10.3 is the submerged volume of the floating ice. Use the buoyant force that was determined using the free-body diagram to solve for the submerged volume of the ice. What fraction of the ice is submerged? Compare this fraction to the specific gravity of ice.

10.15 **[N]** One of the unknowns in Section 10.3 is the mass of the aluminum cube. Calculate the mass of the cube.

10.16 **[N]** One of the unknowns in Section 10.3 is the buoyant force on the aluminum cube. Calculate the buoyant force...

   (a) ...if the cube is still sinking, as illustrated in Figure 10.12.

   (b) ...if the cube is sitting on the bottom of the lake, as illustrated in Figure 10.13.

10.17 **[N]** In Section 10.4, none of the unknown pressures is found. Calculate them.

10.18 **[G]** A curved section of pipe is shown in the image below. Make two sketches of water flowing through this pipe: One with laminar flow and one with turbulent flow.
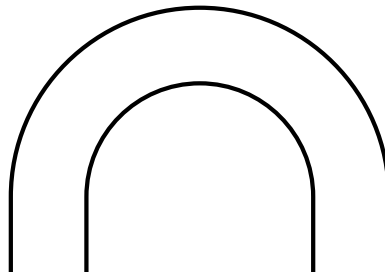


A curved section of pipe.[1]

10.19 **[N]** In Section 10.6 it was shown that there is no solution for an $m = 2$ node. Is there a solution for an $m = 2$ antinode?

## Level 4 - Analyze

10.20 **[N]** How tall would a column of water need to be to apply the same pressure to the ground as the column of sandstone in Section 10.1?

10.21 **[N]** How tall would a column of mercury need to be to apply the same pressure to the ground as the column of sandstone in Section 10.1? The density of mercury, which is a liquid at room temperature, is $13\,500\,\mathrm{kg/m^3}$.

10.22 **[W & N]** The image below shows a measuring cup with a sealable spout. The spout has been sealed such that the level of liquid (assumed to have the same density as water) in the spout is 1 cm below the level of liquid in the main body of the measuring cup. How much external pressure is the seal applying to the liquid?

A measuring cup with two different levels of liquid.[1]

10.23 **[N]** A hydraulic lift is used to raise a 2 500 kg truck off of the floor in a repair shop. The total area of the four cylinders that are actually lifting the truck is 0.5 m$^2$. The external pressure applied to the liquid is created by a cylindrical piston with a radius of 0.03 m. How much force does the piston need to use to lift the truck?

10.24 **[W, G, & N]** A 1-cm-radius solid ball of lead is dropped into a deep pool of mercury.

    (a) What is the volume of the lead ball?

    (b) Look up the density of lead and use it to calculate the mass of the lead ball.

    (c) Look up the density of mercury. Does the lead ball sink or float?

    (d) Draw a free-body diagram of the lead ball.

    (e) Find the buoyant force on the lead ball.

    (f) Find the submerged volume of the lead ball.

10.25 **[W]** We considered examples where the magnitude of the buoyant force was equal to or less than the force of gravity. Is it possible for buoyant force to be greater than the force of gravity? If so, give an example. If not, explain why not.

10.26 **[N]** Consider a horizontal pipe carrying a liquid with high viscosity. What would happen to the volumetric flow rate if everything else remained unchanged except that…

    (a) The radius of the pipe doubled?

    (b) The length of the pipe doubled?

    (c) The pressure difference doubled?

    (d) The viscosity of the liquid doubled?

10.27 ✚ **[W]** "Pitot tubes" like the one in the figure below are used on aircraft to measure the windspeed of the airplane. A Pitot tube depends on the Venturi effect. The Pitot tube has one opening pointing forward on the airplane, and another on the side of the tube. A sensor is used to measure the pressure difference at these two positions. Based on what you know about pressure and fluid flow, explain which of the two openings would be at a higher pressure.

A Pitot tube on the bottom side of an airplane wing.[64]

10.28 **[N]** If the distance between two point sources was increased, would the number of possible nodes and antinodes increase, decrease or stay the same? Explain your reasoning.

## Level 5 - Evaluate

10.29 **[G & N]** Create a line graph that shows the pressure in the column of water in Figure 10.5 as a function of height off of the ground. Note that the height off of the ground is not the same as "h" in the figure. The total height of the column of water is 8 m.

10.30 **[W, G, & N]** Question 10.23 asks about a hydraulic lift that is used to lift a truck. The force needed to lift the truck is very large, but the force that is supplied by the piston to do this is very small in comparison. Work is done on the truck to lift it off of the ground. That work can only be supplied by small force created by the piston. Does this hydraulic lift violate conservation of energy? If so, could such a machine actually exist? If not, explain how such a large force can be used to do work when the only input is a relatively small force.

10.31 **[W, G, & N]** In Figure 10.10, pieces of garlic have sunk through the oil and are floating on the surface of the vinegar. Look up densities of vinegar and olive oil. Assume that the garlic is half submerged in olive oil and half in vinegar in its equilibrium position. If that assumption is correct, what is the density of the garlic?

10.32 **[W]** Oil floats on top of water because water is more dense than oil. Use an argument based on potential energy and stable equilibrium to explain why the less dense material floats on top of the more dense material.

10.33 **[W, G, & N]** In Section 10.4, a pipe with varying thickness is analyzed, with the conclusions that the flow rate would be equal in each section, the pressure in the first and last sections would be equal, and the pressure would be lowest in the center section. Are each of these conclusions still valid if the fluid flow is laminar but the fluid has a high viscosity?

   (a) Is the volumetric flow still equal for all three sections of pipe? If not, where is volumetric flow highest or lowest?

   (b) Is the pressure in the third section of pipe still the same as in the first section? If not, is it lower or higher?

   (c) Is the pressure in the center section still lower than for the other sections of pipe?

10.34 **[N]** If the wavelength of a water wave was 2 m, what minimum distance would two point sources need to be away from each other in order to create nodes where $m = 5$?

### Level 6 - Create

10.35 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Add the main ideas from this chapter to a similar concept map that is specifically for liquids.

10.36 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

10.37 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

# Chapter 11

# Gases

We have already learned about liquids, which are one type of fluid. Now we will focus on a second type of fluid: gases. In the chapter on liquids it may have seemed that the words "fluid" and "liquid" were used interchangeably. In fact, they were not. Each time the word "fluid" was used in that chapter, the ideas that were being discussed were equally applicable to gases.

The molecules in a solid are held together with strong bonds; the bonds between molecules in a liquid are weaker than those for a solid; the bonds between the molecules of a gas are extremely weak. We will only consider ideal gases, for which there are no attractive forces at all between the molecules. As a result, the particles of an ideal gas tend to bounce around freely, and unlike liquids they are compressible.

We will look at the motion of gas particles on a molecular level and how the number of molecules and the volume of the gas are related to pressure. We will also consider work done by a gas and sound waves traveling through a gas, including the Doppler effect.



Figure 11.1: The air pressure in the tires has to be enough to hold up the heavy load on the bike.[65]

## 11.1 An Ideal Monatomic Gas

### Words

Our model for a gas will be molecules that bounce around freely through space, interacting with each other only through collisions. With liquids we began by studying ideal liquids that had no viscosity, and then later added viscosity. For gases we will only consider ideal gases.

An ideal gas is one in which:

- molecules of the gas interact with each other only through perfectly elastic collisions

- size of the molecules of an ideal gas is negligible compared to both the size of the container and the average distance between gas molecules

- all of the gas molecules are identical

We will focus almost completely on "monatomic" gases–that is, gases where each particle is composed of a single atom like helium or neon. Our atmosphere is mostly nitrogen, which is a diatomic gas since each nitrogen gas molecule is composed of two nitrogen atoms, so later we will also consider diatomic gases.

Let's begin with a single helium atom bouncing around in an otherwise empty box that is 0.1 meter long on each side. We will use the average speed of a helium atom in our atmosphere, approximately 1 350 m/s. How much average pressure does the atom apply to the box?

### Graphics
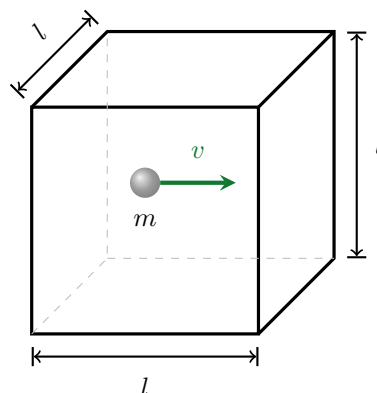


Figure 11.2: A single atom of a monatomic ideal gas with mass $m$ bouncing back and forth with speed $v$ in a box whose sides all have the same length $l$.[1]

In the figure above, the size of the atom has been vastly exaggerated. An actual helium atom has a radius of approximately $1 \times 10^{-14}$ m, so this helium atom is shown approximately 10,000,000,000,000 times larger than it should be! Remember, this is one of the requirements for an ideal gas, that the size of the molecules themselves is very, very small compared to other distances in the physical scenario.

### Numbers

**Assumptions:** ideal gas; ignore gravity

| Knowns | Unknowns |
|---|---|
| $l = 0.1$ m | $P_{avg}$ |
| $v = 1\,350$ m/s | |
| $m = 6.7 \times 10^{-27}$ kg | |

We are looking for a pressure, which is a force per unit area. The atom is striking the walls of the box, and we can find the area of the surface it is hitting. Since we are looking for an average pressure, not just the pressure at one location, we should use the entire surface area of the box. It has six square walls, so…

$$A = 6 \cdot (l \cdot l) = 0.06 \text{ m}^2$$

Now we need to find the average force. Force is a change in momentum over time. Each time the atom hits a wall it reverses direction, so there is a change in the sign of its momentum. So each collision creates a change in momentum with a magnitude of…

$$\Delta p = m \cdot (v - (-v)) = 2m \cdot v = 1.8 \times 10^{-23} \text{ kg} \cdot \text{m/s}$$

…pointing toward the center of the box. We can use the path length $s$ and the speed $v$ to find the time $\Delta t$ between collisions with the walls:

$$\Delta t = \frac{s}{v} = \frac{0.1 \text{ m}}{1\,350 \text{ m/s}} = 7.4 \times 10^{-5} \text{ s}$$

If the atom only interacts with the walls of the container with elastic collisions and it starts out moving as shown in Figure 11.2, it will continually bounce back and forth, hitting the right and left walls over and over again at regular intervals. That means the atom will apply an average outward force on each of these walls. If we ignore the effects of gravity it will never hit the bottom, and it will also never hit any of the other walls. So perhaps our model is a little bit too simple. Let's add two more atoms, one bouncing up and down and the other bouncing forward and back while the original atom continues to bounce left and right.

In fact, this is also an unrealistically small number of gas particles in a volume of this size. In a real gas, the atoms are colliding with each other, not just with the walls of the container, so they are scattering around randomly in all different directions. These constant collisions at various angles also mean that the speeds of individual gas molecules change often, although the average speed remains constant. Because of these constant changes, when we deal with gases we are really always talking about average values, not values that are correct for any given gas particle.

The pressure in the box increases proportionally with the number of gas molecules. To give a sense for the huge numbers of atoms involved when talking about gases, if we added a million helium atoms to this box every second, and we had started adding them at the time when the universe was created, the pressure in the box would only now be reaching the same pressure as earth's atmosphere!
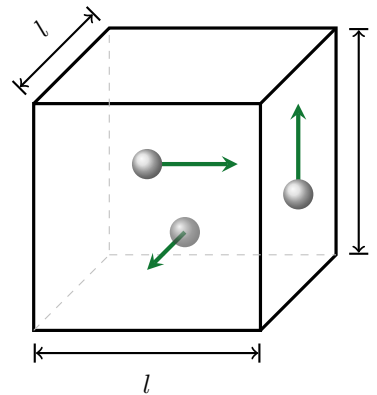


Figure 11.3: Three atoms of a monatomic ideal gas with mass $m$ bouncing around with speed $v$ in a box whose sides all have the same length $l$.[1]

This relationship between path length, time, and speed has not appeared before as a mathematical model in this text, although you worked it out for yourself when learning about units way back in Section 2.1. It is valid whenever you want to find the average speed over a given path:

$$v_{avg} = \frac{s}{\Delta t}$$

Now that we have the time between collisions with the walls and the change in momentum of the atom in each of those collisions, we can find the average force:

$$F_{avg} = \frac{\Delta p}{\Delta t} = 2.4 \times 10^{-19}\,\text{N}$$

Since the force of the walls on the atom is equal in magnitude and opposite in direction to the force of the atom on the walls, we can find the average pressure created by the atom on the walls:

$$P_{avg} = \frac{F_{avg}}{A} = \frac{2.4 \times 10^{-19}\,\text{N}}{0.06\,\text{m}^2} = 4.0 \times 10^{-18}\,\text{Pa}$$

If the number of atoms were increased to three, collisions with the walls would happen three times as often, so the pressure would triple. This shows that when the volume and the speed of the molecules remains the same the pressure is proportional to the number of molecules of gas. Atmospheric pressure on earth is $1.01 \times 10^5\,\text{Pa}$, so the box in this scenario would need to contain approximately $4 \times 10^{23}$ atoms of helium to be at atmospheric pressure. That's 400,000,000,000,000,000,000,000 atoms!

**Words**

What happens to the pressure when we reduce the volume of an ideal monatomic gas? We can use the same scenario as at the end of Section 11.1 with three helium atoms, but this time we will reduce the height of the box from 0.1 m to 0.05 m.

For the two atoms that are moving horizontally, either left and right or forward and back, nothing has really changed. But the atom that is moving vertically now runs into the walls of the container twice as often, since it is moving at the same speed as before but only can only go half of the distance.

Since the atom that is moving vertically hits the walls twice as often as before, the average force and therefore the average pressure that this atom exerts on the walls of the container has doubled since the volume was reduced by half.

The atoms that are moving horizontally still travel the same distance, applying the same amount of average force to the side walls as they did before. But they also apply twice as much pressure as before, because the areas of the walls they are hitting has decreased, and pressure is force per area.

Our simple model accurately predicts what happens when volume is reduced while keeping the speed of the molecules constant: a smaller surface area and a higher rate of collisions combine to increase the pressure in a way that is inversely proportional to the change in volume. So the pressure doubles if the volume decreases by half.

**Graphics**
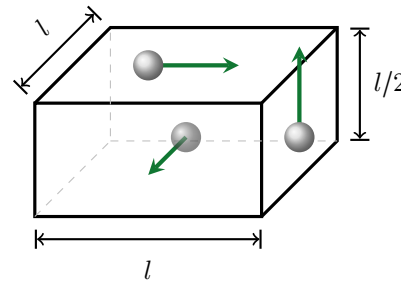


Figure 11.4: Three atoms of a monatomic ideal gas with mass $m$ bouncing around with speed $v$ in a box with one side half the length of the other two sides.[1]



Figure 11.5: A scuba diver blowing bubbles.[66]

**Numbers**

**Assumptions:** ideal gas; average speed of molecules is constant

| Knowns | Unknowns |
|---|---|
| $V_i = 6\,\text{liters}$ | $V_f$ |
| $P_{atm} = 1.01 \times 10^5\,\text{Pa}$ | |
| $\rho_{m,seawater} = 1\,026\,\text{kg/m}^3$ | |
| $h_i = 40\,\text{m}$ | |
| $h_f = 0\,\text{m}$ | |
| $g = 9.8\,\text{m/s}^2$ | |

Technically, liters are not SI units. But they are commonly used throughout the world (just about everyone, even in the USA, knows how big a 2-liter bottle is). Since we are given a known volume in liters and we are looking for a volume, in this situation we will find the answer in this same non-SI unit. The SI unit for volume is $m^3$, and the density is given with this unit, so we will need to be careful to make sure our units are canceled correctly as we work through the question.

The density of seawater was found with an online search; the same can be done for atmospheric pressure, or the value can be found in Appendix C.

Pressure can affect volume in the same way that volume affects pressure. Let's consider the volume of the air bubbles exhaled by a scuba diver. Air is not an ideal gas, since it is composed of different types of molecules, most of which are not monatomic. But ideal gas laws still work remarkably well for any gas in most situations. The recommended maximum depth for conventional scuba diving is 40 meters, and the average lung capacity of an adult human is approximately six liters. If a diver exhales six liters of air in the ocean at a depth of 40 meters, what will the volume of the exhaled air be as it reaches the surface?

To a good approximation, pressure under water increases by one atmosphere every ten meters. So at a depth of 40 meters the pressure would be about five times the pressure at the surface. And since volume is inversely proportional to pressure that means the volume of the exhaled air would increase by roughly a factor of five from the time it leaves the diver's lungs until it reaches the surface. That means the volume of the air at the surface would be nearly half the volume of the diver's entire body! This explains why divers are taught not to hold their breath while surfacing, but to continually breathe out.
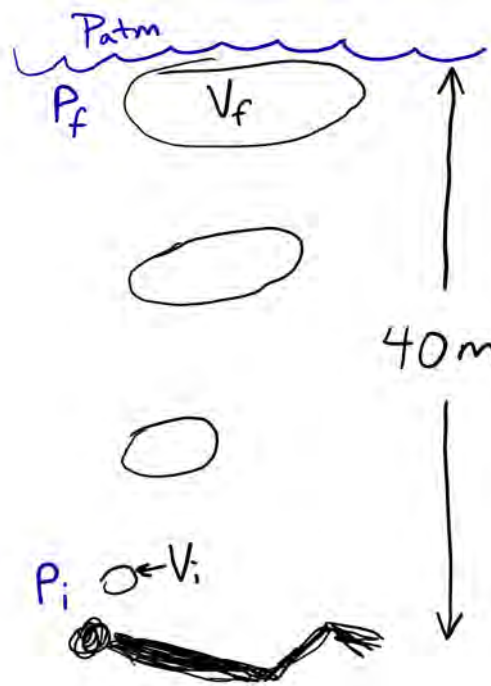


Figure 11.6: Sketch of the scuba diver blowing bubbles. [1]

There is a lot of information that we don't know, for example the types of gas molecules and the number of each. But remembering that pressure and volume are inversely related, as long as the number and type of molecules and the average speed of the molecules all remain constant, we can use this simple relationship between pressure and volume:

$$\boxed{P_{tot,1} \cdot V_1 = P_{tot,2} \cdot V_2} \qquad (11.1)$$

This relationship is referred to as "Boyle's Law."

At the surface of the water, the depth is zero so the final pressure is just that of the atmosphere, $1.01 \times 10^5$ Pa. At the depth of the diver, the seawater is pressing down on the exhaled air, so the initial pressure is given by Pascal's Law:

$$
\begin{aligned}
P_i &= \rho_{m,seawater} \cdot g \cdot h_i + P_{atm} \\
&= \left(1\,026 \, \text{kg/m}^3\right) \cdot \left(9.8 \, \text{m/s}^2\right) \cdot \left(40 \, \text{m}\right) + P_{atm} \\
&= 5.03 \times 10^5 \, \text{Pa}
\end{aligned}
$$

Now we can rearrange Equation 11.1 to find the final volume:

$$
\begin{aligned}
V_f &= \frac{P_i \cdot V_i}{P_f} \\
&= \frac{\left(5.03 \times 10^5 \, \text{Pa}\right) \cdot \left(6 \, \text{liters}\right)}{1.01 \times 10^5 \, \text{Pa}} \\
&= 30 \, \text{liters}
\end{aligned}
$$

## 11.3    Gauge Pressure

### Words

If you have used a bicycle or a car, you have likely put air in the tires. Before air is added to a new tire, it is already full of air at the same pressure as the surrounding atmosphere. But even though the tire actually contains $1.01 \times 10^5 \, \text{Pa}$ of pressure, if you put a tire pressure gauge on it the gauge will read "0." That is because the gauge does not measure the total (or "absolute") pressure in the tire–it measures the "gauge pressure," the pressure difference between the inside and the outside.

Whether it is better to think in terms of gauge pressure or absolute pressure depends on the scenario. If our concern is that a balloon may pop if we try to put too much air into it, then the important factor is the difference between the pressure inside the balloon and the pressure outside of the balloon.

When thinking about the volume of the scuba diver's bubbles, it was the absolute pressure that was important, because the volume of the bubble depended on the spacing of the air molecules themselves, not on how that spacing would compare to the spacing of the air molecules outside of the bubbles.

Absolute pressure is always positive, because every time a molecule inside a container bounces off of one of the walls it creates an outward force on the wall. But gauge pressure is the difference between two pressures, so it could be positive, negative, or zero.

### Graphics
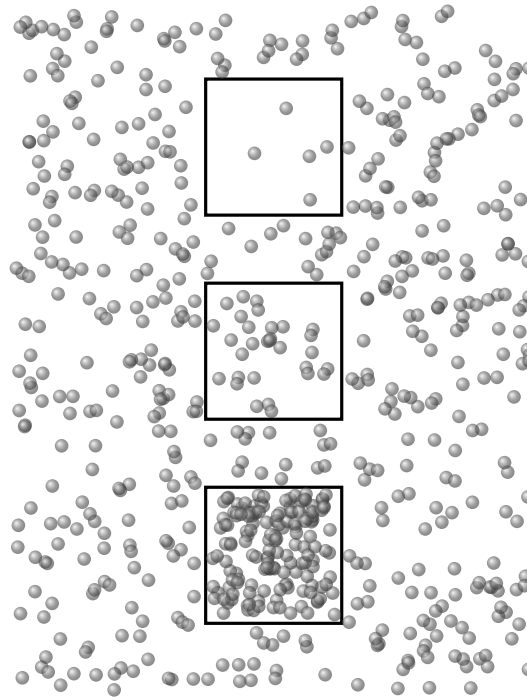


Figure 11.7: Three boxes containing a gas, surrounded by the same gas. The density of gas molecules in the center box is the same as the density of gas molecules outside of the boxes. [1]

The total (or absolute) pressure is positive inside each box. But the gauge pressure is negative in the top box, zero in the center box, and positive in the bottom box.

### Numbers

**Assumptions:** rectangular area; same pressure in each tire; same area for each tire; at sea level

| Knowns | Unknowns |
|---|---|
| $m_{system}$ = 198 kg | $P$ |
| $A_{front} = A_{back} = 0.002 \, \text{m}^2$ | |
| $g = 9.8 \, \text{m/s}^2$ | |
| $P_{atm} = 1.01 \times 10^5 \, \text{Pa}$ | |

If we remember that pressure is force per area then mathematically it is relatively easy to find the pressure for one of the tires:

$$P = \frac{F}{A}$$
$$= \frac{\frac{1}{2}F_g}{A_{back}}$$
$$= \frac{\frac{1}{2}m_{system} \cdot g}{A_{back}}$$
$$= \frac{\frac{1}{2}(198 \, \text{kg}) \cdot (9.8 \, \text{m/s}^2)}{0.002 \, \text{m}^2}$$
$$= 4.85 \times 10^5 \, \text{Pa}$$

We can check to see if this is a reasonable answer by doing a quick internet search of bike tire pressure. The maximum pressure for a medium bike tire is 70 psi, or 70 pounds per square inch. That converts to $4.8 \times 10^5 \, \text{Pa}$, so these tires are right on the edge of being overinflated.

Let's consider a bicycle in the picture at the beginning of this chapter, assuming that the bicycle is moving at constant speed, carrying a person and three bags of charcoal. The bike's mass is 8 kg, the person's mass is 70 kg, and the mass of each bag of charcoal is 40 kg. We will take the area of the ground touched by each tire to be approximately a 2 cm by 10 cm rectangle. We can use this information to find the air pressure (assumed equal) in the tires. We also need to remember to state whether the pressure we have found is the gauge pressure or the absolute pressure.

If we think of the bike, rider, and charcoal as a single system, all of its weight is pressing down on the road, and the road is pushing back up on the system with the same amount of force. If we now think of just that part of the tires that is touching the road, the road pushes up on the tires and the air inside the tires pushes down. So the pressure on this part of the tires has to be equal to the weight of the bike. The higher the pressure is in the tires, the smaller the area that would have to be touching the ground.

Should we be thinking about gauge pressure or absolute pressure? That depends on whether atmospheric pressure is pushing upward on the bottom of the tire while it is touching the road. Unless something has been done specifically to remove the air from under the tire, then there is still atmospheric pressure there, so it is the gauge pressure that we should be considering. To convince yourself of this, think about picking up a suction cup that is just sitting on a smooth surface, compared to picking up the same suction cup after air has been pressed out from underneath.
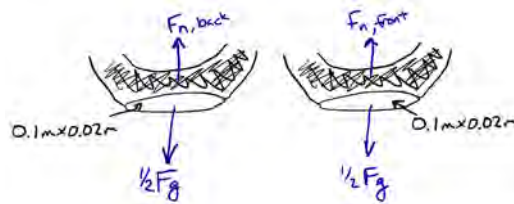


Figure 11.8: Sketch of the bottoms of the two bike tires, including the forces on each of them. [1]

If we assume that each tire is supporting half of the weight then we can just think about the pressure and forces for one tire and then apply the same answer to both tires. In the text it said we are assuming that the area touching the ground is a rectangle, but it is shown in the sketch as more of an ellipse, which is probably more accurate.

In our calculation we did not take into consideration the outside pressure of the atmosphere pushing up on the bottom of the tire, so our number is the gauge pressure of the tire. To find the total pressure in the tire we would have to add the external pressure to the gauge pressure:

$$P_{tot} = P_{external} + P_{gauge} \qquad (11.2)$$

In this situation the external pressure is atmospheric pressure, so…

$$\begin{aligned} P_{tot} &= P_{atm} + P_{gauge} \\ &= 1.01 \times 10^5 \text{ Pa} + 4.85 \times 10^5 \text{ Pa} \\ &= 5.86 \times 10^5 \text{ Pa} \end{aligned}$$

## 11.4   Temperature of Ideal Gases

### Words

Gases are made up of huge numbers of particles that are bouncing around randomly in every direction. Their positions and velocities are constantly changing as they collide with each other and with the walls of the container. And yet one of the assumptions we have been using a lot is that the average speed of the gas particles doesn't change. But how can we possibly know that is true, or have any control over it? The answer is simple: temperature.

The average velocity of the gas particles increases as the temperature increases. Since the velocity increases, the particles hit the sides of the container with higher force, increasing the pressure. If the walls of the container are able to move then the walls could be pushed out, increasing the volume. Or if there is a way for gas particles to enter and exit the container then particles will tend to leave the container, decreasing the number of particles in the container. So with a gas there are multiple factors that all affect the way the gas behaves.

Let's consider what happens when the temperature of the gas inside a 2 200-cubic-meter hot air balloon is heated to $80°C$ and the outside air is $15°C$. The density of air at sea level when the temperature is $15°C$ is approximately $1.23 \text{ kg/m}^3$. The total mass of the balloon itself, including a basket holding two passengers but not including the air inside the balloon, is 400 kg.

### Graphics



Figure 11.9: Hot air balloons are filled with air that is at a higher temperature than the surrounding air. [67]

The higher speed of the air molecules inside the balloon create a higher pressure. Since the bottom of the balloon is open, the higher pressure inside tends to push air molecules out of the bottom of the balloon. This reduces the density of the air inside the balloon, creating an upward buoyant force on the balloon.

### Numbers

**Assumptions:** ideal gas; volume of the basket is negligible; no air resistance; atmospheric pressure

| Knowns | Unknowns |
|---|---|
| $V = 2,200 \text{ m}^3$ | $F_{net}$ |
| $T_{outside} = 15°C$ | |
| $T_{inside} = 80°C$ | |
| $\rho_{outside} = 1.23 \text{ kg/m}^3$ | |
| $m_{balloon} = 400 \text{ kg}$ | |

The *ideal gas law* describes the relationship between the total pressure $P_{tot}$, volume $V$, number of molecules $N$, and temperature $T$ of an ideal gas:

$$\boxed{P_{tot} \cdot V = N \cdot k_B \cdot T} \qquad (11.3)$$

$k_B$ is the Boltzmann constant, $1.38 \times 10^{-23}$ J/K. Celsius [°C] and kelvin [K] are both SI units for temperature, but only kelvin can be used with the ideal gas law. We can convert temperatures as follows:

$$\boxed{T_K - 273\text{K} = T_C = \frac{5}{9}\left(T_F - 32°\text{F}\right)} \qquad (11.4)$$

...where $T_K$, $T_C$, and $T_F$ are temperature measured in kelvin, Celsius, and Fahrenheit, respectively. Note that the degree symbol is used only with Celsius and Fahrenheit, not with kelvin.

Since the hot air balloon is open at the bottom, gas particles can flow in and out. That keeps the pressure inside the balloon the same as the pressure outside. The balloon also stays the same volume as the air inside gets warmer or cooler. So when we increase the temperature of the air inside some of the air will flow out of the balloon, decreasing the number of air molecules inside. That means the air inside the balloon is less dense than the air outside of the balloon.

Remember that gases are fluids, so the buoyant force that we learned about when studying liquids is also relevant here. The less-dense balloon is completely submerged in the more-dense air outside of the balloon, so that creates a buoyant force that pushes the balloon upward.
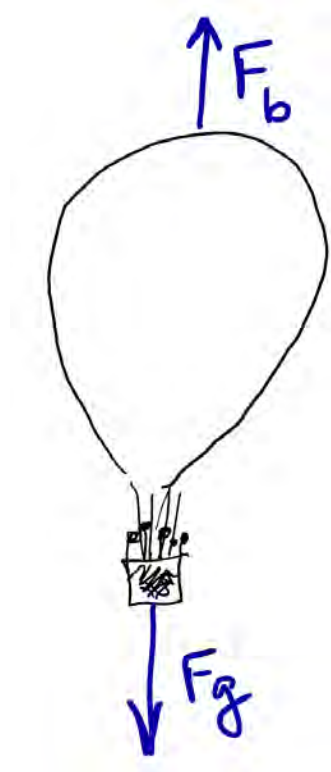


Figure 11.10: Forces on the hot air balloon, assuming no air resistance. [1]

First, we should convert temperatures to kelvin: $T_{outside}$ = 288K; $T_{inside}$ = 353K

There are two forces we need to find: gravitational and buoyant. Buoyant force is present in any fluid, not just liquids, and we can think of the balloon as being completely submerged in the air. Using Archimedes' Principle…

$$F_b = \rho_{m,fluid} \cdot g \cdot V_{submerged}$$
$$= \left(1.23 \,\text{kg/m}^3\right) \cdot \left(9.8 \,\text{m/s}^2\right) \cdot 2,200 \,\text{m}^3$$
$$= 26\,500 \,\text{N}$$

To find the gravitational force, we need to know the mass of the balloon. We are given the mass of everything except the air, so we just need to find the mass of the hot air inside the balloon. Looking at Equation 11.3, we can see that if the pressure and the volume are constant, as they are in this case when the container is open at one end, then the number of molecules of the ideal gas is inversely proportional to the temperature. And since all of the molecules of an ideal gas are identical, that means the mass is also inversely proportional to the temperature. So the gravitational force on the $80°$C air inside the balloon is…

$$F_{g,air} = m_{80°\text{C}} \cdot g = m_{15°\text{C}} \cdot \frac{T_{inside}}{T_{outside}} \cdot g$$

…where $m_{15°\text{C}}$ is the mass of the air inside the balloon if it were at $15°$C. We can find that mass from the volume of the balloon and the density of the outside air.

$$F_{g,air} = \rho_{outside} \cdot V \cdot \frac{T_{inside}}{T_{outside}} \cdot g = 21\,600 \,\text{N}$$

## 11.5 Sound Waves

### Words

We found that transverse and longitudinal waves can travel through solids; and a combined transverse/longitudinal wave, commonly called a water wave, can travel through liquids. Transverse waves require an attractive force between neighboring particles, so transverse waves cannot travel through ideal gases at all. Longitudinal waves, however, can travel through gases; they are commonly called sound waves.

Compression and rarefaction is created in the gas molecules that bounce off of an oscillating surface. These areas of compression and rarefaction move away from the surface as pressure waves traveling at the speed of sound through the gas. The speed of sound in dry air at atmospheric pressure is approximately 330 m/s.

As with other types of waves, sound waves can constructively and destructively interfere with each other. And as in the case of longitudinal waves on a string, constructive and destructive interference can create standing waves with nodes and antinodes. Nodes on a string are positions where total destructive interference occurs in the transverse motion of the string, resulting in no movement of the string. Nodes in a sound wave are different. They are positions where total destructive interference of the pressure waves occur, resulting in no change in pressure. Similarly, antinodes of sound waves are positions where the largest pressure changes are happening, not positions where

### Graphics
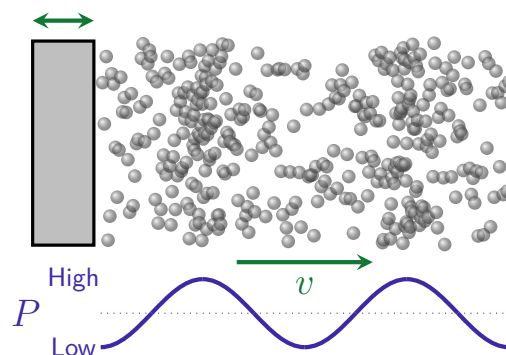


Figure 11.11: Longitudinal waves created in a gas by an object that is oscillating right and left. The waves are regions of higher and lower pressure that move away from the object. [1]

Sound waves are often shown graphically simply as lines that represent areas of high pressure.
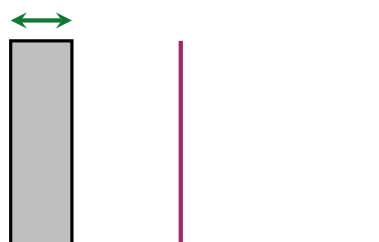


Figure 11.12: A simpler way to sketch the same scenario as in the figure above.[1]

### Numbers

**Assumptions:** dry air at atmospheric pressure

| Knowns | Unknowns |
|---|---|
| $v = 330 \text{ m/s}$ | $l$ |
| $f_1 = 262 \text{ Hz}$ | $f_2$ |

We can use the relationship between wavelength, speed, and frequency to find the wavelength of middle C in air:

$$\lambda_1 = \frac{v}{f_1}$$
$$= \frac{330 \text{ m/s}}{262 \text{ Hz}}$$
$$= 1.26 \text{ m}$$

Looking at Figure 11.14 we can see that this wavelength is four times the length of the tube:

$$l = \frac{\lambda_1}{4} = \frac{1.26 \text{ m}}{4} = 0.315 \text{ m}$$

Figure 11.14 also helps us to find the wavelength of the second harmonic for this tube:

$$\frac{3\lambda_2}{4} = l$$
$$\lambda_2 = \frac{4l}{3} = 0.42 \text{ m}$$

the most movement is happening.

Many musical instruments are designed to control the wavelengths of the sound created by changing the length of a hollow tube or by opening and closing holes in the tube. If a sound wave is created inside a tube that is closed at one end, the open end, where the pressure remains the same as the pressure outside the tube, is a node; the closed end is an antinode. Interestingly, the tube can be curved around into different shapes because pressure waves travel easily around the curves.

Higher notes can be made by decreasing the length of the tube (and thus decreasing the length of the wavelength of the first harmonic) as is done with a trombone. Different notes can also be created by using tubes of different lengths, as with a pipe organ or pan flute.

Some instruments, like a clarinet, have holes along the length of the tube that can be opened or closed. Any open hole creates a node at that position.

The first and second harmonics of one tube of a pan flute are shown in Figure 11.14. We can use what we learned in Chapter 9 to find the length that a tube of a pan flute should be to produce a "middle C," a note that is near the middle of a piano keyboard, which has a frequency of 262 Hz. We should also be able to find out what the second harmonic would be for that tube.



Figure 11.13: Different lengths of tubes of a pan flute produce different notes. [68]
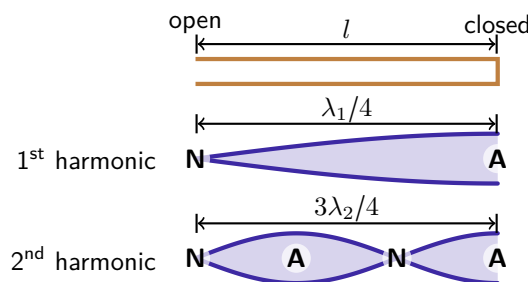


Figure 11.14: The first two harmonics in a tube that is closed at one end. "**N**" indicates a node and "**A**" indicates an antinode.[1]

From this we can find the frequency of the second harmonic in air:

$$f_2 = \frac{v}{\lambda_2}$$

$$= \frac{330\,\text{m/s}}{0.42\,\text{m}} = 786\,\text{Hz}$$

233

## 11.6   Sound Level

### Words

When sound waves aren't confined to small volumes of gas, they tend to spread out in spherical shells, in much the same way that ripples move outward in circles when still water is disturbed. The energy in each pressure wave spreads out over more and more area as the sphere moves outward, so the intensity of the sound drops with distance. That is why something sounds louder when it is closer. The intensity of sound waves can also drop, or "attenuate," as sound travels through a material, primarily because of intermolecular forces acting against the vibration in the material.

The waves in Figure 11.15 are constantly moving outward; wherever a person is standing relative to the drum set, the waves travel past them at the same speed and the same frequency.

We perceive sounds in terms of pitch and loudness. We have seen already that pitch is related to the frequency of the sound wave. Loudness is related to the amplitude of the sound wave. From a physical perspective, what we call "volume" or "loudness" is measured in terms of a variation of pressure compared to the average pressure in the air. This is called the "sound pressure level" $SPL$, and it is usually measured in decibels (dB). Zero dB is typically near the minimum threshold of human hearing, but that is a rough measurement, because the threshold of human hearing varies across different people and different pitches, and changes as we age.

### Graphics
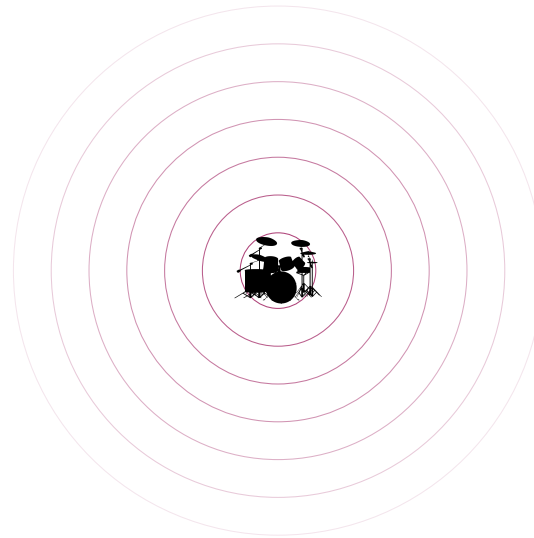


Figure 11.15: Sound waves move outward from the source in spherical shells, getting weaker as their energy spreads out more and more.[69]

### Numbers

Sound pressure level $SPL$ is related to the pressure differences created by sound waves:

$$SPL = 20 \, \log_{10}\left(\frac{P_{\mathrm{s}}}{P_0}\right) \mathrm{dB} \qquad (11.5)$$

...where the sound pressure $P_{\mathrm{s}}$ is a measure of the variation in pressure caused by sound in a medium, and $P_0$ is a reference sound pressure.

Compared to the sound pressure $P_{\mathrm{s},1}$ at a distance $r_1$ from the source, the sound pressure at a distance $r_2$ from the source is given by:

$$P_{\mathrm{s},2} = \frac{r_1}{r_2} \cdot P_{\mathrm{s},1} \qquad (11.6)$$

...if the sound is propagating in open air.

234

A logarithmic scale is used for sound because many human senses, including hearing, do not respond to stimuli in a linear fashion. For example, while it may be possible for a person to hear a whisper in a quiet room, it would not be possible for the same person to hear the same whisper while standing next to a roaring jet engine. Our ability to hear is dependent not just on the sound pressure created by something, but on the size of that sound pressure compared to the sound pressures created by other simultaneous sounds.

For sounds traveling in open air, the sound pressure decreases with the distance from the source. A good "rule of thumb" is that the sound pressure level decreases by 6 dB with every doubling of distance from the source of the sound.

The $SPL$ of a whisper is approximately 25 dB; the $SPL$ of a typical car is approximately 60 dB when measured from a distance of 1 m. Let's find the distance we would need to be away from a car for it to be at the same loudness as a whisper.

There is a difference of 35 dB between the whisper and the car. That is almost 6 times 6 dB, so the distance should have to double about 6 times. That means the distance from the car would have to go from 1 m to about $2^6$ m, or 64 meters. The calculation in the numbers column shows that the actual value is 56 m, so this is a good approximation.

**Assumptions:** open air

| Knowns | Unknowns |
|---|---|
| $SPL_1 = 60\,\text{dB}$ | $r_2$ |
| $r_1 = 1\,\text{m}$ | |
| $SPL_2 = 25\,\text{dB}$ | |

$$\Delta SPL = SPL_2 - SPL_1$$
$$= 25\,\text{dB} - 60\,\text{dB} = -35\,\text{dB}$$

$$\Delta SPL = \left(20\,\log_{10}\left(\frac{P_{s,2}}{P_0}\right) - 20\,\log_{10}\left(\frac{P_{s,1}}{P_0}\right)\right)\text{dB}$$

$$-35 = 20\left(\log_{10}\left(\frac{P_{s,2}}{P_0}\right) - \log_{10}\left(\frac{P_{s,1}}{P_0}\right)\right)$$

$$\frac{-35}{20} = \log_{10}\left(\frac{P_{s,2}}{P_0}\cdot\frac{P_0}{P_{s,1}}\right) = \log_{10}\left(\frac{P_{s,2}}{P_{s,1}}\right)$$

$$-1.75 = \log_{10}\left(\frac{r_1}{r_2}\right)$$

$$10^{-1.75} = \frac{r_1}{r_2}$$

$$r_2 = r_1 \cdot 10^{1.75} = (1\,\text{m})\cdot 56 = 56\,\text{m}$$

So it should still be possible to hear a whisper when a car is 56 m away.

## 11.7    Doppler Effect and Echos

### Words

In the discussion about Figure 11.15, it was stated that wherever a person is standing relative to the drum set, the waves will reach their ears at the same frequency. And that is true if the person is not moving relative to the drumset. But what if the source of the sound were moving at high speed relative to the observer? This could mean that the source is moving, the observer is moving, or both are moving.

Let's consider what a person hears when a train passes by at 25 m/s with its whistle making a single, 400 Hz sound wave.

If you are standing by a train track the pitch of a train whistle drops as the engine goes past. The reason for this is illustrated in Figure 11.16. The large outer circle is from when the train was in the center of the image, and is centered on the position of the train at that time. The next circle as we move inward came from the time when the train was slightly to the left of center. And so forth until the smallest, darkest circle, which came from the train when it was farthest to the left.

Notice what happens to the spacing between the high-pressure parts of the waves. They are closer together in front of the train and farther apart behind the train. The speed of sound is the same everywhere, so that means the frequency (and therefore the pitch that we can hear) in front of the train is higher than it is behind the train. This is called the "Doppler Effect."

### Graphics
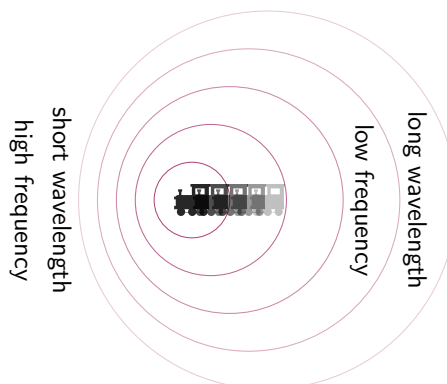


Figure 11.16: Sound waves move outward from the source in spherical shells, but if the source is moving then the center of the shells moves along with the object producing the sound. This creates shorter wavelengths in front of the moving objects and longer wavelengths behind the moving object.[70]

### Numbers

**Assumptions:** dry air at atmospheric pressure

| Knowns | Unknowns |
|---|---|
| $v_{\text{sound}} = 330 \, \text{m/s}$ | $f_{\text{initial}}$ |
| $v_{\text{train}} = 25 \, \text{m/s}$ | $f_{\text{final}}$ |
| $v_{\text{person}} = 0$ | |
| $f_{\text{emitted}} = 400 \, \text{Hz}$ | |

The relationship between the observed frequency of sound and the actual emitted frequency of sound is given by:

$$f_{\text{observed}} = \left( \frac{v_{\text{sound}} \pm v_{\text{observer}}}{v_{\text{sound}} \mp v_{\text{source}}} \right) \cdot f_{\text{emitted}} \quad (11.7)$$

...if neither the source nor the observer is moving faster than the speed of sound through the medium. A plus sign is used in the numerator if the observer is moving toward the source and a minus sign is used if the observer is moving away from the source. A minus sign is used in the denominator if the source is moving toward the observer and a plus sign is used if the source is moving away from the observer.

Something even more interesting happens when the object is moving faster than the speed of sound, as illustrated in Figure 11.17. The sound waves are completely behind the airplane shown in that figure, so if you are in front of the airplane you will not be able to hear it coming at all. Even if you are directly above or below the airplane you will not be able to hear it until after it has passed you. And when the sound finally does come, the sound waves at the leading edge all interfere with each other constructively, producing a powerful pressure wave sometimes called a shock wave, which creates a "sonic boom."

An object that is not otherwise producing sound can reflect sound waves; this is what causes an echo. Ultrasound, which is simply sound waves that are too high in pitch for humans to hear, uses echos to map different types of tissue within a living body. If the object is moving relative to the source of the sound, the Doppler Effect shifts the frequency of the echo. A "Doppler ultrasound" test measures not only the amplitude of the echo but also its frequency, so it can create an image that shows relative velocity in addition to just position. This makes it possible, for example, to observe internal blood flow in a non-invasive way.
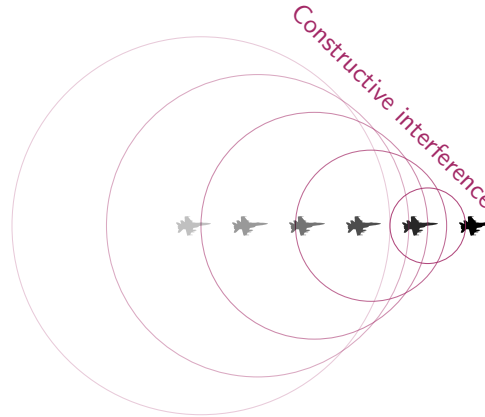


Figure 11.17: An object that is moving faster than sound produces a shock wave of pressure that creates a "sonic boom."[71]



Figure 11.18: The black area near the center of this ultrasound image shows that no sound was reflected by the amniotic fluid. The black area near the bottom left is a "shadow" created by the head of the unborn child.[72]

In this case,

$$f_{\text{initial}} = \left( \frac{v_{\text{sound}} + v_{\text{observer}}}{v_{\text{sound}} - v_{\text{source}}} \right) \cdot f_{\text{emitted}}$$

$$= \left( \frac{v_{\text{sound}} + v_{\text{person}}}{v_{\text{sound}} - v_{\text{train}}} \right) \cdot f_{\text{emitted}}$$

$$= \left( \frac{330 \,\text{m/s} + 0}{330 \,\text{m/s} - 25 \,\text{m/s}} \right) \cdot 400 \,\text{Hz}$$

$$= 433 \,\text{Hz}$$

$$f_{\text{final}} = \left( \frac{v_{\text{sound}} - v_{\text{observer}}}{v_{\text{sound}} + v_{\text{source}}} \right) \cdot f_{\text{emitted}}$$

$$= \left( \frac{v_{\text{sound}} - v_{\text{person}}}{v_{\text{sound}} + v_{\text{train}}} \right) \cdot f_{\text{emitted}}$$

$$= \left( \frac{330 \,\text{m/s} - 0}{330 \,\text{m/s} + 25 \,\text{m/s}} \right) \cdot 400 \,\text{Hz}$$

$$= 372 \,\text{Hz}$$

The emitted pitch is close to the G above middle C on a piano; the pitch heard as the train is approaching is close to the A above middle C (often called "A440"); the pitch heard after the train passes is close to the F# above middle C. So the pitch drops by approximately three half-notes as the train passes.

## 11.8 Non-Ideal Gases                                                                    ✚

### Words

This chapter began with a definition of an ideal gas:

- molecules of the gas interact with each other only through perfectly elastic collisions

- size of the molecules of an ideal gas is negligible compared to both the size of the container and the average distance between gas molecules

- all of the gas molecules are identical

This model of an ideal gas works well in a great many situations, but in some situations it is necessary to consider the deviation of a real gas from the model of an ideal gas. We will consider the effects of violating each of the parts listed above for the definition of an ideal gas.

First, gases in fact don't interact only through perfectly elastic collisions. If gases are highly reactive then they will react chemically with other molecules. But even gases that are completely non-reactive still experience attractive "Van der Waals" forces when the gas particles are close enough to each other. These small attractive forces create a slight decrease in the pressure of the gas compared to what would be expected for an ideal gas. When the gas particles get even closer together, they experience repulsive forces.

### Graphics
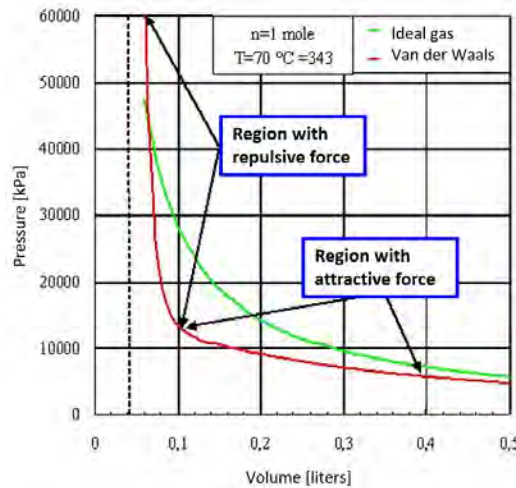


Figure 11.19: This P-V plot shows a sample isotherm for a non-ideal gas compared to an ideal gas.[73]

### Numbers

For non-ideal gases, where the interactions between particles and the total volume of the particles themselves cannot be ignored, the ideal gas law cannot be used. It is instead modified with the introduction of two additional parameters $\mathbf{a}$ and $\mathbf{b}$. These parameters are experimentally determined for particular gases. The modified ideal gas law is called "Van der Waals' equation:"

$$\left(P + \mathbf{a} \cdot \frac{n^2}{V^2}\right) \cdot (V - n \cdot \mathbf{b}) = n \cdot R \cdot T \qquad (11.8)$$

The parameter $\mathbf{a}$ is related to the interactions between molecules of the gas. The parameter $\mathbf{b}$ is the volume occupied by one mole of the gas molecules. Note that when $\mathbf{a}$ and $\mathbf{b}$ are both zero, the Van der Waals' equation reduces to a version of the ideal gas law that is written in terms of moles instead of molecules.

All of the variables in the ideal gas law–pressure, volume, number of moles, and temperature–are referred to as "state functions," because they describe the state of the gas at a particular point in time, without giving any information about how the gas came to be in its current state.

Second, ideal gases assume that the size of the gas molecules themselves is negligible compared to the volume of the container. But when the number of molecules is increased or the volume is decreased enough, the volume of the particles themselves can no longer be neglected. As a result, the effective volume of the container is reduced by the volume of the gas molecules. This reduction in the effective volume causes the pressure to be larger than would be expected for an ideal gas.

And finally, it is much more common to find a mixture of different gases rather than a gas that is made up of only one type of molecule. Our atmosphere is an example of this, made up of almost 80% nitrogen and 21% oxygen, with many other gases mixed in. For a gas like this that is made up of many different molecules, the total pressure of the gas is made up of the sum of all of the partial pressures of each gas. This is called "Dalton's Law."

Daltons' Law states that the total pressure of a gas that is made up of a mixture of multiple molecules is the sum of the partial pressures of each constituent gas:

$$P_{tot} = P_1 + P_2 + \cdots \qquad (11.9)$$
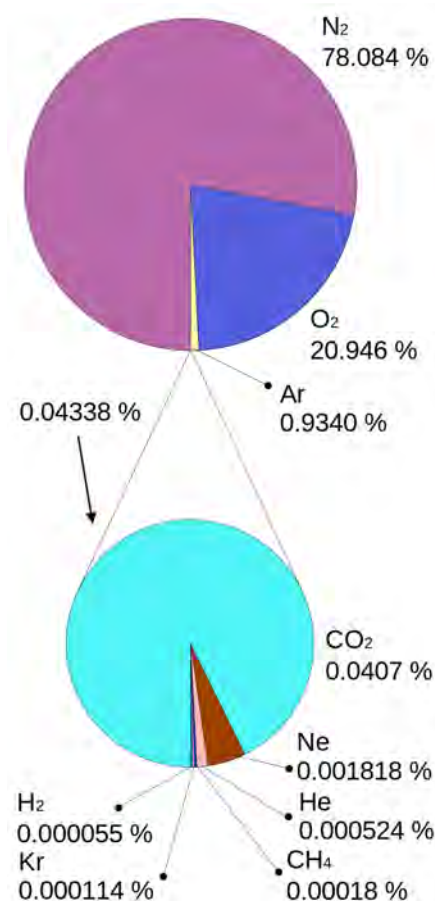


Figure 11.20: These pie charts show the distribution of gases in the earth's atmosphere.[74]

239

## 11.9 Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

### General

- Gases are a type of fluid.

- Monatomic gases are gases in which each molecule is composed of a single atom.

- Molecules of a gas tend to bounce around freely, interacting with each other only through collisions.

- The size of the molecules of an ideal gas is negligible compared to both the size of the container and the average distance between gas molecules.

- All of the molecules of an ideal gas are identical.

- Sound waves, which are longitudinal waves with areas of compression and rarefaction, can travel through gases.

- Transverse waves cannot travel through gases.

- Sound waves can interfere with each other.

### Forces

- When volume and temperature of gas molecules remain constant, the pressure of a gas is proportional to the number of molecules of gas.

- When number and temperature of gas molecules remain constant, the pressure of a gas is inversely proportional to the volume of the gas.

- Gauge pressure is the difference between total pressure and external pressure, *e.g.* atmospheric pressure.

- Gauge pressure can be positive, negative or zero; total pressure cannot be negative.

- Buoyant force is present in gases as well as liquids.

- Nodes in a sound wave are positions where the pressure is constant.

- Antinodes in a sound wave are positions where the largest pressure changes are happening.

- Nodes occur at open places in a hollow tube; antinodes occur at closed ends of a hollow tube.

- When multiple ideal gases are mixed together, the total pressure is the sum of the partial pressures of each individual gas.

### Motion

- The average speed of gas molecules increases with temperature

- The frequency of a sound wave changes depending on the motion of the object creating the sound and the observer.

## Momentum

- (Nothing!)

## Energy

- Molecules of an ideal gas interact with each other only through perfectly elastic collisions.

- When sound waves aren't confined, their energy spreads out in spherical shells, and the amplitude of the sound decreases with distance.

## Mathematical Models

| equation | | restrictions on the validity of the equation |
|---|---|---|
| $P_{tot,1} \cdot V_1 = P_{tot,2} \cdot V_2$    (11.1) <br><br> "Boyle's Law" | | ideal gas; temperature & number of molecules constant |
| $P_{tot} = P_{external} + P_{gauge}$    (11.2) | | -none- |
| $P_{tot} \cdot V = N \cdot k_B \cdot T$    (11.3) <br><br> "Ideal gas law" | | ideal gas; temperature must be in kelvin |
| $T_K - 273\mathrm{K} = T_C = \frac{5}{9}\left(T_F - 32°\mathrm{F}\right)$    (11.4) | | -none- |
| $SPL = 20 \log_{10}\left(\frac{P_s}{P_0}\right) \mathrm{dB}$    (11.5) | | -none- |
| $P_{s,2} = \frac{r_1}{r_2} \cdot P_{s,1}$    (11.6) | | sound is propagating in open air |
| $f_{observed} = \left(\frac{v_{sound} \pm v_{observer}}{v_{sound} \mp v_{source}}\right) \cdot f_{emitted}$    (11.7) | | source and observer are moving slower than the speed of sound |
| $\left(P + \mathbf{a} \cdot \frac{n^2}{V^2}\right) \cdot (V - n \cdot \mathbf{b}) = n \cdot R \cdot T$    (11.8) <br><br> "Van der Waals' equation" | | all gas molecules are identical |
| $P_{tot} = P_1 + P_2 + \cdots$    (11.9) <br><br> "Dalton's Law" | | multiple ideal gases mixed together |

# 11.10  Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

## Level 1 - Remember

11.1 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

11.2 **[W, G, & N]** Which of the following can be negative?

   (a) absolute pressure

   (b) atmospheric pressure

   (c) gauge pressure

   (d) total pressure

## Level 2 - Understand

11.3 **[W & G]** Figure 11.6 shows bubbles getting larger and larger as they rise to the surface. Is the way that they are drawn reasonable? Is there anything about that sketch that is not reasonable?

11.4 **[N]** An internet search was used in Section 11.3 to find bike tire pressure. Are tire pressures reported on the internet gauge pressures or total pressures? Explain your reasoning.

11.5 **[W & G]** The image below shows an unusual kind of pressure gauge. Does it measure total pressure or gauge pressure? How can you tell? What do the numbers to the left of the zero represent?



An unusual pressure gauge.[75]

11.6 **[G]** Figure 11.14 shows harmonics for a pipe of a pan flute that is open at one end. If the pipes were open at both ends, would the ends of the pipes have pressure nodes or antinodes?

## Level 3 - Apply

**11.7** **[N]** The calculations for net force were not completed in Section 11.4. What is the net force on the balloon, including the direction? Don't forget about the mass of the balloon itself.

**11.8** **[N]** Practice converting temperatures between the various scales mentioned in Section 11.4:

   (a) What is 0K in degrees Celsius?

   (b) What is $0°C$ in degrees Fahrenheit?

   (c) What is $0°F$ in degrees Celsius?

   (d) Is there any temperature where the temperature measured in degrees Celsius has the same numerical value as the temperature measured degrees Fahrenheit? If so, find that temperature. If not, explain why not.

   (e) Is there any temperature where the temperature measured in degrees Celsius has the same numerical value as the temperature measured in kelvin? If so, find it. If not, explain why not.

**11.9** **[G]** Figure 11.14 shows the first two harmonics for a pipe of a pan flute that is open at one end. Draw the third harmonic for this pipe.

**11.10** **[G]** Figure 11.14 shows the first two harmonics for a pipe of a pan flute that is open at one end. Draw the first harmonic for a pipe that is open at both ends.

**11.11** **[G]** Figure 11.14 shows the first two harmonics for a pipe of a pan flute that is open at one end. Draw the first harmonic for a pipe that is closed at both ends.

**11.12** **[N]** Sounds louder than 70 dB can start to damage hearing. A jackhammer creates a sound pressure level of approximately 100 dB at a distance of 1 m. At what distance is it safe to be exposed to the sound of a jackhammer without hearing protection?

## Level 4 - Analyze

**11.13** **[W & N]** The average speed of the helium atoms in Section 11.1 is given as 1 350 m/s. What is the average *velocity* of the helium atoms in the box if it is measured over a long period of time? Explain your answer.

**11.14** **[N]** How would the final volume change in Section 11.2 if the diver were in fresh water instead of seawater?

**11.15** **[W, G, & N]** Use an internet search to find the mass of an automobile and its recommended tire pressure, and use this information to find the area of each tire that is touching the ground, assuming the mass is distributed equally between the tires. Is your result reasonable? Explain.

**11.16** **[W]** Some hospital rooms are kept at "positive pressure" for patients who are particularly susceptible to airborne illnesses Does this refer to gauge pressure or total pressure? In what direction would air flow around the edges of the door of a room that is kept at positive pressure? Explain why this would be advantageous.

**11.17** **[W]** Some hospital rooms are kept at "negative pressure" for patients who have illnesses that are easily spread through the air. Does this refer to gauge pressure or total pressure? In what direction would air flow around the edges of the door of a room that is kept at negative pressure? Explain why this would be advantageous.

**11.18** ✚ **[W,G, & N]** A simple mercury barometer can be used to measure atmospheric pressure. It is composed of something like a long test tube, closed at the top end and open at the bottom. The tube is partially filled with liquid mercury, and the open end of the tube is submerged in a pool of mercury.

In the space above the mercury in the tube is a near-total vacuum. Because of this, there is zero air pressure pushing down on the top of the mercury in the tube. Look up the density of mercury, and use that to find the height $h$ of the column of mercury if there is one atmosphere of pressure pushing down on the pool of mercury.
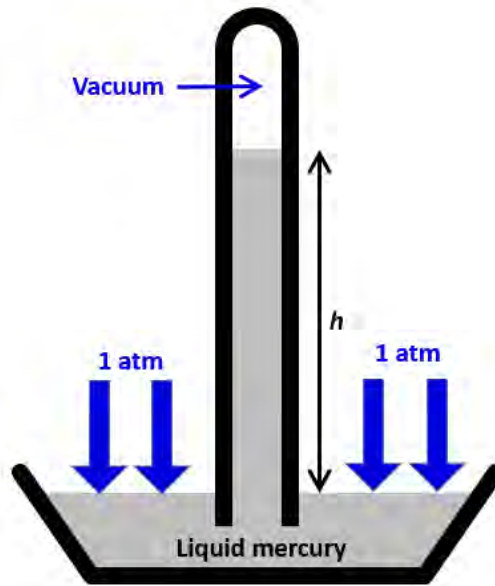


A simple mercury barometer[1]

11.19 **[W, G, & N]** One of the assumptions made in Section 11.3 is that the area touching the ground is a rectangle. In fact the area is probably more like an ellipse. Look up the formula for the area of an ellipse. Compare the areas of a rectangle and an ellipse with the same dimensions as the rectangle. What is the percentage difference between the two?

11.20 **[W & G]** Section 11.7 doesn't describe what happens when the observer is moving relative to the object that is creating the sound. Describe, using Figure 11.16, how the sound of the drums would change if a person were moving quickly toward the drums or quickly away from the drums. Remember that the waves drawn in the figure are not stationary but are moving outward.

11.21 **[N]** In Section 11.7 a situation was analyzed where a train was moving past a stationary person. Re-analyze that situation with the train still blowing its 400 Hz whistle, but this time with the train motionless and the person passing the train at a speed of 25 m/s.

## Level 5 - Evaluate

11.22 **[W, G, & N]** A 100-ml syringe is half-filled with water and half-filled with air at atmospheric pressure. If the plunger is pushed in, increasing the pressure inside the syringe to four times atmospheric pressure...

 (a) ...what is the new volume of the water?

 (b) ...what is the new volume of the air in the syringe?

 (c) ...what assumptions are you making in answering this question?

11.23 **[W & N]** Are the assumptions made in Section 11.4 reasonable? Are there any other assumptions that are made but that aren't listed? Explain your answers.

11.24 **[N]** A statement is made in Section 11.4 that it is important to use kelvin to measure temperature when dealing with ideal gases. Explain why that is true. *Hint: Look at the ideal gas law (Equation 11.3) and think about what happens to the pressure or the volume when the temperature goes to zero, and compare with your personal experience at those temperatures.*

11.25 **[W & G]** The caption for Figure 11.7 makes statements about the density of the gas molecules in the figure, and the text below the figure depends on those statements. Are there other assumptions that are needed in order for the text below the figure to correctly compare the gauge pressures in the figure? If so, what are they?

11.26 **[W, G, & N]** One of the assumptions made in Section 11.3, the one about the area of each tire that is touching the ground, is probably not a very good one. If the assumption about the air pressure being the same in each tire was correct, and that pressure was $4.85 \times 10^5 \, \text{Pa}$, what would the area be for each tire if the charcoal is positioned directly over the back tire and the person's center of mass is centered horizontally halfway between the two tires?

11.27 **[N]** What temperature would the air in the hot air balloon have to be in Section 11.4 so that the balloon would just hover in the air without going up or down?

11.28 **[N]** Equation 11.1 is really just an application of Equation 11.3 in a situation where the temperature and number of particles is constant. Create a similar mathematical model that is an application of Equation 11.3 in a situation where the number of particles is constant but the temperature is not.

11.29 **[W, G, & N]** The calculations in Section 11.5 show that the length of a single pipe of a pan flute should be 0.315 meters long to produce a note near the center of a piano. Looking at Figure 11.13, does it appear that any of the pipes may be close to this length? Are most of the pipes longer or shorter than 0.315 meters? What does that tell you about their frequency or pitch compared to the center keys of a piano?

11.30 **[W & N]** Compare the speed of sound in air to the average speed of a gas particle in air. Are they roughly the same size (which in physics often means within a factor of 10 of each other)? Which is larger? Does this make sense to you? Explain why or why not.

11.31 **[W & N]** Equation 11.7 says that it is only valid when the observer and source are both moving as speeds lower than the speed of sound. Try to use the equation with one or both moving at the speed of sound, toward and away from each other. What happens? In some situations you should get a reasonable answer. In others you will not. Explain what is happening physically in the situations when the mathematical answer is not reasonable.


## Level 6 - Create

11.32 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Add the main ideas from this chapter to a similar concept map that is specifically for gases. Remember that many ideas about fluids were introduced in the chapter on liquids, so many ideas from that chapter can be used in this concept map for gases.

11.33 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

11.34 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have

written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

# Chapter 12

# Temperature and Heat

The word "heat" hasn't been used before in this book, but you know it by a different name: thermal energy. So we have already looked at both temperature and heat, and now we will study them more closely and also see how they interact with each other. In everyday English, "heating something up" means the same as increasing the temperature of the thing. And it is true that adding heat to an object often increases its temperature, but that is not always the case. There are some situations where temperature can change without adding heat, for example if a gas is rapidly compressed. There are other situations where heat can be added to an object without changing its temperature, for example when ice melts in a glass of water.

Temperature affects the volume of solids and liquids. And we have also seen that temperature can affect the volume of a gas, but gas is more complicated since it is compressible. The interplay of volume, pressure, temperature, and heat in gases has spawned an entire field of physics, called "thermodynamics." Up until this point we have always considered heat (which, remember, is just another name for thermal energy) to be lost. But thermodynamics shows us that this lost thermal energy can be used, for example in a steam engine, to do work. Thermodynamics also shows that there are ways to use energy to reduce the temperature of objects, for example in a refrigerator.



Figure 12.1: A steam engine converts heat into work.[76]

## 12.1　Heating a Solid

### Words

In most cases we can think of thermal energy as the kinetic energy of atoms moving around randomly on a microscopic level. This kinetic energy isn't making the object move, because the random motion means that the total momentum of the object is zero. But adding heat to one part of a solid object increases the motion of the molecules in that part. The molecules in a solid are tightly bound to each other, so they stay in the same position within the solid, but they oscillate back and forth, pushing and pulling on the molecules next to them. This oscillation affects nearby molecules, making them oscillate–in this way, the thermal energy spreads through the solid. This method of heat transfer through a solid is called "conduction." Some solids conduct heat better than others–metals are known for being good conductors of heat, and glass is known for being a poor conductor of heat. That is why cooking pots are usually made of metal and insulation is often made of glass fibers.

These individual microscopic oscillations of the molecules can be observed as a change in the temperature of the object: adding thermal energy increases the temperature. The amount of the temperature change depends on the mass and on the material itself. In addition, the total size of the object changes with temperature. Most materials expand slightly as their temperature increases.

### Graphics



Figure 12.2: A cast iron skillet on a stove.[77]
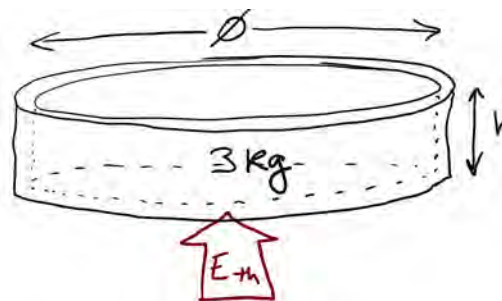


Figure 12.3: Sketch of the skillet showing heat coming in the bottom.[1]

### Numbers

**Assumptions:** physical "constants" are valid over the whole temperature range; iron remains solid over the whole temperature range; final temperature is the same throughout the iron skillet; skillet shape remains the same

| Knowns | Unknowns |
|---|---|
| $m_i = 3\,\mathrm{kg}$ | $m_f$ |
| $T_i = 20°\mathrm{C}$ | $\varnothing_f$ |
| $T_f = 90°\mathrm{C}$ | $h_f$ |
| $\varnothing_i = 0.25\,\mathrm{m}$ | $V_f$ |
| $h_i = 0.04\,\mathrm{m}$ | $\Delta E_{th}$ |
| $\alpha_{iron} = 1.0 \times 10^{-5}\,°\mathrm{C}^{-1}$ | |
| $c_{iron} = 450\,\mathrm{J/(kg \cdot °C)}$ | |

The coefficient of linear thermal expansion $\alpha_{iron}$ and the specific heat capacity $c_{iron}$ were found using an internet search. The specific heat capacity describes the relationship between temperature change $\Delta T$ and heat $\Delta E_{th}$ (or $Q$ is often used as a symbol for heat) added to a material with mass $m$:

$$\boxed{\Delta E_{th} = m \cdot c \cdot \Delta T} \qquad (12.1)$$

We can use this to determine the heat (thermal energy) needed to increase the temperature of the skillet.

$$\Delta E_{th} = (3\,\mathrm{kg}) \cdot \left(450\,\frac{\mathrm{J}}{\mathrm{kg} \cdot °\mathrm{C}}\right) \cdot (90°\mathrm{C} - 20°\mathrm{C})$$

$$= 94\,500\,\mathrm{J}$$

We will consider what happens to an empty cast iron skillet that is placed on a hot stove. The skillet is initially at 20°C, with a mass of 3 kg, an inside diameter of 25 cm, and an inside depth of 4 cm. For simplicity, we will ignore the handle of the skillet and approximate its shape as a hollow cylinder that is closed on one end. If the skillet is heated to a temperature of 90°C, what is its new mass, diameter, height, and volume? How much heat is needed to reach this temperature?

To answer these questions, we need to know something about the properties of iron. One important property is the coefficient of linear thermal expansion $\alpha$, which describes the change in the linear dimensions of an object as the temperature changes. These coefficients can be found in tables online or in reference books, but in every case they are only approximations that are valid over a certain temperature range. We will assume that the values found online are valid for all temperatures unless the reference source includes the temperature range. The skillet gets slightly larger as the temperature increases. But from your own experience you probably know the change in the size of the skillet is extremely small compared to the size of the skillet itself. And the mass should not change at all, because no iron is being added or removed.

Thermal expansion is not much of an issue for an iron skillet, but if you consider miles of iron railroad tracks, the change in length can be significant enough to warp the tracks and derail a train. Large structures often include expansion joints to protect from damage caused by temperature changes.
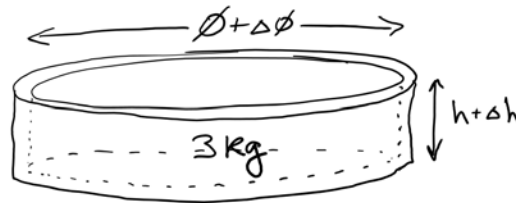


Figure 12.4: Sketch of the skillet after it has increased in size.[1]



Figure 12.5: This expansion joint allows the length of the rails on either side to expand or contract by several centimeters without losing alignment with each other.[78]

The change in any linear dimension of a solid can be found using the coefficient of linear thermal expansion:

$$\boxed{\Delta l = l \cdot \alpha \cdot \Delta T} \qquad (12.2)$$

…where $l$ is any linear dimension, *e.g.* height, width, radius, or diameter. We can use this to determine the height and diameter of the skillet at 90°C.

$$\Delta h = h_i \cdot \alpha \cdot \Delta T$$
$$= (0.04\,\mathrm{m}) \cdot \left(1.0 \times 10^{-5}\,^\circ\mathrm{C}^{-1}\right) \cdot \left(90^\circ\mathrm{C} - 20^\circ\mathrm{C}\right)$$
$$= 2.8 \times 10^{-5}\,\mathrm{m}$$

Similarly, the change in the diameter of the skillet is $\Delta\varnothing = 1.75 \times 10^{-4}$ m.

The volume of the cylinder is simply the area of the bottom multiplied by the height, so the original volume was…

$$V_i = \pi \cdot r_i^2 \cdot h_i$$
$$= \pi \cdot (0.25/2\,\mathrm{m})^2 \cdot (0.04\,\mathrm{m})$$
$$= 1.9635 \times 10^{-3}\,\mathrm{m}^3$$

The final volume, using the final diameter and final height, is $1.9676 \times 10^{-3}\,\mathrm{m}^3$. These extremely small changes in lengths and volume are probably too small to notice or even to measure without using specialized equipment. But since the change in length is proportional to the original length, thermal expansion can easily be seen in very long structures like bridges or railroad tracks.

## 12.2 Heating a Liquid

### Words

Let's continue the scenario from Section 12.1, by removing the skillet from the heat source and pouring 1.5 liters of cold water into it. The skillet starts at $90°$C and the water starts at $10°$C. Assuming that there is no flow of heat into or out of the skillet-water system, what final temperature will they reach? What is the final volume of the water?

When water is poured into the skillet, heat flows from the skillet into the water. This is because the temperature of the skillet is higher than the temperature of the water. Heat flows from regions of higher temperature to regions of lower temperature unless something is actively working on the system to prevent or even reverse the flow of heat.

Heat will continue to flow from the skillet into the water until they are both at the same temperature, at which point the flow of heat will stop. When heat is no longer flowing the system is said to be in "thermal equilibrium." This is a way of defining temperature: two objects are at the same temperature if heat does not flow between them when they are brought into thermal contact with each other.

This physical scenario can be approached in terms of conservation of energy. Some thermal energy leaves the skillet, and the same amount of thermal energy enters the water. The final temperature of the system should be somewhere between $90°$C, the initial temperature of the skillet and $10°$C, the initial temperature of the water.

### Graphics



Figure 12.6: Schematic of the skillet just after the water has been added, showing heat transferring from the skillet to the water.[1]
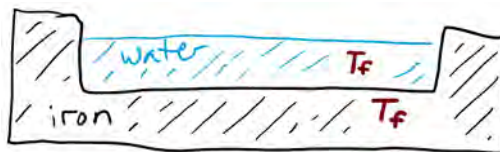


Figure 12.7: Schematic of the skillet and water after they have reached thermal equilibrium.[1]

### Numbers

**Assumptions:** isolated system; specific heat capacities and coefficient of bulk thermal expansion are valid over the whole temperature range; iron remains solid and water remains liquid

| Knowns | Unknowns |
|---|---|
| $m_{iron} = 3\,\text{kg}$ | $T_f$ |
| $V_{water,i} = 0.0015\,\text{m}^3$ | $V_{water,f}$ |
| $T_{water,i} = 10°\text{C}$ | |
| $T_{iron,i} = 90°\text{C}$ | |
| $\beta_{water} = 2.1 \times 10^{-4}\,°\text{C}^{-1}$ | |
| $c_{iron} = 450\,\text{J}/\left(\text{kg} \cdot °\text{C}\right)$ | |
| $c_{water} = 4182\,\text{J}/\left(\text{kg} \cdot °\text{C}\right)$ | |

Using conservation of energy for this isolated system where the only type of energy is thermal, we have...

$$0 = \Delta E_{tot} = \Delta E_{th,tot} = \Delta E_{th,water} + \Delta E_{th,iron}$$
$$= m_{water} \cdot c_{water} \cdot \Delta T_{water} + m_{iron} \cdot c_{iron} \cdot \Delta T_{iron}$$
$$= m_{water} \cdot c_{water} \cdot \left(T_f - T_{water,i}\right) +$$
$$m_{iron} \cdot c_{iron} \cdot \left(T_f - T_{iron,i}\right)$$

Now we can rearrange and solve for $T_f$:

$$\left(m_{water} \cdot c_{water} + m_{iron} \cdot c_{iron}\right) \cdot T_f =$$
$$m_{water} \cdot c_{water} \cdot T_{water,i} + m_{iron} \cdot c_{iron} \cdot T_{iron,i}$$

At first, one might think that the temperature should be halfway between the two, so $50°C$. But remember that the skillet has twice the mass of the water, so that would tend to make the final temperature closer to $90°C$ than to $10°C$. But it isn't as simple as that. Another important factor is the specific heat capacity of the materials. Specific heat capacity describes the amount of heat needed to increase the temperature of a material. And the specific heat capacity of water is ten times larger than that of iron. So in fact the final temperature is much closer to the initial temperature of the water than it is to the initial temperature of the iron.

Like solids, most liquids also expand when they are heated, but liquids don't have linear dimensions, so their expansion is described as a change in volume. They have a coefficient of "bulk" or "volumetric" thermal expansion, $\beta$. The value of $\beta$ is usually reasonably constant for different temperatures, but water is unusual. Its coefficient of bulk thermal expansion decreases at low temperatures and actually becomes negative below approximately $4°C$. So at low temperatures water actually expands as the temperature decreases!

The water is physically touching the skillet, so the heat transfers into the water through conduction. Heat can also transfer through fluids by conduction, but if the heat is coming from the bottom then the primary method of heat transfer is convection. Convection is circulation of the molecules that occurs because the warmer fluid at the bottom expands and its density decreases. This creates a buoyant force that lifts the warmer fluid up, and it is replace by cooler fluid flowing down.
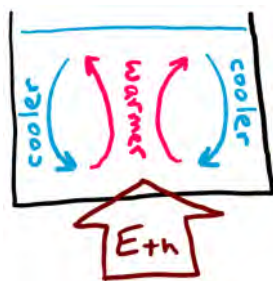


Figure 12.8: As heat enters the bottom of the fluid, the fluid at the bottom gets warmer and expands. The warmer fluid rises since it is less dense than the cooler fluid above it, and the cooler fluid flows down to the bottom. This is called convection. [1]
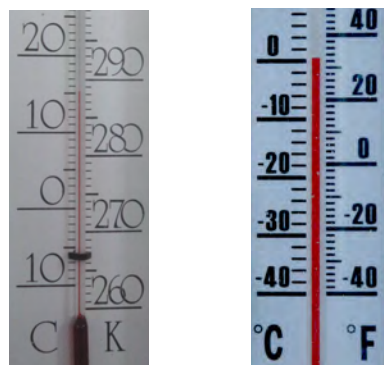


Figure 12.9: On the left the numbers are different but the spacing is the same, so a change of $10°C$ is a change of 10K. On the right, a change of $10°C$ is a change of $18°F$.[79],[80]

We were not given the mass of the water, but it can be found from the density of fresh water, which is $1\,000\,\text{kg/m}^3$.

$$T_f = \frac{m_{water} \cdot c_{water} \cdot T_{water,i} + m_{iron} \cdot c_{iron} \cdot T_{iron,i}}{m_{water} \cdot c_{water} + m_{iron} \cdot c_{iron}}$$
$$= \frac{(62\,730 + 121\,500)\ \text{J}}{(6\,273 + 1\,350)\ \text{J/°C}}$$
$$= 24°C$$

The bulk thermal expansion coefficient $\beta_{water}$ was found using an internet search. The value found online was in units of $K^{-1}$, so K was replaced with $°C$, with no mathematical conversion needed. That is because thermal expansion is calculated using change in temperature, not the temperature itself. Conversions are different for change in temperature than they are for temperature:

$$\boxed{\Delta T_K = \Delta T_C = \frac{5}{9}\left(\Delta T_F\right)} \qquad (12.3)$$

The bulk thermal expansion coefficient is used in exactly the same way as the linear thermal expansion coefficient:

$$\boxed{\Delta V = V \cdot \beta \cdot \Delta T} \qquad (12.4)$$

So the final volume of the water is...

$$V_{water,f} = V_{water,i} + \Delta V_{water}$$
$$= V_{water,i} \cdot \left(1 + \beta_{water} \cdot \Delta T_{water}\right)$$
$$= \left(0.0015\,\text{m}^3\right)\left(1 + \left(2.1 \times 10^{-4}\,°\text{C}^{-1}\right)\left(14°C\right)\right)$$
$$= 0.001504\,\text{m}$$

Again, this is an extremely small change in volume.

## 12.3 Changing State

### Words

Solid, liquid, and gas are the three most common "states" of matter that we interact with on a daily basis. It is possible for a material to change from one of these states to another when heat is added or removed. If heat is added to a solid until its temperature reaches the "melting point," any additional heat will not increase the temperature but will melt the solid into a liquid. Once all of the solid has melted, adding more heat will increase the temperature of the liquid. Then, if heat is added to the liquid until its temperature reaches the "boiling point," any additional heat will not increase the temperature but will evaporate the liquid into a gas. After the liquid has all been evaporated, additional heat can then increase the temperature of the gas.

The same process can also proceed in reverse. If heat is removed from a gas that is already at the boiling point, the gas will condense into a liquid. Once all of the gas has condensed, removing heat will lower the temperature of the liquid until it reaches the melting point. At the melting point, removing heat from the liquid freezes it into a solid without changing the temperature. Once all of the liquid has frozen, removing more heat will lower the temperature of the solid.

For our scenario we will start with 1.2 kg of ice at a temperature of −20°C. How much heat needs to be added to melt all of the ice and then boil away half of the water?

### Graphics
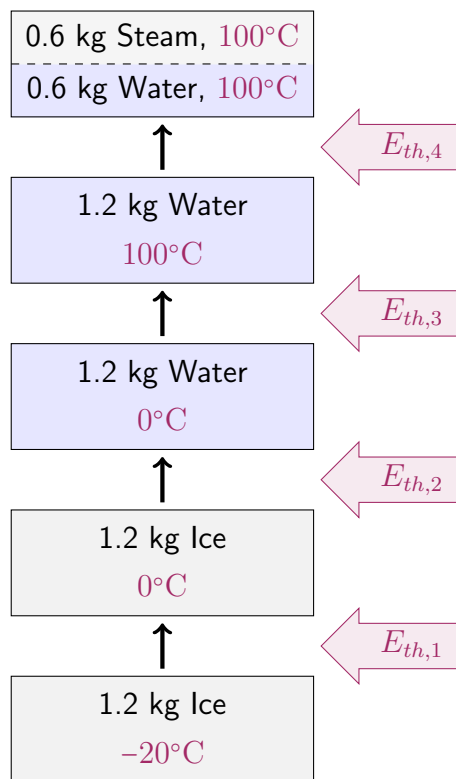


Figure 12.10: $E_{th,1}$ brings the ice to its melting point. Then $E_{th,2}$ melts the ice. Then $E_{th,3}$ brings the water to its boiling point. Finally, $E_{th,4}$ boils away half of the water.[1]

### Numbers

**Assumptions:** isolated system; specific heat capacities are valid over the whole temperature range; atmospheric pressure

| Knowns | Unknowns |
|---|---|
| $m_{ice,i} = 1.2\,\text{kg}$ | $E_{th,tot}$ |
| $T_i = -20°\text{C}$ | |
| $T_{water,melting} = 0°\text{C}$ | |
| $T_{water,boiling} = 100°\text{C}$ | |
| $c_{water} = 4\,182\,\text{J}/(\text{kg}\cdot°\text{C})$ | |
| $c_{ice} = 2\,090\,\text{J}/(\text{kg}\cdot°\text{C})$ | |
| $L_{f,water} = 3.34 \times 10^5\,\text{J/kg}$ | |
| $L_{v,water} = 2.23 \times 10^6\,\text{J/kg}$ | |
| $m_{water,f} = 0.6\,\text{kg}$ | |

The different physical processes here (changing temperature, melting, and boiling) require different mathematical models. The heat needed to change temperature has already been introduced. The amount of heat required to melt a solid is…

$$\boxed{E_{th} = m \cdot L_f} \tag{12.5}$$

…where $L_f$ is the latent heat of fusion for the material and $m$ is the amount of the material that melts. Similarly, the amount of heat required to evaporate a liquid is…

$$\boxed{E_{th} = m \cdot L_v} \tag{12.6}$$

The amount of heat required to melt a solid is called the "latent heat of fusion," and the amount of heat required to evaporate a liquid is called the "latent heat of vaporization." These latent heats vary depending on the material.

It is also possible to change the state of a material by changing the pressure while keeping temperature constant. This means that melting and boiling points are in fact dependent on pressure; they are normally reported at atmospheric pressure. At some combinations of temperature and pressure it is possible for a solid to change phase directly to a gas without first changing to liquid. This is called "sublimation." A common example of this is "dry ice," solid carbon dioxide, which undergoes sublimation at atmospheric pressure.

The combinations of temperature and pressure at which phase transitions occur are often shown on a phase diagram like the one in Figure 12.12. "Phase" is often used interchangeably with "state" when discussing transitions between solids, liquids, and gases, but some materials have multiple phases within a single state. Water, for example, forms into at least nine unique crystal phases at various temperatures and pressures within its solid state.

Solid lines in a phase diagram indicate the transition points between the states. The near-vertical dotted line in Figure 12.12 is representative of water, which has an unusual slope: solid water can melt as pressure increases! Every material has a "triple point" on its phase diagram at which it can exist in solid, liquid, or gas state; and a "critical point" beyond which there is no clear distinction between the liquid and gas states.
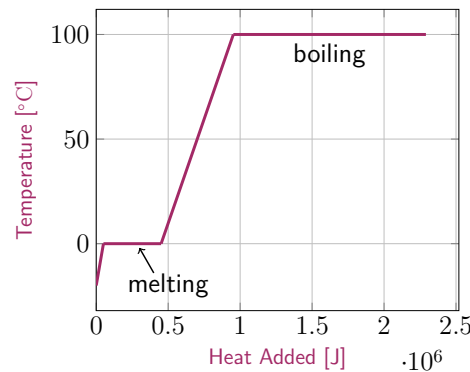


Figure 12.11: Temperature as a function of heat added. The plateaus in the graph occur when the material is changing state, not changing temperature.[1]
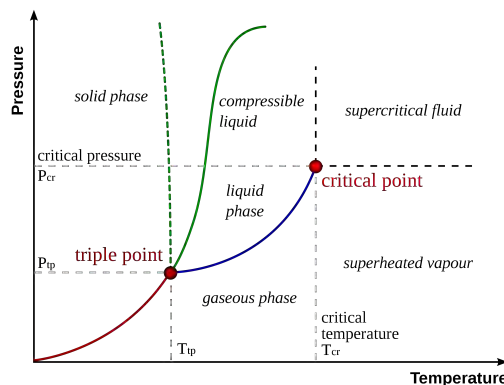


Figure 12.12: This generic phase diagram shows the general trend followed by most materials, with the temperatures and pressures at which a material exists in solid, liquid, or gas states.[81]

...where $L_v$ is the latent heat of vaporization for the material and $m$ is the amount of the material that evaporates. $L_f$ also describes the heat that has to be removed to freeze a liquid into a solid, and $L_v$ describes the heat that has to be removed to condense a gas into a liquid. The melting point, boiling point, latent heats, and specific heats in the "knowns" were found using an internet search.

In order to find the total heat required, we need to find the heat required for each physical process and add them together. Using Equation 12.1, $E_{th,1}$, the heat required to bring the ice up to its melting point, is...

$$E_{th,1} = m_{ice} \cdot c_{ice} \cdot \Delta T_{ice} = 50\,160 \text{ J}$$

Using Equation 12.5, $E_{th,2}$, the heat required to melt the ice is...

$$E_{th,2} = m_{ice} \cdot L_{f,water} = 400\,800 \text{ J}$$

Using Equation 12.1 again, $E_{th,3}$, the heat required to bring the water up to its boiling point, is...

$$E_{th,3} = m_{water} \cdot c_{water} \cdot \Delta T_{water} = 501\,840 \text{ J}$$

Using Equation 12.6, $E_{th,4}$, the heat required to boil away *half of* the water is...

$$E_{th,4} = (m_{water,i} - m_{water,f}) \cdot L_{v,water} = 1\,338\,000 \text{ J}$$

So the total amount of heat required is...

$$E_{th,tot} = E_{th,1} + E_{th,2} + E_{th,3} + E_{th,4}$$
$$= 2\,290\,000 \text{ J}$$

## 12.4   Gas at Constant Volume

### Words

As with solids and liquids, adding heat to a gas can increase its temperature, but the compressibility of gases make them much more complicated than solids or liquids. So we will begin by looking at gases that are in a container with a fixed volume, regardless of the pressure, temperature, or number of gas molecules. An air compressor is an example of just such a container.

When heat goes into a gas that is held at constant volume, the energy goes into the molecules of the gas. For an ideal monatomic gas, the thermal energy tranforms into translational kinetic energy of the gas particles. But the air in the earth's atmosphere is almost completely made up of the *diatomic* gases (gases with two atoms per molecule) nitrogen and oxygen. Ideal diatomic gases can carry other types of energy besides just translational kinetic energy. They can also have rotational kinetic energy. And the bond between the two atoms can act like a spring, so they can also oscillate with spring potential energy. For molecules with more than two atoms, like water vapor and carbon dioxide, there are even more ways that they can rotate and oscillate. It is impossible to measure all of these different types of energy across all of the gas molecules in a typical gas, so this energy is put together into a single category, the "internal energy" of the gas.

### Graphics



Figure 12.13: Air compressors are designed to hold several atmospheres of air pressure. They are often used for inflating tires and running pneumatic equipment like staplers and nail guns.[1]

### Numbers

**Assumptions:** ideal monatomic gas

| Knowns | Unknowns |
|---|---|
| $P_{i,gauge} = 2\,\text{atm}$ | $P_{f,gauge}$ |
| $V = 0.023\,\text{m}^3$ | $\Delta E_{th}$ |
| $T_i = 20°\text{C}$ | $N$ |
| $T_f = 80°\text{C}$ | |

The ideal gas law can be used to find the number of molecules in the gas from the initial conditions:

$$P_{i,tot} \cdot V = N \cdot k_B \cdot T_i$$

This can be rearranged to solve for $N$, but we need to remember first to convert the gauge pressure to absolute pressure and then convert it into pascals, and convert the temperature to kelvin.

$$N = \frac{(P_{i,gauge} + P_{atm}) \cdot V}{k_B \cdot T_i}$$
$$= \frac{\left(3\,\text{atm} \cdot \frac{1.01 \times 10^5\,\text{Pa}}{1\,\text{atm}}\right)(0.023\,\text{m}^3)}{(1.38 \times 10^{-23}\,\text{J/K})\,(293\text{K})}$$
$$= 1.7 \times 10^{24}$$

The numbers of molecules $N$ involved with gases in most real-world situations are huge, so the number of "moles" $n$ is often used instead, where one mole is $6.02 \times 10^{23}$ molecules. The abbreviation commonly used for for mole is mol. $6.02 \times 10^{23}$ is called "Avogadro's number," $N_A$

We will consider a physical scenario where a 23-liter air compressor is filled with an ideal monatomic gas at a temperature of 20°C and a gauge pressure of 2 atmospheres. How many molecules of gas are in the air compressor? How much heat needs to be added to the gas to increase its temperature to 80°C, and what would the pressure of the gas be at that temperature?

As was mentioned in Chapter 11, the number of molecules will be huge compared to numbers that we work with on a daily basis. More than 1,000,000,000,000,000,000,000,000 molecules, in fact! That's a trillion trillions, or one septillion–a number larger than our minds can really comprehend. So another unit is often used when dealing with gases: the mole. One mole is $6.02 \times 10^{23}$, and if we count gas molecules in moles then that makes the numbers much easier to work with. A mole is just a number of something, much like a dozen is 12 of something. One mole of gas molecules in our atmosphere has a volume of 22 liters at sea level at a temperature of 0°C, so we should expect that this 23-liter volume that is higher than atmospheric pressure and higher than 0°C should contain more than one mole of gas molecules.

As for the pressure, we know that the average speed of the molecules increases as the temperature increases. So the molecules will have higher changes in momentum when colliding with the walls of the container. That means a larger force on the walls of the container, so the pressure will be higher at the higher temperature.

## P-V Plot
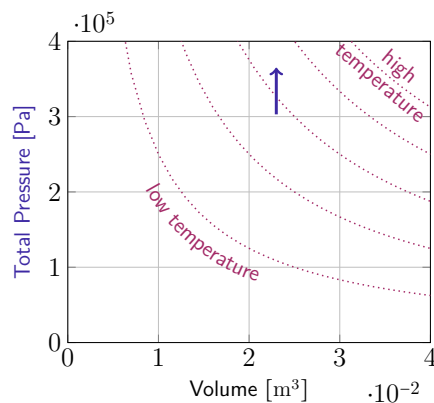


Figure 12.14: A line graph of total pressure versus volume is often called a "P-V plot." The blue arrow indicates the path followed by the gas in the air compressor as the temperature increased. Pressure increased but volume remained constant. The dotted red lines are isotherms. [1]

P-V plots are helpful in understanding the operation of thermodynamic systems. Lines on the graph that follow curves of constant temperature are called "isotherms." These are similar to the "isobars" that follow curves of constant pressure on a weather map, or the contour lines that follow curves of constant elevation on a topographical map. On a P-V plot, higher temperatures are up and to the right.

So the amount of gas in this scenario could also be described as 2.86 moles. The ideal gas law can be rewritten in terms of moles, using the ideal gas constant $R$ in place of the Boltzmann constant $k_B$:

$$\boxed{P_{tot} \cdot V = n \cdot R \cdot T} \tag{12.7}$$

$R$ has a value of $8.31 \frac{\text{J}}{\text{K} \cdot \text{mol}}$. Since we have the final values for everything except pressure, we can use Equation 12.7 to find the final pressure:

$$P_{f,tot} = \frac{n \cdot R \cdot T_f}{V} = 3.65 \times 10^5 \, \text{Pa}$$

Subtracting atmospheric pressure gives a final gauge pressure of $2.64 \times 10^5 \, \text{Pa}$ or 2.6 atmospheres.

In this scenario, all of the heat that goes into the gas is stored as internal energy $U_{int}$ of the gas. For an ideal monatomic gas, the mathematical model relating temperature to internal energy is...

$$\boxed{U_{int} = \frac{3}{2} n \cdot R \cdot T = \frac{3}{2} N \cdot k_B \cdot T} \tag{12.8}$$

For a diatomic gas, the fraction changes from 3/2 to 5/2, and for more complicated molecules the fraction becomes even higher. The change in temperature can be used to find the change in internal energy, which in this scenario is equal to the heat added to the system $\Delta E_{th}$:

$$\Delta E_{th} = \Delta U_{int} = \frac{3}{2} n \cdot R \cdot \Delta T$$
$$= \frac{3}{2} (2.86 \, \text{mol}) \left( 8.31 \frac{\text{J}}{\text{K} \cdot \text{mol}} \right) (60 \text{K})$$
$$= 2\,140 \, \text{J}$$

## 12.5    Gas at Constant Pressure

### Words

Instead of a container with a fixed volume, this time we will study what happens inside a container with a fixed pressure. Pistons are used in many different types of machines, so for this scenario we will look at a sealed vertical piston with negligible friction and a cross-sectional area of $0.02\,\mathrm{m^2}$. The mass of the plunger is 70 kg, and the plunger is initially motionless and in static equilibrium. The initial volume of the ideal monatomic gas inside the piston is $0.05\,\mathrm{m^3}$ and the initial temperature of the gas is $15°C$. The outside pressure is 1 atm.

Find the initial pressure and the number of moles of gas inside the cylinder. Then find the final pressure, temperature, and volume of the gas inside the cylinder when the system again reaches static equilibrium after 10 kJ of heat is added to the gas.

Initially, the pressure inside the gas has to be just large enough to hold the piston in place against the force of gravity and the atmospheric pressure, which both are pushing down on the piston.

When heat is added to the gas, we should expect that the temperature of the gas will increase, which means that the momenta of the gas molecules would increase, increasing the pressure on the walls of the container. But this time one of the walls, the top surface, is movable. The upward force on the piston would be higher than the downward force, so the piston is forced upward. That increases the volume of the container.

### Graphics
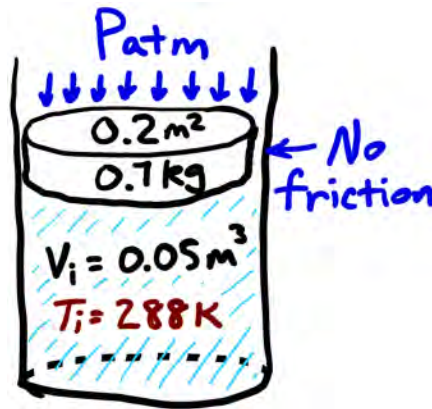


Figure 12.15: Sketch of the vertical piston. The plunger seals in the gas and is able to move up and down. [1]
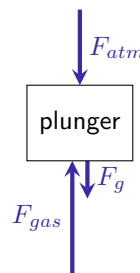


Figure 12.16: Free-body diagram of the plunger[1]

### Numbers

**Assumptions:** ideal monatomic gas; negligible friction; near the earth's surface

| Knowns | Unknowns |
|---|---|
| $A = 0.02\,\mathrm{m^2}$ | $P_{tot}$ |
| $m = 70\,\mathrm{kg}$ | $n$ |
| $V_i = 0.05\,\mathrm{m^3}$ | $T_f$ |
| $T_i = 288\mathrm{K}$ | $V_f$ |
| $\Delta E_{th} = 1 \times 10^4\,\mathrm{J}$ | |
| $P_{ext} = 1.01 \times 10^5\,\mathrm{Pa}$ | |

Since the plunger starts in static equilibrium, we can use the free-body diagram to find the force that the gas needs to apply to the plunger to hold it up, and from that find the pressure of the gas.

$$F_{net} = 0 = F_{gas} - F_{atm} - F_g$$

$$F_{gas} = F_{atm} + F_g$$

$$P_{gas} = \frac{F_{gas}}{A} = \frac{F_{atm} + F_g}{A}$$

$$P_{tot} = \frac{(P_{atm} \cdot A) + (m \cdot g)}{A} = 135\,300\,\mathrm{Pa}$$

Now we can solve for the number of moles using Equation 12.7:

$$n = \frac{P_{tot} \cdot V_i}{R \cdot T_i} = 2.83\,\mathrm{moles}$$

When the volume increases, two things happen to the gas molecules inside the container. First, the time between collisions with the top surface increases, which lowers the pressure. And second, since the wall is moving away from the particles that are colliding with it, the magnitude of the momentum of the particles is lower after each collision than it was before the collision, which decreases the temperature. So allowing the volume to change affects both the pressure and the temperature of the gas.

The piston is in equilibrium when the inside pressure is equal to the outside pressure, so the pressure is the same at the beginning as it was at the end. What about during the time between the beginning and the end? The change in pressure that starts the piston moving is small compared to the total pressure, and if the piston is moving at a constant rate once it starts moving then the pressure during that time is the same as the initial pressure. So the pressure for the whole time is essentially constant.

When the pressure is held constant, the final temperature of the gas is lower than it would have been if the volume had been held constant. That means the final internal energy of the gas is lower than it would have been at constant volume. Where did the rest of the thermal energy go if it didn't go into the gas? It was used to do work on the cylinder, lifting it up against the force of gravity and the atmospheric pressure. This is simply an application of conservation of energy: The total amount of heat that goes into a gas is equal to the sum of the change in the internal energy of the gas and the work done by the gas.
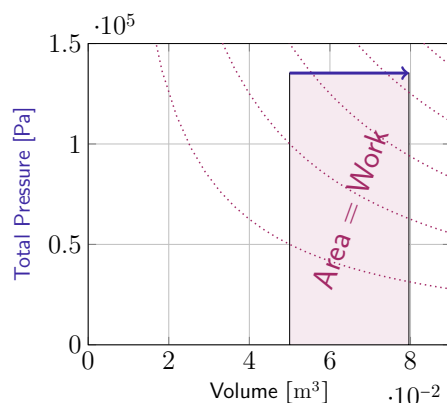


Figure 12.17: A P-V plot for a gas expanding at constant volume. The blue arrow indicates the path followed by the gas in the piston. The area under the curve shows the amount of work done by the gas. [1]

As in a graph of force vs position, the area under the curve shows the work done on a graph of pressure vs volume of a gas.

The gas does work on the plunger, pushing it up. As long as the force is constant, work is force times displacement. A little bit of algebra lets us apply the same principle to gases:

$$W = F_{net} \cdot \Delta x \cdot \cancel{\cos\theta}^{\,1}$$
$$= \left(\frac{F_{net}}{A}\right)(\Delta x \cdot A)$$
$$= P_{tot} \cdot \Delta V$$

So as long as the pressure is constant,

$$\boxed{W = P \cdot \Delta V} \tag{12.9}$$

If $n$ is also constant, then another expression for work can be found using Equation 12.7 again:

$$W = P \cdot \Delta V = n \cdot R \cdot \Delta T$$

Conservation of energy in the gas tells us that the thermal energy that goes *into* the gas $\Delta E_{th}$ is equal to the change in the internal energy of the gas $\Delta U_{int}$ plus the work done *by* the gas $W$:

$$\boxed{\Delta E_{th} = \Delta U_{int} + W} \tag{12.10}$$

Combining this with Equation 12.8 and the expression found above…

$$\Delta E_{th} = \Delta U_{int} + n \cdot R \cdot \Delta T$$
$$= \left(\frac{3}{2}n \cdot R \cdot \Delta T\right) + (n \cdot R \cdot \Delta T)$$
$$= \frac{5}{2}n \cdot R \cdot \Delta T$$

Solving for $\Delta T$ for this scenario gives a value of 170K, so the final temperature is 458K.

## 12.6  Heat Engines and Thermodynamic Cycles

### Words

Now we will consider 0.4 moles of an ideal monatomic gas in a sealed container that goes through four different processes:

1. volume is held constant at $0.01\,\mathrm{m}^3$ while the pressure of the gas increases from 100 000 Pa to 300 000 Pa

2. pressure is held constant while the volume increases to $0.03\,\mathrm{m}^3$

3. volume is held constant while the pressure returns to 100 000 Pa

4. pressure is held constant while the volume returns to $0.01\,\mathrm{m}^3$

This is called a "thermodynamic cycle," in which the gas traces out a closed path on a P-V plot. In the first process, the volume is not changing so the gas is not doing any work. In the second process, the gas is expanding. The pressure that the gas applies to the walls of its container is always pointing outward, and if the volume is expanding then the walls are moving outward (or at least one wall is moving outward), in the same direction as the force on the wall. So the gas is doing work on the container. In the third process, again no work is being done, because the volume is not changing. In the fourth process the walls are moving inward, against the pressure caused by the gas. That means the container is doing work on the gas.

### Graphics
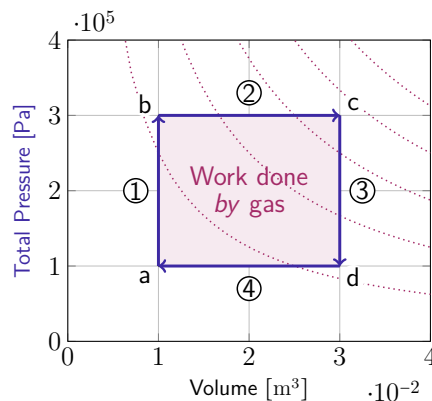


Figure 12.18: P-V plot for a gas going through one complete cycle in four numbered steps. First from "a" to "b," then "b" to "c," "c" to "d," and finally back to "a." The area inside the loop shows the amount of work done *by* the gas as it completes one complete cycle. [1]

Vertical paths on a P-V plot describe "isovolumetric" ("same-volume") processes. Horizontal paths on a P-V plot describe "isobaric" ("same-pressure") processes.

The path along the bottom of the rectangle in the figure above goes from right to left, in the negative direction, so the work done along that path is also negative. So the area under the rectangle would be added to the total work for the path from b to c and subtracted for the path from d to a. This is equivalent to not including it at all.

### Numbers

**Assumptions:** ideal monatomic gas

| Knowns | Unknowns |
|---|---|
| $n = 0.4\,\mathrm{mol}$ | $W_{net}$ |
| $V_a = V_b = 0.01\,\mathrm{m}^3$ | $\Delta E_{th}$ |
| $V_c = V_d = 0.03\,\mathrm{m}^3$ | |
| $P_a = P_d = 1 \times 10^5\,\mathrm{Pa}$ | |
| $P_b = P_c = 3 \times 10^5\,\mathrm{Pa}$ | |

The work for each of the four processes can be added together to get the total work done for one cycle. For the second and fourth processes we can use Equation 12.9. But P is not constant for the first and third processes, so we can't use that Equation. But the change in V is zero for those processes, so the work done by each of those processes is just zero. That leaves...

$$
\begin{aligned}
W &= \cancelto{0}{W_1} + W_2 + \cancelto{0}{W_3} + W_4 \\
&= (P_b \cdot (V_c - V_b)) + (P_d \cdot (V_a - V_d)) \\
&= (6\,000\,\mathrm{J}) - (2\,000\,\mathrm{J}) = 4\,000\,\mathrm{J}
\end{aligned}
$$

We can also find the change in the internal energy of the gas for each of the four processes using the relationships found at the end of Section 12.4 & 12.5. But first we would need the temperatures at each of the starting and ending points for each process. These can be found by rearranging the ideal gas law:

So during the expansion the gas is doing work, and during the contraction work is being done on the gas. The change in volume is the same both times in this example, but the force is higher when the pressure is higher, in the second process. So in one complete cycle the gas does a net positive amount of work on the container. The gas seems to have created useful energy. But energy can't come from nowhere–it is coming from the heat that is going into the gas as it goes through the cycle. This is a "heat engine," which converts heat into work. Which is remarkable. Up until now, every time we have looked at thermal energy it has been energy that was lost from whatever system we have looked at. This time, heat is being captured and put to use.

We can find the work done and the heat for each process it goes through one cycle.

The net thermal energy going into the gas is equal to the amount of work it does, but if we consider each of the different processes, we can see that not all of the thermal energy that goes into the gas is converted to work. In the third process, for example, the volume isn't changing–that means no work is being done. But the volume is decreasing, so the temperature is decreasing. That means the internal energy of the gas is decreasing, so some of the internal energy is being lost as heat instead of being used for work. So this heat engine is not 100% efficient. In fact, no heat engine can be 100% efficient, but some thermal energy is lost just as with the other physical scenarios we have considered.

The operation of a heat engine is often drawn schematically as shown below.
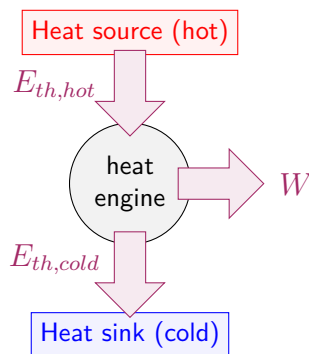


Figure 12.19: Diagram of energy flow through a heat engine.[1]

The circle in the middle represents the machine itself, and the arrows show the energy flowing into or out of the machine. Heat naturally flows from high temperature to low temperature, so the heat going into the heat engine has to come from a high-temperature heat source. Similarly, the heat leaving the heat engine has to go into a low-temperature "heat sink."

Heat engines are designed to do work. That work is shown as an output coming from the side of the heat engine.

$$T = \frac{P \cdot V}{n \cdot R}$$

This gives the following temperatures: $T_a = 301\text{K}$; $T_b = T_d = 903\text{K}$; and $T_c = 2709\text{K}$

For the first and third processes we should use the relationship found in Section 12.4 for constant volume:

$$\Delta E_{th,1} = \frac{3}{2} n \cdot R \cdot (T_b - T_a) = 3\,000 \text{ J}$$

$$\Delta E_{th,3} = \frac{3}{2} n \cdot R \cdot (T_d - T_c) = -9\,000 \text{ J}$$

For the second and fourth processes we should use the relationship found in Section 12.5 for constant pressure:

$$\Delta E_{th,2} = \frac{5}{2} n \cdot R \cdot (T_c - T_b) = 15\,000 \text{ J}$$

$$\Delta E_{th,4} = \frac{5}{2} n \cdot R \cdot (T_a - T_d) = -5\,000 \text{ J}$$

So the net thermal energy going into the gas is...

$$\Delta E_{th,net} = (3\,000 + 15\,000 - 9\,000 - 5\,000) \text{ J} = 4\,000 \text{ J}$$

...exactly as expected. But in order to get 4 000 J of work out of the heat engine, 18 000 J of heat had to be put in. The other 14 000 J of energy left the system as heat.

Since only 4 000 of 18 000 J went where we wanted them to go, we say that the efficiency of this heat engine is 4 000 J/18 000 J, or 22%.

## 12.7 Refrigerators and Efficiency

### Words

A heat engine converts thermal energy into work while transferring thermal energy from a high-temperature heat source to a low-temperature heat sink. This generally increases the temperature of the heat sink and decreases the temperature of the heat source. What if such a machine could be run in reverse, so that it removes heat from a low-temperature heat sink by transferring it to a high-temperature heat source? Then it could use energy to remove heat from something that is already cold, making it even colder. A refrigerator!

We will consider the same 0.4 moles of an ideal monatomic gas in a sealed container that goes through four different processes, in reverse order compared to Section 12.6:

1. pressure is held constant at 100 000 Pa while the volume of the gas increases from $0.01\,\mathrm{m}^3$ to $0.03\,\mathrm{m}^3$

2. volume is held constant while the pressure increases to 300 000 Pa

3. pressure is held constant while the volume returns to $0.01\,\mathrm{m}^3$

4. volume is held constant while the pressure returns to 100 000 Pa

We can find the amount of work that needs to be done on the gas in one cycle and the efficiency of the system at removing heat from the cold side.

### Graphics
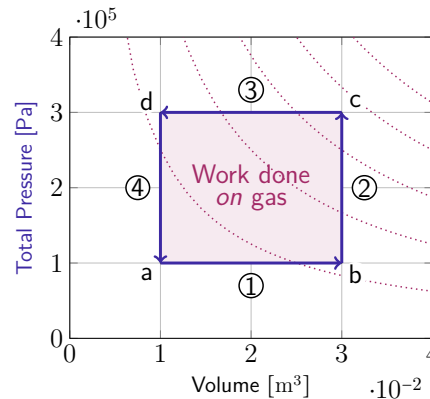


Figure 12.20: P-V plot for a gas going through a cycle. Since the path goes counter-clockwise, the area inside the loop shows the amount of work done *on* the gas as it completes one cycle. [1]
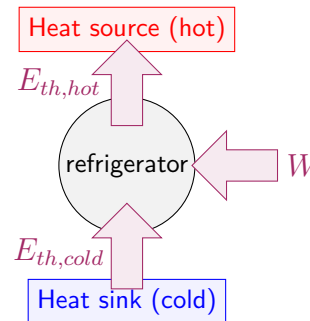


Figure 12.21: Energy flow through a refrigerator.[1]

### Numbers

**Assumptions:** ideal monatomic gas

| Knowns | Unknowns |
|---|---|
| $n = 0.4\,\mathrm{mol}$ | $W_{net}$ |
| $V_a = V_d = 0.01\,\mathrm{m}^3$ | $e$ |
| $V_b = V_c = 0.03\,\mathrm{m}^3$ | |
| $P_a = P_b = 1 \times 10^5\,\mathrm{Pa}$ | |
| $P_c = P_d = 3 \times 10^5\,\mathrm{Pa}$ | |

As in Section 12.6, the work for each of the four processes can be added together to get the total work done for one cycle:

$$W = W_1 + \cancelto{0}{W_2} + W_3 + \cancelto{0}{W_4}$$
$$= (P_a \cdot (V_b - V_a)) + (P_c \cdot (V_d - V_c))$$
$$= (2\,000\,\mathrm{J}) - (6\,000\,\mathrm{J}) = -4\,000\,\mathrm{J}$$

The amount of work done by the gas is negative–in other words, 4 000 J of work was done *on* the gas.

A general expression for efficiency that can be used in any situation is

$$e = \frac{\text{What we want}}{\text{What we put in}} \tag{12.11}$$

260

The efficiency of any machine depends on what it is made to do and on what has to be put in to make it work. One of the most common examples of efficiency is related to vehicles. When buying a vehicle, many people want to know its fuel efficiency. Vehicles are designed to move–that is their primary purpose–so distance moved is part of the efficiency. And if the vehicle uses gasoline or diesel fuel then in order to make the vehicle move, fuel has to be put in. So the amount of fuel is also part of the efficiency. The most efficient vehicles will go the longest distance with the smallest amount of fuel. In US customary units, we want a lot of miles from few gallons of fuel, so we want a large number of miles per gallon, "mpg."

For a heat engine we want a large amount of work to be done compared to the heat that goes in. Both work and heat are measured in joules, so the units are joules/joule. In other words, the efficiency is unitless. So energy efficiency is often given as a number or a percentage.

For a refrigerator, we want to remove a large amount of heat compared to the work that goes in, so again the efficiency is unitless. For a refrigerator, the efficiency is often called the "coefficient of performance."

Isobaric and isovolumetric are not the only options for thermodynamic processes. Two other common processes are "isothermal," in which the temperature of the gas is held constant, and "adiabatic," in which there is no heat transferred into or out of the gas. Machines that use isothermal and adiabatic processes are usually more efficient than those that use isobaric and isovolumetric processes.
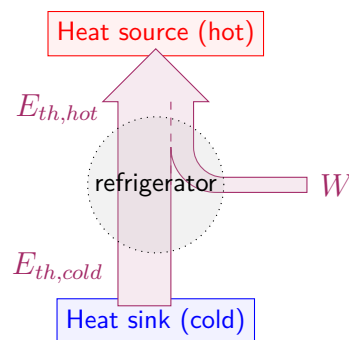


Figure 12.22: Another way to draw the energy flow through a refrigerator. This drawing shows that both the work and the heat from the cold side are transferred as heat to the hot side.[1]
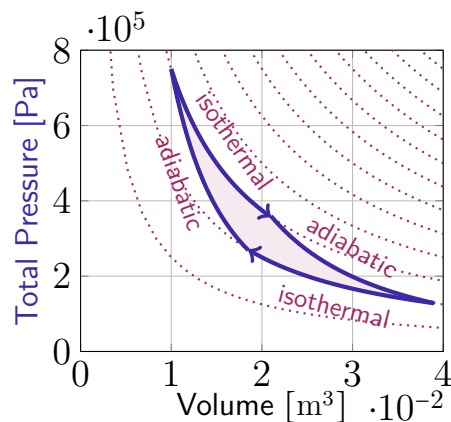


Figure 12.23: P-V plot for a gas going through a clockwise cycle of isothermal and adiabatic processes. This path maximizes the efficiency of a heat engine. [1]

In the case of a refrigerator, we want to remove heat from the heat sink and we put in work. Using the numbers from calculations of the change in thermal energy for each part of the cycle from Section 12.6, the efficiency of this refrigerator is

$$e = \frac{14\,000 \text{ J}}{4\,000 \text{ J}} = 3.5$$

Converting to a percentage gives 350%! The number can be more than 100% because in this case none of the energy that we put in is actually going where we want it to go–all of it is going into the heat source. And as it does that, it draws an additional amount of heat out of the heat sink and sends it into the heat source as well. This is shown in Figure 12.22. In a situation like this, the efficiency of a system is often instead called the "coefficient of performance."

In the case of a heat engine, we want work and we put in heat from the heat source. The maximum efficiency can be achieved using isothermal and adiabatic processes in what is called a "Carnot cycle." That maximum efficiency depends on the temperatures (in K) of the heat source and the heat sink:

$$e_{\text{max}} = 1 - \frac{T_{\text{cold}}}{T_{\text{hot}}} \tag{12.12}$$

For the heat source and heat sink in Section 12.6, a Carnot engine would have an efficiency of

$$e_{\text{max}} = 1 - \frac{301 \text{K}}{2709 \text{K}} = 89\%$$

## 12.8   Entropy

### Words

Potential energy stored in a spring can transform into kinetic energy of a mass. And the kinetic energy of that mass can transform back into spring potential energy. We have also seen that thermal energy can flow from a hot skillet into cold water, cooling the skillet while heating the water. If the skillet and water behaved in the same way as the spring and mass, the flow of thermal energy could then reverse, returning the skillet to its original high temperature and the water to its original cold temperature. But *entropy* prevents this from happening; it is an irreversible process.

"Entropy" can be understood in terms of temperature and heat or in terms of a number of available states. Systems with higher temperature or more available states generally have higher entropy. Entropy is often described as the amount of disorder in a system, but "disorder" is difficult to identify or even define. In the specific example of solids, liquids, and gases it can be said that solids are held in place in an ordered structure; liquids can move around but the molecules are still close to one another so they are less ordered; the molecules of gases are free to move around so they have the least order. As such, matter in liquid form tends to have more entropy than matter in solid form, and matter in gaseous form tends to have more entropy than matter in liquid form.

If you have a driveway with one parking space, and you have one car, then the available states for

### Graphics
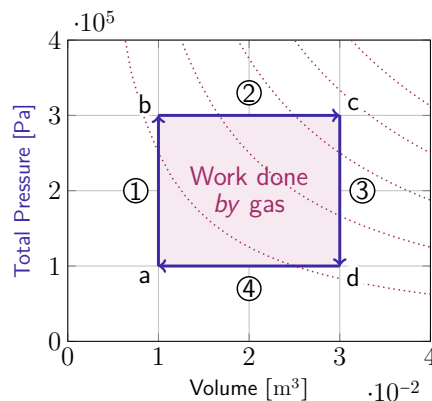


Figure 12.24: P-V plot for a gas going through one complete cycle in four numbered steps. First from "a" to "b," then "b" to "c," "c" to "d," and finally back to "a." [1]

### Numbers

The change in entropy $\Delta S$ of a system that is at constant temperature is given by

$$\Delta S = \frac{\Delta E_{th}}{T} \qquad (12.13)$$

We can use this relationship to find the change in entropy created by the heat engine from Section 12.6 as it goes through one cycle.

**Assumptions:** ideal monatomic gas; heat source and heat sink with constant temperatures

| Knowns | Unknowns |
|---|---|
| $n = 0.4\,\mathrm{mol}$ | $\Delta S$ |
| $V_a = V_b = 0.01\,\mathrm{m}^3$ | |
| $V_c = V_d = 0.03\,\mathrm{m}^3$ | |
| $P_a = P_d = 1 \times 10^5\,\mathrm{Pa}$ | |
| $P_b = P_c = 3 \times 10^5\,\mathrm{Pa}$ | |
| $\Delta E_{th,1} = 3\,000\,\mathrm{J}$ | |
| $\Delta E_{th,2} = 15\,000\,\mathrm{J}$ | |
| $\Delta E_{th,3} = -9\,000\,\mathrm{J}$ | |
| $\Delta E_{th,4} = -5\,000\,\mathrm{J}$ | |

The entropy of the heat engine is the same at the beginning as it was at the end, since the pressure, temperature, volume, and number of molecules is the same at the end as it was at the beginning. So to find the change in entropy we need to consider

the car/driveway system are either (1) your car is parked in the driveway or (2) it isn't. If you have a car and a truck, then there are more available states–now either your car, your truck, or nothing can be parked in the driveway. If you have an entire parking lot then there are more possible places for either vehicle to park. So entropy increases when you have more types of vehicle or more spaces.

In a gas, calculation of the number of states is complicated–possible states are affected, for example, by the number and types of gas molecules, their possible momenta (which is related to temperature) and their possible positions (which is related to volume).

Back to the water and skillet, as they came to equilibrium temperature the entropy of the skillet decreased, but the entropy of the water increased even more. So the total entropy of the system is higher when their temperatures are the same. And the flow of energy stops when entropy is at a maximum. There is no physical process that can reduce the entropy of any isolated system. In an ideal situation a physical process could possibly result in no change in entropy. But ideal situations don't actually exist in the real world. So every possible physical process results in an increase in entropy.

Since the universe can be considered an isolated system, that means the entropy of the universe is always increasing. Unless an interaction with something outside of the physical universe intervenes, the entropy of the universe will eventually reach a maximum. At that point, all meaningful activity will stop–this is called the "heat death" of the universe.
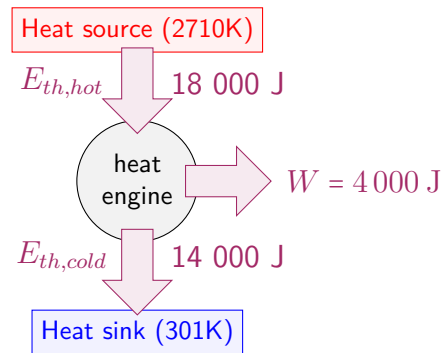


Figure 12.25: Diagram of energy flow through a heat engine.[1]



Figure 12.26: Stars contain vast amounts of energy, but if allowed to continue burning for a few hundred million years they will eventually run out of fuel. Their temperature would reach the same as that of their surroundings, entropy would be at a maximum, and no more useful energy would be available for any meaningful activity. This is called "heat death." [2]

the entropy of the heat source and the heat sink. Heat is flowing into the heat engine when $\Delta E_{th}$ is positive, so during the first and second processes. Heat is flowing out during the third and fourth processes. Since the heat is flowing in from the heat source and out to the heat sink…

$$\Delta E_{th,hot} = \Delta E_{th,1} + \Delta E_{th,2} = 18\,000 \text{ J}$$

$$\Delta E_{th,cold} = \Delta E_{th,3} + \Delta E_{th,4} = -14\,000 \text{ J}$$

That will allow us to find the change in entropy of the heat source and the heat sink if we can find their temperatures. For heat to flow into the engine from the heat source, the temperature of the heat source has to be higher than the temperature of the gas, which is highest at point c. We can find it by rearranging the ideal gas law:

$$T_{source,min} = \frac{P_c \cdot V_c}{n \cdot R} = 2710\text{K}$$

Similarly,

$$T_{sink,max} = \frac{P_a \cdot V_a}{n \cdot R} = 301\text{K}$$

If the heat source is at its minimum possible temperature then the change in entropy of the heat source is…

$$\Delta S_{source} = \frac{-\Delta E_{th,hot}}{T_{source,min}} = -6.6 \text{ J/K}$$

The negative sign is because we want heat leaving the source, not heat entering the engine. Similarly,

$$\Delta S_{sink} = \frac{-\Delta E_{th,cold}}{T_{sink,max}} = 46.5 \text{ J/K}$$

So this heat engine creates a net entropy of at least 39.9 J/K for every 4 000 J of output work.

## 12.9  Summary

Chapter summaries in this book are ordered by concept, not necessarily in the order in which they are presented in the chapter. Mathematical models are grouped together at the end of each summary. See the appendices for the meanings of all symbols used in this book.

### General

- A mole is a number: $6.02 \times 10^{23}$. Gas molecules are often counted in moles.

- The volume of a gas does not change during an isovolumetric process.

- Every possible physical process increases the entropy of an isolated system.

### Forces

- The pressure of a gas does not change during an isobaric process.

### Motion

- (Nothing!)

### Momentum

- (Nothing!)

### Energy

- Heat is another name for thermal energy.

- Thermal energy is related to the kinetic energy of atoms moving randomly on a microscopic level.

- Heat often travels by conduction through solids and liquids. In conduction, it is microscopic vibrations, not molecules, that carry heat through a material.

- Heat often travels by convection through fluids. In convection, it is the bulk motion of material caused by gravity and density changes, that carries heat through a material.

- The lengths of most solids increase with temperature.

- The volumes of most liquids increase with temperature.

- Heat flows from regions of higher temperature to regions of lower temperature unless something is actively working on the system to prevent or even reverse the flow of heat.

- Thermal equilibrium is a state in which heat is not flowing, so all objects are at the same temperature.

- The change in the temperature of a material when heat is added to it is described by the specific heat capacity of the material.

- The melting point of a material is the temperature at which a material in the solid state can melt into the liquid state, and a material in the liquid state can freeze into the solid state.

- The boiling point of a material is the temperature at which a material in the liquid state can evaporate into the gas state, and a material in the gas state can condense into the liquid state.

- The amount of heat required to melt a solid is called the latent heat of fusion. The same amount of heat must be removed from a liquid to freeze it into a solid.

- The amount of heat required to evaporate a liquid is called the latent heat of vaporization. The same amount of heat must be removed from a gas to condense it into a liquid.

- The internal energy of a gas includes the translational and rotational kinetic energies of the molecules and also the potential energy related to vibrations of molecules that have more than one atom.

- When heat goes into a gas that is held at constant volume, all of the heat is transformed into the internal energy of the gas.

- The total amount of heat that goes into a gas is equal to the sum of the change in the internal energy of the gas and the work done by the gas.

- The area under the path on a pressure-vs-volume graph is the amount of work done by the gas if the path goes from left to right, or the amount of work done on the gas if the path goes from right to left.
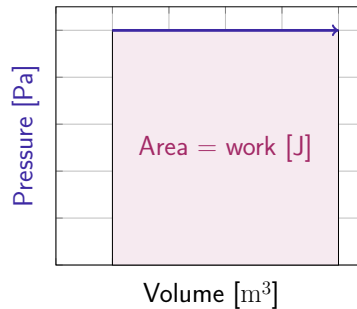


Pressure vs volume graph showing work done *by* the gas[1]

- A heat engine is able to convert thermal energy into usable work while transferring thermal energy from a high-temperature heat source to a low-temperature heat sink.

- No heat engine can be 100% efficient at converting thermal energy into work.

- The energy flow through a heat engine is often drawn schematically:
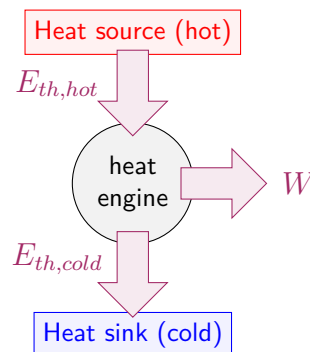


Diagram of energy flow through a heat engine[1]

- Work is done on a gas in a thermodynamic cycle if the path goes counterclockwise on a P-V plot, and work is done by the gas if the path goes clockwise. The amount of work is the area enclosed by the path.

- The temperature of a gas does not change during an isothermal process.

- No heat is transferred into or out of a gas during an adiabatic process.

## Mathematical Models

| equation | | restrictions on the validity of the equation |
|---|---|---|
| $\Delta E_{th} = m \cdot c \cdot \Delta T$ | (12.1) | (approximation over a limited temperature range) |
| $\Delta l = l \cdot \alpha \cdot \Delta T$ | (12.2) | (approximation over a limited temperature range) |
| $\Delta T_K = \Delta T_C = \frac{5}{9}\left(\Delta T_F\right)$ | (12.3) | -none- |
| $\Delta V = V \cdot \beta \cdot \Delta T$ | (12.4) | (approximation over a limited temperature range) |
| $E_{th} = m \cdot L_f$ | (12.5) | -none- |
| $E_{th} = m \cdot L_v$ | (12.6) | -none- |
| $P_{tot} \cdot V = n \cdot R \cdot T$ <br><br> "Ideal gas law" | (12.7) | ideal gas |
| $U_{int} = \frac{3}{2}n \cdot R \cdot T = \frac{3}{2}N \cdot k_B \cdot T$ | (12.8) | ideal monatomic gas |
| $W = P \cdot \Delta V$ | (12.9) | ideal gas, constant pressure |
| $\Delta E_{th} = \Delta U_{int} + W$ | (12.10) | -none- |
| $e = \frac{\text{What we want}}{\text{What we put in}}$ | (12.11) | -none- |
| $e_{\max} = 1 - \frac{T_{\text{cold}}}{T_{\text{hot}}}$ | (12.12) | Ideal Carnot engine |
| $\Delta S = \frac{\Delta E_{th}}{T}$ | (12.13) | constant temperature |

## 12.10 Questions

Questions are ordered according to Bloom's Taxonomy, progressing from regurgitating information (Level 1) to synthesizing new information with previous knowledge to create something new (Level 6). The bold letters at the beginning of each question indicate whether the question involves Words **[W]**, Graphics **[G]**, and/or Numbers **[N]**. See the appendices for conversion factors.

### Level 1 - Remember

12.1 **[W]** What is another name for heat?

12.2 **[W & N]** In what units is heat measured?

12.3 **[W & N]** Add labels to each equation in the "Mathematical Models" section of the summary that tell what the symbol to the left of the = sign represents.

### Level 2 - Understand

12.4 **[W]** What is the difference between heat and temperature?

12.5 **[N]** An assumption was made in Section 12.1 that the iron would remain solid over the entire temperature range. Look up the temperature at which iron melts. Is the assumption valid for this physical scenario?

12.6 **[W]** Can convection occur inside a solid material? Explain your answer.

12.7 **[W & N]** $0°C$ is the temperature at which ice melts. What is the temperature at which water freezes?

12.8 **[W & N]** $100°C$ is the temperature at which water boils. What is the temperature at which water vapor condenses?

12.9 **[W]** Describe the difference between latent heat of fusion and latent heat of vaporization.

12.10 **[W]** The size and shape of a container is held constant while the pressure of the gas inside the container increases. What type of process is this (adiabatic, isobaric, isothermal, or isovolumetric)?

12.11 **[W]** The pressure inside a container is held constant while the temperature of the gas inside the container increases. What type of process is this (adiabatic, isobaric, isothermal, or isovolumetric)?

12.12 **[W]** The temperature of a gas is held constant while the volume of the container increases. What type of process is this (adiabatic, isobaric, isothermal, or isovolumetric)?

12.13 **[W]** A container is heavily insulated to prevent heat from entering or leaving it. The volume of the container increases. What type of process is this (adiabatic, isobaric, isothermal, or isovolumetric)?

12.14 **[W]** List five things you have done today that have resulted in a net increase in the entropy of the universe. (Don't think about this one for too long–thinking requires a great deal of electrical activity in the brain, which increases the entropy of the universe!)

12.15 **[W]** List everything you have done today that has resulted in a net decrease in the entropy of the universe.

## Level 3 - Apply

12.16 ✚ **[W]** This question has multiple parts about three systems "1," "2," and "3." . Step through these parts, and you will discover what is known as the "Zeroth Law of Thermodynamics."

(a) System 1 is brought into thermal contact with System 2, and it is found that no heat transfers between the two systems when they are brought into contact with each other. Are they in thermal equilibrium with each other?

(b) System 1 and System 2 are separated and then System 2 is brought into thermal contact with System 3. It is found that no heat transfers between them. Are they in thermal equilibrium with each other?

(c) System 2 and System 3 are separated. If System 3 were now brought into thermal contact with System 1, would you expect to find that they are in thermal equilibrium with each other?

(d) If you answered yes to all of the above parts then you just discovered the Zeroth Law of Thermodynamics! Incidentally, what does this tell you about the temperatures of these three systems?

12.17 **[N]** Assuming that the coefficient of linear thermal expansion that is used in Section 12.1 is valid for any temperature, what would the diameter of the skillet be at a temperature of absolute zero? What would the diameter be at the melting point of iron?

12.18 **[N]** The final temperature of cold water poured into a hot skillet was analyzed in Section 12.2. How much $10°C$ water would you need to pour into the $90°C$ skillet so that the final temperature of the water and skillet would be $50°C$?

12.19 **[N]** The final temperature of 1.5 kg of $10°C$ water poured into a hot skillet was analyzed in Section 12.2. What if instead of water 1.5 kg of $10°C$ lead had been placed in the hot skillet? What would the final temperature be in that situation? You will need to look up the specific heat capacity of lead to answer this question.

12.20 **[N]** Practice converting temperature changes between the various scales mentioned in Section 12.2:

(a) What is a change of 273K in degrees Celsius?

(b) What is a change of $0°C$ in degrees Fahrenheit?

(c) What is a change of $10°F$ in degrees Celsius?

12.21 **[G & N]** The scenario in Section 12.3 starts with ice at a temperature of $-20°C$ and ends with water half boiled away. Re-do the calculations for the energy needed, starting at *absolute zero* and ending when *all* of the water has boiled away.

12.22 **[W, G, & N]** In Section 12.4, an ideal monatomic gas was assumed. In fact, air compressors are generally filled with air, which is mostly composed of diatomic gases. Re-do the analysis using the same volume, temperatures, and initial pressure, but assuming that the air compressor is filled with a diatomic gas instead of a monatomic gas. Do the values change for $N$, $P_{f,tot}$, and $\Delta E_{th}$?

12.23 **[N]** The final volume is never calculated in Section 12.5, although it is shown in the P-V plot. Find a way to calculate it from the information in the numbers column. Does your answer match reasonably well with the value in the P-V plot?

12.24 **[G & N]** Use the P-V plot in Section 12.5 to find the work done by the gas in the cylinder. The work isn't actually calculated in the numbers section of this chapter. Find a way to use the information there to calculate the work and then compare with the answer that you got using the graph.

12.25 **[N]** What is the maximum efficiency possible for a heat engine that is operating between a heat source at $100°$ and a heat sink at $0°$?

12.26 **[N]** What is the change in entropy of 0.5 kg of ice at $0°C$ ice melting completely into $0°C$ water?

## Level 4 - Analyze

12.27 **[W & N]** By what percentage did the height and the diameter of the iron skillet change in Section 12.1? Did the volume change by the same percentage? Explain why or why not.

12.28 **[N]** The distance between the parallel rails of a standard train track is 1.4 meters. Use that information to estimate the length of the horizontal gap in the expansion joint in the bottom rail of Figure 12.5. Look up the coefficient of linear thermal expansion of steel, and also search for the maximum and minimum outdoor temperatures where you live. If this type of expansion joint were used in railroad tracks where you live, what is the maximum possible length of each solid piece of rail that will not exceed the limits of the expansion joints?

12.29 **[N]** An iron skillet containing water is analyzed in Sections 12.1 & 12.2. What would happen if the same skillet, at $10°C$, was filled with water, also at $10°C$, and then this system was heated? Both the iron and the water would expand. But which would expand faster? Would the water overflow from the skillet, or would the water level actually drop?

12.30 **[W, G, & N]** The scenario in Section 12.3 assumes that all of the heat is going into the water and the ice. But the ice and water must have been in some kind of container. Re-do the analysis for the same situation, but including a 2-kg copper pot that is holding the ice and water. You will need to look up the specific heat capacity of copper. Assume that the copper pot starts at $-20°C$ and ends at $100°C$. Does including the copper pot significantly change the amount of heat required?

12.31 **[W & N]** Consider the two different mathematical models of the internal energy that are given in Equation 12.8, one after each equals sign. Use these mathematical models to find the relationship between $k_B$ and $R$. Describe whether that relationship is supported by the units and numerical values for these two constants.

12.32 **[W & N]** Gases don't have simple expressions for specific heats like solids and liquids do. But sometimes it is convenient to find an expression of specific heat that will work for a gas in a given situation. For example, the specific heat of an ideal gas at constant volume $C_v$ or at constant pressure $C_p$, both symbolized with an uppercase $C$ instead of a lowercase $c$. Use the information given in the numbers columns to find the values of $C_v$ and $C_p$ for ideal monatomic gases *and* for ideal diatomic gases. Show your work. The associated mathematical models are of the following form:

$$\Delta E_{th} = n \cdot C_v \cdot \Delta T$$

and

$$\Delta E_{th} = n \cdot C_p \cdot \Delta T$$

12.33 **[G & N]** Figure 12.21 could just as easily have been for a heat pump whose primary purpose is to increase the temperature of the heat source. What would be the efficiency of that heat pump?

12.34 **[G]** Make an alternative diagram of energy flow like that in Figure 12.22 for Figure 12.19.

12.35 ✚ **[N]** Boyle's Law, which states that $P \cdot V$ is constant for an ideal gas in a sealed container when the temperature is held constant, is a consequence of the ideal gas law:

$$P \cdot V = n \cdot R \cdot T$$
$$P \cdot V = (\text{constant}) \cdot (\text{constant}) \cdot (\text{constant})$$
$$P \cdot V = (\text{constant})$$

$n$ is constant, since the container is sealed. $R$ is always a constant, and if $T$ is also held constant, then the entire right side of the equation is constant. Boyle's Law. There are two other named laws that are also consequences of the ideal gas law. Do the same sort of analysis using the ideal gas law to determine each of the following:

(a) "Charles' Law," which is valid for an ideal gas in a sealed container when the pressure is held constant.

(b) "Avogadro's Law," which is valid for an ideal gas in an *unsealed* container when the pressure and temperature are both held constant

## Level 5 - Evaluate

12.36 **[W, G, & N]** In Section 12.1 the linear expansion of iron was used to find measurements. But the volume that was calculated was actually an empty space inside an iron container. If the iron expands uniformly, would that make the empty inside volume larger or smaller? Explain your reasoning.

12.37 **[W & N]** In Section 12.2, water is described as being unusual because at some temperatures it has a negative coefficient of bulk thermal expansion. How unusual is this property? Do an internet search for other materials that also have a negative coefficient of bulk thermal expansion. Are there any other common materials with this property? Does there seem to be much engineering or scientific interest in this topic?

12.38 **[N]** Find a table of specific heat capacities for various materials. Can you find any general trends? Do some types of materials generally have higher specific heat capacities than some other types of materials?

12.39 **[N]** Find tables of latent heats for various materials. Can you find any general trends? Is latent heat of fusion generally higher or lower than latent heat of vaporization?

12.40 **[W & N]** 1 kg of liquid gold at its melting point is dropped into a bucket of 5 kg of liquid water at its melting point. What is the temperature of the water-gold system when it reaches thermal equilibrium, assuming that the system is thermally isolated from its surroundings? In what state (solid, liquid, or gas) are the water and the gold at thermal equilibrium? You will need to look up the thermal properties of gold to answer this question.

12.41 **[G & N]** When P-V plots were introduced in Figure 12.14, the figure showed the relationship between pressure, volume, and temperature for an ideal gas. There is a necessary assumption that was made but not mentioned when the graph was created. Think about the ideal gas law in relation to PV plots. What unmentioned assumption is being made in the plots?

12.42 **[W, G, & N]** Can the efficiency of a heat pump be greater than 100%? Explain your answer.

## Level 6 - Create

12.43 **[W, G, & N]** At the beginning of Chapter 1 in Figure 1.1 was a template for a concept map. Add the main ideas from this chapter to the appropriate concept maps for earlier chapters.

12.44 **[W, G, & N]** Imagine you are writing a test question related to this chapter. Think of your own example of a situation that you can analyze using the concepts, graphics, and mathematical analyses described in this chapter. Describe the situation, and use the tools from this chapter to analyze the situation as completely as you can, including motion, forces, energy, and momentum.

12.45 **[W, G, & N]** Think about possible misconceptions about the material in this chapter. Write a question and an *incorrect* solution to it that demonstrates a student making such a conceptual error. *This cannot be a simple misuse of a vocabulary word, a unit error, or a mathematical error* like making an addition error or multiplying when addition was needed, unless the error is rooted in a real misunderstanding about the physics behind the calculation or the misuse of a word. After you have written the question and incorrect solution, explain what is wrong with the student's solution, and write a correct solution to the problem. *Note: You may use a question from this chapter that you got wrong the first time, and explain the initial error in your thinking and how you corrected it.*

# Appendix A

# Symbols, Subscripts, & Abbreviations

| Symbol | Quantity | SI Unit | |
|---|---|---|---|
| $a$ | acceleration (magnitude or component) | meters per second squared | $\left[\mathrm{m/s^2}\right]$ |
| $\vec{a}$ | acceleration (vector) | meters per second squared | $\left[\mathrm{m/s^2}\right]$ |
| **a** | [attraction between particles] | newton-meters to the 4th power | $\mathrm{N \cdot m^4}$ |
| $A$ | area | square meters | $\left[\mathrm{m^2}\right]$ |
| $\mathcal{A}$ | amplitude | (whatever is being measured) | (any) |
| **b** | [volume of particles] | meters cubed per mole | $\mathrm{m^3/mol}$ |
| $c$ | specific heat capacity | joules per kilogram - degree Celsius | $\left[\mathrm{J/\left(kg \cdot {}^\circ C\right)} \quad \text{or} \quad \mathrm{J/\left(kg \cdot K\right)}\right]$ |
| $e$ | efficiency | usually a decimal or percentage | (usually none) |
| $E$ | energy | joules | [J] |
| $\hat{E}$ | the East direction | -none- | |
| $f$ | frequency | hertz | [Hz] |

| Symbol | Quantity | SI Unit | |
|---|---|---|---|
| $F$ | force (magnitude or component) | newtons | [N] |
| $\vec{F}$ | force (vector) | newtons | [N] |
| $g$ | acceleration of gravity at earth's surface | (this is a constant) | $9.8\,\mathrm{m/s^2}$ |
| $h$ | height | meters | [m] |
| $I$ | rotational inertia | kilogram meters squared | $\left[\mathrm{kg \cdot m^2}\right]$ |
| $k_B$ | Boltzmann constant | (this is a constant) | $1.38 \times 10^{23}\ \mathrm{J/K}$ |
| $k_s$ | spring constant | newtons per meter | [N/m] |
| $l$ | length | meters | [m] |
| $L$ | latent heat | joules per kilogram | [J/kg] |
| $L$ | angular momentum | kilogram meters squared per second | $\left[\mathrm{kg \cdot m^2/s}\right]$ |
| $m$ | mass | kilograms | [kg] |
| $\mathbf{m}$ | an integer (0, ±1, ±2, *etc.*) | -none- | |
| $MA$ | mechanical advantage | -none- | |
| $n$ | an ordinal number (like 2 for 2nd) | -none- | |
| $N$ | a large number | -none- | |
| $N_A$ | Avogadro's number | (this is a constant) | $6.02 \times 10^{23}$ |
| $\hat{N}$ | the North direction | -none- | |
| $p$ | momentum (magnitude or component) | kilogram-meters per second | $\left[\mathrm{kg \cdot m/s}\right]$ |
| $\vec{p}$ | momentum (vector) | kilogram-meters per second | $\left[\mathrm{kg \cdot m/s}\right]$ |
| $P$ | power | watts | [W] |
| $P$ | pressure | Pascals | [Pa] |
| $r$ | radius or distance from center | meters | [m] |
| $R$ | range | meters | [m] |

| Symbol | Quantity | SI Unit | |
|---|---|---|---|
| $s$ | path length | meters | [m] |
| $S$ | entropy | joules per kelvin | [J/K] |
| $SPL$ | Sound Pressure Level | decibels | [dB] |
| $t$ | time | seconds | [s] |
| $T$ | period | seconds | [s] |
| $T$ | temperature | degrees Celsius or kelvin | [°C] or [K] |
| $U$ | potential energy | joules | [J] |
| $v$ | speed or component of velocity | meters per second | [m/s] |
| $\vec{v}$ | velocity | meters per second | [m/s] |
| $W$ | work | joules | [J] |
| $x$ | horizontal position or component of $\vec{x}$ | meters | [m] |
| $\vec{x}$ | position (vector) | meters | [m] |
| $\hat{x}$ | the $x$ direction | -none- | |
| $y$ | vertical position | meters | [m] |
| $\hat{y}$ | the $y$ direction | -none- | |
| $Y$ | Young's modulus | newtons per meter | [N/m] |
| $\alpha$ | angular acceleration | radians per second squared | $\left[\text{rad/s}^2\right]$ |
| $\alpha$ | coefficient of linear thermal expansion | reciprocal kelvin or reciprocal degrees Celsius | $\left[\text{K}^{-1} \quad \text{or} \quad °\text{C}^{-1}\right]$ |
| $\beta$ | coefficient of bulk thermal expansion | reciprocal kelvin or reciprocal degrees Celsius | $\left[\text{K}^{-1} \quad \text{or} \quad °\text{C}^{-1}\right]$ |
| $\Delta$ | "Change in …" | -none- | |
| $\overrightarrow{\Delta x}$ | displacement | meters | [m] |
| $\theta$ | angle | degrees or radians | [° or rad] |
| $\lambda$ | wavelength | meters | [m] |

| Symbol | Quantity | SI Unit | |
|--------|----------|---------|---|
| $\mu_k$ | coefficient of kinetic friction | -none- | |
| $\mu_m$ | linear mass density | kilograms per meter | $[\mathrm{kg/m}]$ |
| $\mu_s$ | coefficient of static friction | -none- | |
| $\mu_v$ | viscosity | -none- | |
| $\rho_m$ | mass density | kilograms per cubic meter | $[\mathrm{kg/m^3}]$ |
| $\sum$ | "Sum of …" | -none- | |
| $\tau$ | torque | newton meters | $[\mathrm{N \cdot m}]$ |
| $\phi$ | phase | degrees or radians | [° or rad] |
| $\Phi_m$ | volumetric flux | cubic meters per second | $[\mathrm{m^3/s}]$ |
| $\varnothing$ | diameter | -none- | |
| $\omega$ | angular velocity | radians per second | $[\mathrm{rad/s}]$ |
| $\equiv$ | "is defined as…" | -none- | |
| $\hat{}$ | "in the …direction" | -none- | |
| $\rightarrow$ | (Indicates a vector quantity) | -none- | |

| Subscript | Meaning |
|-----------|---------|
| $0$ | at time $t = 0$ |
| $0$ | reference level (for sound pressure) |
| $1, 2, etc.$ | of object #1, #2, etc. |
| $avg$ | average |
| $atm$ | atmospheric |
| $b$ | buoyant |

| Subscript | Meaning |
|---|---|
| $c$ | centripetal |
| $C$ | in degrees Celsius |
| $com$ | Center of Mass |
| $E$ | East |
| $f$ | friction (for force) |
| $f$ | fusion (for latent heat) |
| $f$ | final (for everything except force and latent heat) |
| $F$ | in degrees Fahrenheit |
| $g$ | gravitational |
| $i$ | initial |
| $int$ | internal |
| $k$ | kinetic |
| $K$ | in kelvin |
| $m$ | mass |
| $max$ | maximum |
| $mech$ | mechanical |
| $n$ | normal (for force) |
| $n$ | $n^{\text{th}}$ ($f_3$ is for the 3$^{\text{rd}}$ harmonic, for example) |
| $N$ | North |
| $net$ | net (total) |
| $r$ | rotational |
| $s$ | static (for frictional force) |
| $s$ | sound (for pressure) |

| Subscript | Meaning |
|---|---|
| $s$ | spring |
| $t$ | tension |
| $T$ | tangential |
| $th$ | thermal |
| $tot$ | total |
| $v$ | viscosity (with $\mu$) |
| $v$ | vaporization (with $L$) |
| $V$ | volumetric |
| $x$ | in the $\hat{x}$ direction |
| $y$ | in the $\hat{y}$ direction |
| $\parallel$ | in the parallel direction |
| $\perp$ | in the perpendicular direction |
| $\rightarrow$, for example "$1 \rightarrow 2$" | of the first object acting on the second |
| $\leftarrow$, for example "$1 \leftarrow 2$" | of the first object as seen by the second |

| Abbreviation | Meaning |
|---|---|
| COM (or com) | Center of Mass |
| FBD | Free Body Diagram |
| $MA$ | Mechanical Advantage |
| SHM | Simple Harmonic Motion |

# Appendix B

# Conversion Factors & Metric Prefixes

| | SI Unit = US Customary Unit |
|---|---|
| **Length** | 0.0254 m = 1 inch (in) |
| | 1 m = 3.28 feet (ft) |
| | 1 609 m = 1 mile (mi) |
| **Mass** | 14.6 kg = 1 slug |
| | 1 kg = the mass of an object that weighs 2.2 lb on earth |
| **Volume** | 1 m$^3$ = 1 000 liters (l) |
| **Speed** | 1 m/s = 2.24 miles per hour (mph) |
| **Force** | 1 N = 0.225 pound (lb) |
| **Pressure** | 133 Pa = 1 millimeter of mercury (mm Hg) |
| | 6 895 Pa = 1 pound per square inch (psi) |
| | $1.01 \times 10^5$ Pa = 1 atmosphere (atm) |
| **Energy** | 4.186 J = 1 calorie (cal) |
| | 4 186 J = 1 Calorie (Cal) = 1 kilocalorie (kcal) |
| | $3.6 \times 10^6$ J = 1 kilowatt-hour (kWh) |
| | 1 055 J = 1 British thermal unit (Btu) |
| **Power** | 746 W = 1 horsepower (hp) |
| **Angle** | $\pi$ rad = 180° |
| **Angular speed** | $\frac{\pi}{30}$ rad/s = 1 rotation per minute (rpm) |

| Metric prefix | Abbreviation | Meaning |
|---|---|---|
| Tera- | T | $10^{12}$ |
| Giga- | G | $10^9$ |
| Mega- | M | $10^6$ |
| kilo- | k | 1000 |
| centi- | c | 0.01 |
| milli- | m | 0.001 |
| micro- | $\mu$ | $10^{-6}$ |
| nano- | n | $10^{-9}$ |
| pico- | p | $10^{-12}$ |
| femto- | f | $10^{-15}$ |

# Appendix C

# Physical Constants

| Symbol | Description | Approximate value |
|:---:|:---:|:---:|
| $g$ | acceleration of gravity at the earth's surface | $9.81 \, \text{m/s}^2$ |
| $G$ | Newton's Universal Gravitation Constant | $6.67 \times 10^{-11} \, \text{N} \cdot \text{m}^2/\text{kg}^2$ |
| $k_B$ | Boltzmann constant | $1.38 \times 10^{-23} \, \text{J/K}$ |
| $N_A$ | Avogadro's number | $6.02 \times 10^{23}$ |
| $R$ | Ideal gas constant | $8.31 \, \frac{\text{J}}{\text{K} \cdot \text{mol}}$ |
| $P_{atm}$ | Atmospheric pressure at sea level | $1.01 \times 10^5 \, \text{Pa}$ |

# Appendix D

# Geometrical Shapes

Area: $A = l \cdot w$

Volume: $V = l \cdot w \cdot h$

Area: $A = \frac{1}{2} b \cdot h$

Volume: $V = \pi r^2 \cdot h$

Area: $A = \pi r^2$
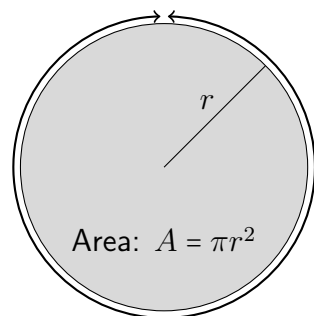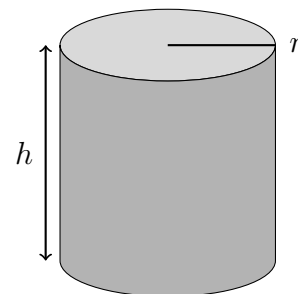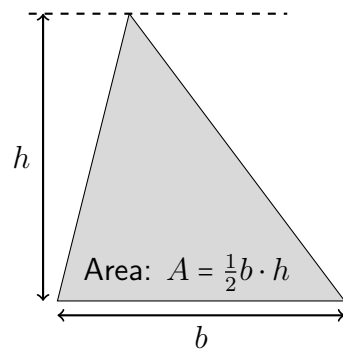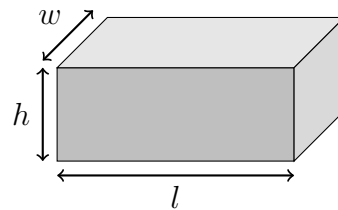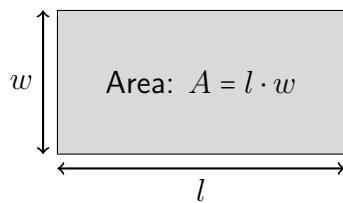
Circumference: $C = 2\pi r$

Volume: $V = \frac{4}{3} \pi r^3$
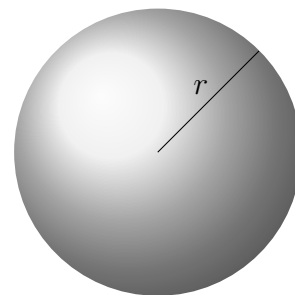
Surface Area: $A = 4\pi r^2$

[]

# Appendix E

# Answers to End-of-Chapter Questions

The following are not intended to be complete answers for any of the end-of-chapter questions. Not all questions are included. For those that are included, these partial answers are only intended to help you know whether you are on the right track.

## Chapter 1

| | |
|---|---|
| **Question 1.16:** | 784 |
| **Question 1.17:** | 0.638 |
| **Question 1.19:** | No |
| **Question 1.21:** | -8 |
| **Question 1.22(b):** | 6.26 |

## Chapter 2

| | |
|---|---|
| **Question 2.11:** | 1.3 |
| **Question 2.12:** | 3.9 |
| **Question 2.13:** | (The graph should feature a straight line.) |
| **Question 2.14:** | (All three graphs should feature a straight line.) |
| **Question 2.15:** | (Two of the three graphs should feature a straight line.) |
| **Question 2.17(b):** | 40.2 |
| **Question 2.17(c):** | 80.4 |
| **Question 2.17(d):** | 8.93 |
| **Question 2.17(e):** | 0 |

| Question 2.18: | (The graph should feature a straight line.) |
| --- | --- |
| Question 2.19: | (Changes are needed in three of the six parts to this question.) |
| Question 2.20: | (Changes are needed in four or five, depending on your reasoning, of the six parts to this question.) |

# Chapter 3

| Question 3.5: | ("Yes" is the answer to four of the six parts to this question.) |
| --- | --- |
| Question 3.6: | 1.67 |
| Question 3.7(b): | 312 & 520 |
| Question 3.9: | 11 |
| Question 3.10: | (All of them are 0, +50, or -50) |
| Question 3.12: | The energy of the ball is not conserved in this scenario. |
| Question 3.13: | 194 000 |
| Question 3.14: | 27 |
| Question 3.15(b & d): | (One of these is "larger" and the other is "smaller.") |
| Question 3.16: | Time and displacement both increase. |
| Question 3.19: | Ignoring friction does not cause a significant error in this calculation. |
| Question 3.20(a): | $1.65 \times 10^4$ |

# Chapter 4

| Question 4.5: | 2 190 |
| --- | --- |
| Question 4.7: | (Remember to include directions.) |
| Question 4.9: | It is not an error in the text. |
| Question 4.10: | 38.7 or 51.3 (depending on the orientation chosen for the angle) |
| Question 4.12: | 3.77 |
| Question 4.13: | 7.95 |
| Question 4.15: | 20 & $1.6 \times 10^4$ |
| Question 4.16: | 10 |
| Question 4.19: | 8609 & either 65.4 or 24.6 |
| Question 4.20: | (One is larger and the other is smaller.) |
| Question 4.21: | Yes, there is such a direction. |
| Question 4.22: | 3 460, 4.5 & 23.4, & 631 |

# Chapter 5

| | |
|---|---|
| **Question 5.10:** | 0 & 79 |
| **Question 5.11:** | 39.2 |
| **Question 5.13:** | It does agree. |
| **Question 5.17:** | (The force of friction remains the same for two out of the four parts to this question, and decreases for the other two.) |
| **Question 5.18:** | Yes, such an angle exists. |
| **Question 5.19:** | (Of the five parts of this question: two increase, one decreases, and two remain the same.) |
| **Question 5.20:** | The amount of energy stored in the spring would decrease. |
| **Question 5.21:** | The amount of energy stored in the spring would increase. |
| **Question 5.25:** | (Your calculation should give a difference that is less than two percent. The actual value will depend on your assumptions.) |

# Chapter 6

| | |
|---|---|
| **Question 6.4:** | (Remember to consider both horizontal and vertical directions.) |
| **Question 6.6:** | This was not an error in the textbook. |
| **Question 6.9:** | (Two planets have a higher tangential velocity than earth.) |
| **Question 6.11:** | (There are four different types of energy that should appear at some point in the bar graphs.) |
| **Question 6.14:** | $2.7 \times 10^{40}$ |
| **Question 6.15:** | $2.7 \times 10^{33}$ |
| **Question 6.16(a):** | $v_f = 15.9 \, \mathrm{m/s}$ |
| **Question 6.16(b):** | $t = 2.12 \, \mathrm{s}$ |
| **Question 6.16(c):** | The horizontal displacement could be larger, the same, or smaller. |

# Chapter 7

| | |
|---|---|
| **Question 7.7:** | 8.1 & 6.48 |
| **Question 7.8:** | 6 000, 62 400, 312 000, & $8.11 \times 10^6$ |
| **Question 7.9:** | (Only one remains constant.) |
| **Question 7.13:** | (It should have the same unit.) |
| **Question 7.15:** | $5.4 \times 10^6$ |

| Question 7.16: | 10.8 |
| --- | --- |
| Question 7.19: | The rotational inertia, final angular momentum, torque, final rotational kinetic energy, work, and applied force should all either increase or decrease. |
| Question 7.20: | The final angular velocity, angular acceleration, rotational inertia, final angular momentum, and torque should all either increase or decrease. |
| Question 7.21: | Yes. |

# Chapter 8

| Question 8.8(a): | $3.65 \times 10^5$ |
| --- | --- |
| Question 8.9(a): | $1.9 \times 10^9$ |
| Question 8.9(e): | $3.5 \times 10^{-2}$ |
| Question 8.10(a): | $1.6 \times 10^{36}$ |
| Question 8.18(c): | $73.5°$ |
| Question 8.19(a): | Yes. |
| Question 8.19(b): | Yes. |

# Chapter 9

| Question 9.10: | $6.67 \times 10^{-7}$ |
| --- | --- |
| Question 9.12: | (Only one beat frequency should be audible) |
| Question 9.15: | $2.5 \times 10^{-4}$ |
| Question 9.17: | (One should be more than 10x faster than the other.) |
| Question 9.19: | (There are two types of energy involved.) |
| Question 9.22: | There should be no forces on any of the pieces. |
| Question 9.24: | It was reasonable to neglect the force of gravity on the thumbtack. |
| Question 9.25(d): | Rebar should be used on the bottom side of the concrete. |
| Question 9.26(d): | Rebar should be used on the top side of the concrete. |

# Chapter 10

| Question 10.14: | $7.36 \times 10^{-3}$ |
| --- | --- |
| Question 10.16(a): | 78.4 |

| | |
|---|---|
| **Question 10.17:** | $P_{1,top} = 3.995 \times 10^5$ Pa; $P_{2,bottom} = 2.127 \times 10^5$ Pa; $P_{3,center} = 4.000 \times 10^5$ Pa; |
| **Question 10.23:** | 139 |
| **Question 10.24(b):** | $4.73 \times 10^{-2}$ |
| **Question 10.24(e):** | 0.464 |
| **Question 10.25:** | Yes. |
| **Question 10.26:** | (Of the four parts of this question, all but one results in a flow rate that is either doubled or decreased by half.) |
| **Question 10.28:** | Increase. |
| **Question 10.31:** | The actual density of garlic can vary between 920 and 990 kg/m³. |
| **Question 10.33(b):** | No. |

# Chapter 11

| | |
|---|---|
| **Question 11.2:** | (Of the four parts to this question, only one can be negative.) |
| **Question 11.5:** | Gauge pressure. |
| **Question 11.7:** | 980 |
| **Question 11.8(d):** | Yes. |
| **Question 11.13:** | Zero. |
| **Question 11.19:** | (The difference should be less than 30%.) |
| **Question 11.21:** | 430 & 370 |
| **Question 11.22(a & b):** | (The total volume of water + air should be 62.5 ml.) |
| **Question 11.26:** | The area for the front tire is $7.88 \times 10^{-4}$ m². |

# Chapter 12

| | |
|---|---|
| **Question 12.6:** | No. |
| **Question 12.17:** | (The diameters found should not differ from the original diameter by more than 3%.) |
| **Question 12.19:** | 80 |
| **Question 12.23:** | $7.98 \times 10^{-2}$ |
| **Question 12.25:** | 27% |
| **Question 12.27:** | Volume does not change by the same percentage as diameter. |
| **Question 12.28:** | (For Cincinnati, Ohio, USA, this comes out to approximately 250 m. The answer will vary based on local climate.) |

| | |
|---|---|
| **Question 12.30:** | Including the pot in the analysis does not significantly affect the total heat required. |
| **Question 12.32:** | ($C_{p,monatomic}$ has the same value as $C_{v,diatomic}$.) |
| **Question 12.40:** | After reaching thermal equilibrium, the water is liquid and the gold is solid. |
| **Question 12.42:** | Yes. |

# Appendix F

# Alignment of Learning Objectives

The learning objectives listed here were collected from the websites of the Ohio Department of Higher Education and the Association of American Medical Colleges on November 10th, 2023.

| Section | MCAT Foundational Concepts | Ohio TAG Learning Objectives |
|---|---|---|
| 1.1 | **4A1.1:** Vectors, components<br>**4A1.4:** Speed, velocity (average and instantaneous)<br>**4A1.5:** Acceleration | |
| 1.2 | **4A1.1:** Units and Dimensions | **I-2a:** Make accurate verbal, graphical, and mathematical descriptions of translational and rotational motion in one and two dimensions<br>**I-2b:** Use algebra and graphical methods to link displacement, velocity, and acceleration |
| 1.3 | **4A1.3:** Vector addition<br><br>**4A3.1:** Vector analysis of forces acting on a point object | **I-2a:** Make accurate verbal, graphical, and mathematical descriptions of translational and rotational motion in one and two dimensions |

| Section | MCAT Foundational Concepts | Ohio TAG Learning Objectives |
|---|---|---|
| | **4A5.2:** Potential Energy: PE = mgh (gravitational, local); PE = $1/2\ kx^2$ (spring) | |
| 1.4 | **4A2.1:** Newton's First Law, Inertia | **I-2a:** Make accurate verbal, graphical, and mathematical descriptions of translational and rotational motion in one and two dimensions |
| | **4A5.1:** Kinetic Energy: KE = $1/2\ mv^2$; units | **I-3a:** Use Newton's laws of motion (1st, 2nd, and 3rd) to explain or predict the motion of translating and rotating objects. |
| | | **I-4b:** Explain or predict the motion of translating and/or rotation of objects in 1D using the conservation of momentum. |
| 1.5 | **4A1.5:** Acceleration | **I-2c:** Solve 1D kinematic problems with constant linear and angular acceleration. |
| | **4A2.2:** Newton's Second Law (F=ma) | **I-3a:** Use Newton's laws of motion (1st, 2nd, and 3rd) to explain or predict the motion of translating and rotating objects. |
| | **4A5.3:** Conservation of energy | **I-4a:** Explain or predict the motion of translating and/or rotating objects using conservation of energy. |
| 2.1 | **4A1.1:** Units and dimensions | **I-2a:** Make accurate verbal, graphical, and mathematical descriptions of translational and rotational motion in one and two dimensions |
| | **4A1.4:** Speed, velocity (average and instantaneous) | |
| 2.2 | **4A2.1:** Newton's First Law, Inertia | **I-2a:** Make accurate verbal, graphical, and mathematical descriptions of translational and rotational motion in one and two dimensions |
| | | **I-3a:** Use Newton's laws of motion (1st, 2nd, and 3rd) to explain or predict the motion of translating and rotating objects. |
| | | **I-4b:** Explain or predict the motion of translating and/or rotation of objects in 1D using the conservation of momentum. |
| 2.3 | **4A1.4:** Speed, velocity (average and instantaneous) | **I-2b:** Use algebra and graphical methods to link displacement, velocity, and acceleration |
| | **4A4.3:** Work Kinetic Energy Theorem | **I-2c:** Solve 1D kinematic problems with constant linear and angular acceleration. |
| | **4A5.3:** Conservation of energy | **I-4a:** Explain or predict the motion of translating and/or rotating objects using conservation of energy. |
| 2.4 | | **I-2b:** Use algebra and graphical methods to link displacement, velocity, and acceleration |

| Section | MCAT Foundational Concepts | Ohio TAG Learning Objectives |
|---|---|---|
|  |  | **I-2c:** Solve 1D kinematic problems with constant linear and angular acceleration. |
| 2.5 |  | **I-2b:** Use algebra and graphical methods to link displacement, velocity, and acceleration |
|  |  | **I-2c:** Solve 1D kinematic problems with constant linear and angular acceleration. |
| 2.6 | **4A1.4:** Speed, velocity (average and instantaneous) | **I-2b:** Use algebra and graphical methods to link displacement, velocity, and acceleration |
| 2.7 | **4A4.1:** Work done by a constant force: W = Fd cos$\theta$ |  |
| 2.8 |  | **I-2b:** Use algebra and graphical methods to link displacement, velocity, and acceleration |
| 3.1 |  | **I-2e:** Relate the motion of two objects relative to each other. |
| 3.2 | **4A2.3:** Newton's Third Law, forces equal and opposite<br><br>**4A3.1:** Vector analysis of forces acting on a point object | **I-3a:** Use Newton's laws of motion (1st, 2nd, and 3rd) to explain or predict the motion of translating and rotating objects. |
| 3.3 | **4A2.3:** Newton's Third Law, forces equal and opposite | **I-3a:** Use Newton's laws of motion (1st, 2nd, and 3rd) to explain or predict the motion of translating and rotating objects. |
| 3.4 |  | **I-4c:** Explain or predict the outcome of collisions. |
| 3.5 | **4A5.3:** Conservation of energy | **I-4a:** Explain or predict the motion of translating and/or rotating objects using conservation of energy.<br><br>**I-4c:** Explain or predict the outcome of collisions. |
| 3.6 |  | **I-4c:** Explain or predict the outcome of collisions. |
| 3.7 | **4A1.4:** Speed, velocity (average and instantaneous)<br>**4A4.1:** Work done by a constant force: W = Fd cos$\theta$ |  |
| 3.8 |  | **I-4c:** Explain or predict the outcome of collisions. |
| 3.9 | **4A4.1:** Work done by a constant force: W = Fd cos$\theta$ |  |
| 4.1 |  | **I-2e:** Relate the motion of two objects relative to each other. |
| 4.2 | **4A1.2:** Vectors, components | **I-2a:** Make accurate verbal, graphical, and mathematical descriptions of translational and rotational motion in one and two dimensions |

| Section | MCAT Foundational Concepts | Ohio TAG Learning Objectives |
|---|---|---|
| | | **I-2e:** Relate the motion of two objects relative to each other. |
| 4.3 | | **I-2a:** Make accurate verbal, graphical, and mathematical descriptions of translational and rotational motion in one and two dimensions |
| 4.4 | | **I-4c:** Explain or predict the outcome of collisions. |
| 4.5 | | **I-4c:** Explain or predict the outcome of collisions. |
| 4.6 | **4A1.3:** Vector addition<br><br>**4A3.1:** Vector analysis of forces acting on a point object | |
| 4.7 | **4A1.3:** Vector addition<br><br>**4A3.1:** Vector analysis of forces acting on a point object<br>**4A5.4:** Power, units | **I-2a:** Make accurate verbal, graphical, and mathematical descriptions of translational and rotational motion in one and two dimensions |
| 4.8 | **4A2.2:** Newton's Second Law (F=ma)<br><br>**4A3.1:** Vector analysis of forces acting on a point object | **I-2a:** Make accurate verbal, graphical, and mathematical descriptions of translational and rotational motion in one and two dimensions<br>**I-3a:** Use Newton's laws of motion (1st, 2nd, and 3rd) to explain or predict the motion of translating and rotating objects. |
| 4.9 | | **I-2a:** Make accurate verbal, graphical, and mathematical descriptions of translational and rotational motion in one and two dimensions |
| 5.1 | **4A2.2:** Newton's Second Law (F=ma)<br><br>**4A4.2:** Mechanical advantage | **I-3a:** Use Newton's laws of motion (1st, 2nd, and 3rd) to explain or predict the motion of translating and rotating objects. |
| 5.2 | **4A4.4:** Conservative forces<br>**4A5.2:** Potential Energy: PE = mgh (gravitational, local); PE = $1/2\ kx^2$ (spring) | |
| 5.3 | **4A5.3:** Conservation of energy | **I-4c:** Explain or predict the outcome of collisions.<br>**I-4a:** Explain or predict the motion of translating and/or rotating objects using conservation of energy. |
| 5.4 | **4A2.2:** Newton's Second Law (F=ma)<br><br>**4A2.4:** Friction, static and kinetic | **I-3a:** Use Newton's laws of motion (1st, 2nd, and 3rd) to explain or predict the motion of translating and rotating objects. |

| Section | MCAT Foundational Concepts | Ohio TAG Learning Objectives |
|---|---|---|
| 5.5 | **4A1.2:** Vectors, components<br><br>**4A2.4:** Friction, static and kinetic<br><br>**4A3.1:** Vector analysis of forces acting on a point object<br>**4A4.4**: Conservative forces<br>**4A5.3:** Conservation of energy | **I-2a:** Make accurate verbal, graphical, and mathematical descriptions of translational and rotational motion in one and two dimensions<br><br>**I-4a:** Explain or predict the motion of translating and/or rotating objects using conservation of energy. |
| 5.6 | | |
| 6.1 | **4A1.2:** Vectors, components | **I-2d:** Solve 2D projectile motion problems with start and end points at different heights. |
| 6.2 | | **I-2a:** Make accurate verbal, graphical, and mathematical descriptions of translational and rotational motion in one and two dimensions |
| 6.3 | | **I-2a:** Make accurate verbal, graphical, and mathematical descriptions of translational and rotational motion in one and two dimensions |
| 6.4 | **4A6.1:** Amplitude, frequency, phase | **I-2a:** Make accurate verbal, graphical, and mathematical descriptions of translational and rotational motion in one and two dimensions<br><br>**I-5a:** Explain or predict motion of objects in simple harmonic motion. |
| 6.5 | | **I-4b:** Explain or predict the motion of translating and/or rotation of objects in 1D using the conservation of momentum. |
| 7.1 | **4A3.2:** Torques, lever arms | **I-2b:** Use algebra and graphical methods to link displacement, velocity, and acceleration<br><br>**I-3a:** Use Newton's laws of motion (1st, 2nd, and 3rd) to explain or predict the motion of translating and rotating objects.<br><br>**I-2a:** Make accurate verbal, graphical, and mathematical descriptions of translational and rotational motion in one and two dimensions |
| 7.2 | **4A3.2:** Torques, lever arms | **I-3a:** Use Newton's laws of motion (1st, 2nd, and 3rd) to explain or predict the motion of translating and rotating objects.<br><br>**I-4e:** Determine the moment of inertia of rigidly connected masses. |
| 7.3 | | **I-2a:** Make accurate verbal, graphical, and mathematical descriptions of translational and rotational motion in one and two dimensions |

| Section | MCAT Foundational Concepts | Ohio TAG Learning Objectives |
|---|---|---|
| | | **I-2c:** Solve 1D kinematic problems with constant linear and angular acceleration. |
| | | **I-4a:** Explain or predict the motion of translating and/or rotating objects using conservation of energy. |
| | | **I-4e:** Determine the moment of inertia of rigidly connected masses. |
| 7.4 | | **I-4b:** Explain or predict the motion of translating and/or rotation of objects in 1D using the conservation of momentum. |
| | | **I-4e:** Determine the moment of inertia of rigidly connected masses. |
| 7.5 | **4A5.4:** Power, units | |
| 7.6 | **4A3.2:** Torques, lever arms | **I-3a:** Use Newton's laws of motion (1st, 2nd, and 3rd) to explain or predict the motion of translating and rotating objects. |
| | | **I-2c:** Solve 1D kinematic problems with constant linear and angular acceleration. |
| 7.7 | **4A2.5:** Center of mass | **I-4d:** Determine the center of mass of extended objects. |
| 7.8 | | **I-4f:** Use the parallel axis theorem in the solution of problems of extended objects of simple symmetries rotating about an axis that is not through their center of mass. |
| 7.9 | **4A3.2:** Torques, lever arms | **I-3a:** Use Newton's laws of motion (1st, 2nd, and 3rd) to explain or predict the motion of translating and rotating objects. |
| 8.1 | **4A5.3:** Conservation of energy | **I-4a:** Explain or predict the motion of translating and/or rotating objects using conservation of energy. |
| 8.2 | **4A6.1:** Amplitude, frequency, phase | **I-5a:** Explain or predict motion of objects in simple harmonic motion. |
| 8.3 | | **I-2a:** Make accurate verbal, graphical, and mathematical descriptions of translational and rotational motion in one and two dimensions |
| 8.4 | **4A6.1:** Amplitude, frequency, phase | **I-5a:** Explain or predict motion of objects in simple harmonic motion. |
| 8.5 | **4A6.1:** Amplitude, frequency, phase | **I-5a:** Explain or predict motion of objects in simple harmonic motion. |
| 8.6 | | **I-5a:** Explain or predict motion of objects in simple harmonic motion. |
| 9.1 | **4A6.1:** Amplitude, frequency, phase | **I-5a:** Explain or predict motion of objects in simple harmonic motion. |

| Section | MCAT Foundational Concepts | Ohio TAG Learning Objectives |
|---|---|---|
| | **4A6.2:** Transverse and longitudinal waves: wavelength and propagation speed | **I-5b:** Explain or predict mechanical wave phenomena in terms of frequency, wavelength, wave speed, and simple harmonic motion. |
| 9.2 | **4A6.2:** Transverse and longitudinal waves: wavelength and propagation speed | **I-5b:** Explain or predict mechanical wave phenomena in terms of frequency, wavelength, wave speed, and simple harmonic motion. |
| 9.3 | **4A6.2:** Transverse and longitudinal waves: wavelength and propagation speed | **I-5b:** Explain or predict mechanical wave phenomena in terms of frequency, wavelength, wave speed, and simple harmonic motion. |
| | **4D1.1:** Production of sound | **I-5d:** Describe standing wave patterns and how their confinement determines the wavelength allowed. |
| | **4D1.6:** Pitch | **I-5c:** Use superposition in solving problems with interference of two waves. |
| | **4D1.7:** Resonance in pipes and strings | |
| 9.4 | | **I-5e:** Describe and predict the addition of two waves of similar but not identical frequency aka the beating of waves |
| 9.5 | **4A6.2:** Transverse and longitudinal waves: wavelength and propagation speed | **I-5b:** Explain or predict mechanical wave phenomena in terms of frequency, wavelength, wave speed, and simple harmonic motion. |
| | **4D1.2:** Relative speed of sound in solids, liquids, and gases | |
| | **4B1.1:** Density, specific gravity | |
| 9.6 | **4B3.2:** Pressure, simple mercury barometer | |
| 10 | | **I-7g:** Describe properties of phases of matter (solid, liquid, and gas) and their transformations. |
| 10.1 | **4B1.1:** Density, specific gravity | **I-6a:** Describe how the pressure in a fluid varies as a function of depth in terms of the pressure relative to the surface of the fluid and the absolute pressure in the fluid. |
| | **4B1.3:** Hydrostatic pressure: Pascal's Law; Hydrostatic pressure; P $= \rho gh$ (pressure vs. depth) | |
| 10.2 | **4B1.3:** Hydrostatic pressure: Pascal's Law; Hydrostatic pressure; P $= \rho gh$ (pressure vs. depth) | **I-6a:** Describe how the pressure in a fluid varies as a function of depth in terms of the pressure relative to the surface of the fluid and the absolute pressure in the fluid. |
| | | **I-6e:** Explain the physics underlying hydraulic lifts. |

| Section | MCAT Foundational Concepts | Ohio TAG Learning Objectives |
|---|---|---|
| 10.3 | **4B1.2:** Buoyancy, Archimedes' Principle | **I-6d:** Predict the apparent weight of objects partially or fully immersed in a fluid. |
| | | **I-6b:** Predict whether an object will sink or float in a fluid. |
| 10.4 | **4B1.5:** Continuity equation ($A \cdot v =$ constant)<br><br>**4B1.8:** Bernoulli's equation<br>**4B1.9:** Venturi effect, Pitot tube | **I-6c:** Predict the variation in velocity and pressure as an incompressible fluid flows though pipes of varying diameter and height |
| 10.5 | **4B1.4:** Viscosity: Poiseuille Flow<br>**4B1.6:** Concept of turbulence at high velocities<br>**4B1.7:** Surface tension | |
| 10.6 | **4D2.1:** Concept of Interference; Young's double-slit experiment | **II-4i:** Relate the interference pattern of a double slit to the separation of slits and wavelength. |
| 11 | | **I-7g:** Describe properties of phases of matter (solid, liquid, and gas) and their transformations. |
| 11.1 | **4B3.4:** Ideal gas: Definition; Ideal Gas Law: PV = nRT; Boyle's Law: PV = constant; Charles' Law: V/T = constant; Avogadro's Law: V/n = constant | **I-7c:** Describe an ideal gas in terms of volume, pressure, temperature, and number of moles. |
| | **4B3.5:** Kinetic Molecular Theory of Gases: Heat capacity at constant volume and at constant pressure; Boltzmann's Constant | **I-7d:** Relate macroscopic and microscopic properties of matter using the kinetic theory of gases. |
| 11.2 | **4B3.4:** Ideal gas: Definition; Ideal Gas Law: PV = nRT; Boyle's Law: PV = constant; Charles' Law: V/T = constant; Avogadro's Law: V/n = constant | **I-7c:** Describe an ideal gas in terms of volume, pressure, temperature, and number of moles. |
| 11.3 | **4B3.2:** Pressure, simple mercury barometer | |
| 11.4 | **4B3.1:** Absolute temperature, K, Kelvin scale<br><br>**4B3.4:** Ideal gas: Definition; Ideal Gas Law: PV = nRT; Boyle's Law: PV = constant; Charles' Law: V/T = constant; Avogadro's Law: V/n = constant | **I-7f:** Describe and predict properties of gases using the ideal gas law |
| 11.5 | **4D1.1:** Production of sound<br><br>**4D1.2:** Relative speed of sound in solids, liquids, and gases | **I-5d:** Describe standing wave patterns and how their confinement determines the wavelength allowed. |

| Section | MCAT Foundational Concepts | Ohio TAG Learning Objectives |
|---|---|---|
| | **4D1.7:** Resonance in pipes and strings | |
| 11.6 | **4D1.3:** Intensity of sound, decibel units, log scale | |
| | **4D1.4:** Attenuation (damping) | |
| 11.7 | **4D1.5:** Doppler Effect: moving sound source or observer, reflection of sound from a moving object | **I-5f:** Solve problems where the frequency of a sound detected is affected by the motion of the source and/or the receiver relative to the medium (Doppler Effect). |
| | **4D1.8:** Ultrasound | |
| | **4D1.9:** Shock waves | |
| 11.8 | **4B3.6:** Deviation of real gas behavior from Ideal Gas Law: Qualitative; Quantitative (Van der Waals' Equation) | |
| | **4B3.7:** Partial pressure, mole fraction | |
| | **4B3.8:** Dalton's Law relating partial pressure to composition | |
| | **5E3.1:** Thermodynamic system - state function | |
| 12.1 | **5E3.6:** Measurement of heat changes (calorimetry), heat capacity, specific heat | **I-7a:** Describe the effect of heat on the properties of materials. |
| | **5E3.12:** Coefficient of expansion | **I-7h:** Transfer of heat by conduction, convection, and radiation. |
| | **5E3.7:** Heat transfer – conduction, convection, radiation | |
| 12.2 | **4B3.1:** Absolute temperature, K, Kelvin scale | **I-7b:** Describe/predict the transfer of energy between a system and its environment using the first law of thermodynamics |
| | **5E3.3:** First Law - conservation of energy in thermodynamic processes | **I-7h:** Transfer of heat by conduction, convection, and radiation. |
| | **5E3.7:** Heat transfer – conduction, convection, radiation | |
| | **5E3.2:** Zeroth Law – concept of temperature | |
| 12.3 | **5E3.3:** First Law - conservation of energy in thermodynamic processes | **I-7b:** Describe/predict the transfer of energy between a system and its environment using the first law of thermodynamics |
| | **5E3.13:** Heat of fusion, heat of vaporization | **I-7g:** Describe properties of phases of matter (solid, liquid, and gas) and their transformations. |
| | **5E3.14:** Phase diagram: pressure and temperature | |

| Section | MCAT Foundational Concepts | Ohio TAG Learning Objectives |
|---|---|---|
| 12.4 | **5E3.3:** First Law - conservation of energy in thermodynamic processes<br><br>**5E3.4:** PV diagram: work done = area under or enclosed by curve<br><br>**4B3.3:** Molar volume at $0°$C and 1 atm = 22.4 L/mol | **I-7b:** Describe/predict the transfer of energy between a system and its environment using the first law of thermodynamics |
| 12.5 | **5E3.4:** PV diagram: work done = area under or enclosed by curve | |
| 12.6 | **5E3.4:** PV diagram: work done = area under or enclosed by curve | **I-7e:** Predict the efficiency of a heat engine and its maximum efficiency. |
| 12.7 | **5E3.4:** PV diagram: work done = area under or enclosed by curve | **I-7e:** Predict the efficiency of a heat engine and its maximum efficiency. |
| 12.8 | **5E3.5:** Second Law – concept of entropy: Entropy as a measure of "disorder"; Relative entropy for gas, liquid, and crystal states | **I-7i:** Distinguish between reversible and irreversible processes. |

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